O-minimal Chow's theorem and its applications in Hodge Theory

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Abstract

We review recent applications of o-minimal theory in Hodge theory. One of key theorems is o-minimal Chow theorem which proves the algebraicity of an analytic subset $X \subseteq \mathbb{C}^n$. Unlike the usual Chow theorem X need not be an analytic subvariety of the projective set.

Proving the definability of period maps and with o-minimal Chow theorem Bakker, Klingler, and Tsimerman showed the algebraicity of Hodge loci in 2018.

On the other hand, Bakker, Brunebarbe and Tsimermam proved in 2018 an O-minimal version of GAGA type theorem which generalizes o-minimal Chow theorem. Applying the definability theorem of period maps, they solved the long-standing Griffiths conjecture with their o-minimal GAGA theorem.

1 Chow's Theorem

First recall

Theorem 1 (Chow's theorem)**.** Any analytic subvariety of projective space $\mathbb{P}^n(\mathbb{C})$ is algebraic. More generaly any closed analytic subset Z of a complex projective variety S is algebraic.

Remark 2. This fails when S is quasi-projective variety i.e., a locally closed subset of a projective variety.

We follow Mumford's arguments in Chapter 4 of [M].

Definition 3 (Analytic set)**.** *Let $U \subseteq \mathbb{C}^n$ be open, and $X \subseteq U$ be closed. For any $x \in X$ there exist an open neighborhood $U_x \subseteq U$ of x and finitely many analytic functions $f_1, \cdots, f_k$ over U_x and*

$$X \cap U_x = \{y \in U_x \mid f_1(y) = \cdots = f_k(y) = 0\}$$

The X is said to be analytic set. This is the external characterization of analyticity.

Definition 4 (n-dimesional complex variety)**.** *Hausdorf sapace M is called a n-dimensional complex variety when it is a union of charts U_α and each chart is homeomorphic to an open subset V_α of $\mathbb{C}^n$ and functions between V_α and V_β are analytic, i.e,*

$$\begin{array}{ccccc} \phi_\alpha & : & U_\alpha & \xrightarrow{\approx} & V_\alpha, \qquad \bigcup_\alpha U_\alpha = M \\ & & \bigcap \textit{open} & & \bigcap \textit{open} \\ & & M & & \mathbb{C}^n \end{array}$$

and each $\phi_\beta \circ \phi_\alpha^{-1} : V_\alpha \longrightarrow V_\beta$ is analytic.

Definition 5 ($*$-analytic set)**.** *Let $U \subseteq \mathbb{C}^n$ be open and $X \subseteq U$ be closed. If we have*

$$X = X^{(r)} \cup X^{(r-1)} \cup \cdots \cup X^{(0)}$$

where each $X^{(i)}$ is i-dimensional complex subvariety of U and

$$\overline{X^{(i)}} \subseteq X^{(i)} \cup X^{(i-1)} \cup \cdots \cup X^{(0)}$$

Then X is said to be $*$-analytic. This is the internal characterization of analyticity.

The following is a key theorem which asserts the equivalence of two definitions of analyticity.

Theorem 6. *analytic* $\Longleftrightarrow$ $*$-*analytic*

Proof : (of Chow's theorem).

Suppose X is an analytic subvariety of $\mathbb{P}^n(\mathbb{C})$. Let

$$\pi : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{P}^n(\mathbb{C})$$

be a projection and $CX := \pi^{-1}(X)$ be a cone over X. Then CX is $*$-analytic with strata of one higher dimension than those of X..

Consider $Z := CX \cup \{0\} \subseteq \mathbb{C}^{n+1}$. Then Z is closed hence *-nalytic with $\{0\}$ its zero-dimensional strata. By Theorem 6, Z is analytic. Hence Z is the zero set of finitely many analytic functions $f_1, \cdots, f_k$ near a convex neighborhood Δ_0 of 0..

Note that Z is invariant under scalar multiplication $\overline{x} \to \lambda \cdot \overline{x}$ (since $X \subseteq \mathbb{P}^n$) .

For any $\lambda \in \mathbb{C}$, $|\lambda| \leq 1$, set $f_i^\lambda(x_0, \cdots, x_n) := f_i(\lambda x_0, \cdots, \lambda x_n)$. Then Z is also the zero set of $f_i^\lambda(\overline{x})$'s, i.e.,

$$Z = \left\{ \overline{x} \in \Delta_0 \;\middle|\; \bigwedge_{i=1}^{k} f_i^\lambda(\overline{x}) = 0 \right\}.$$

Write each analytic function as

$$f_i(\overline{x}) = \sum_\alpha c_\alpha^{(i)} \cdot \overline{x}^\alpha \quad \text{(infinite series)}$$

where $\alpha = (\alpha_0. \cdots, \alpha_n)$. Further let

$$f_{i,r}(\overline{x}) := \sum_{|\alpha|=r} c_\alpha^{(i)} \cdot \overline{x}^\alpha$$

then for all λ and $\overline{x} \in Z$ we have

$$f_i^\lambda(\overline{x}) = \sum_{r=0}^{\infty} \lambda^r \cdot f_{i,r}(\overline{x}) = 0$$

Thus for any λ, s, and $\overline{x} \in Z$ we have that

$$\left(\frac{d}{d\lambda}\right)^s f_i^\lambda(\overline{x}) = \sum_{\lambda=s}^{\infty} \left(\begin{array}{c} r \\ s \end{array} \right) \lambda^{r-s} \cdot f_{i,r}(\overline{x}) = 0.$$

Setting $\lambda = 0$, for all $\overline{x} \in Z$ we have $f_{i,r}(\overline{x}) = 0$. It follows that Z is the zero set of possibly infinitely many homogeneous polynomials

$$f_{i,r}(\overline{x}), \quad (1 \leq i \leq k,\ 1 \leq r < \infty).$$

We now apply Hilbert Basis Theorem to choose finitely many polynomials from $\{f_{i,r}\}$ such that

$$Z = V(\cdots, f_{i,r}, \cdots).$$

Therefore X is a finite union of algebraic varieties. ■

Key points

- Z is invariant under scalar multipication since $X \subseteq \mathbb{P}^n$.
- Hilbert Basis Theorem guaratees that Z is an algebraic variety.

2 O-minimal Chow's theorem

Using the fact that $\mathbb{C} \cong \mathbb{R} \times \mathbb{R}$, we can study complex analysis in o-minimal settings.

Example 7. The following are all o-minimal.

- $\mathbb{R}_{\mathrm{an}} := (\mathbb{R}, +, \cdot, <, 0, 1, \{\text{all restricted analytic functions}\})$ (van den Dries)
- $\mathbb{R}_{\exp} := (\mathbb{R}, +, \cdot, <, 0, 1, \exp)$ (Wilkie)
- $\mathbb{R}_{\mathrm{an},\exp} := \mathbb{R}_{\mathrm{an}} + \mathbb{R}_{\exp}$ (van den Dries, Miller)

From now on in this note, when we say a subset $X \subseteq \mathbb{C}^n$ is definable, we mean that X is definable in an o-minimal structure whose underlyong set is $\mathbb{C}^n$.

Theorem 8 (Corollary 4.5, [PS])**.** Let X be a definable $\mathbb{C}$-analytic subset of $\mathbb{C}^n$. Then X is an algebraic subset of $\mathbb{C}^n$.

There are several different proofs. We discuss here one of key lemmas needed for the proof of o-minimal Chow theorem presented in [YZ] which is the second proof in [B].

Lemma 9. Any definable holomorphic function $f : \mathbb{C}^n \to \mathbb{C}$ is a polynomial.

Proof : By induction on n. When $n = 1$, suppose that f is a definable entire function. If f is bounded then f is a constant function by Liouville. So suppose f is not bounded nor a polynomial. Consider $g(z) := f(1/z)$. Then $z = 0$ is not a pole of $g(z)$ but an essential singularity. Hence $f(z)$ has an essential singularity at infinity. By Picard Big Theorem in an arbitrary small neiborhood Δ of the essential singurarity, for almost all $a \in \mathbb{C}$, $f^{-1}(a)$ is an infinite set. For $a = 0$ we have that $f^{-1}(0)$ is an infinite discrete set since f cannot have accumerated zero, contradicting the o-minimalty of the underlying structure.

Assume the statement is true for $n - 1$. Let $f : \mathbb{C}^n \to \mathbb{C}$ be definable and holomorphic. Let $(z, w) \in \mathbb{C} \times \mathbb{C}^{n-1}$ and consider the Taylor expansion of $f(z, w)$ along with z-axis;

$$f(z, w) = \sum_{k=0}^{\infty} g_k(w) z^k, \text{ where } g_k(w) = \frac{1}{k!} \cdot \frac{\partial^k f}{\partial z^k}(0, w).$$

Each $g_k(w) : \mathbb{C}^{n-1} \to \mathbb{C}$ is holomorphic, hence a polynomial by induction hyothesis,

We need to show that there are only finitely many such $g_k(w)$. Note first that for any $w \in \mathbb{C}^{n-1}$, the unary funcion $f_w(z) := f_(z, w)$ is a complex polynomial. Now consider the graph $\Gamma(f) \subseteq \mathbb{C}^{n+1}$ of f. Project $\Gamma(f)$ on to the last n-coordinates. Then the fiber at $(w, v) \in \mathbb{C}^{n-1} \times \mathbb{C}$ is the set

$$F_{w,v} = \{z \in \mathbb{C} \mid f(z, w) = v\}.$$

When $f_w \not\equiv 0$, then $|F_{w,v}| \leq \deg(f_w)$. On the other hand if $f_w \equiv 0$, then $F_{w,0} = \mathbb{C}$ and $F_{w,v} = \emptyset$ $(v \neq 0)$.

Applying the cell decomposition theorem we see that the number of connected components of the projection are uniformly bounded, Thus for any $w \in \mathbb{C}^{n-1}$, $\deg(f_w)$ is uniformly bounded, say by $N \in \mathbb{N}$. Thus for $k > N$ we have $g_k \equiv 0$. It follows that

$$f(z, w) = \sum_{k=0}^{N} g_k(w) z^k$$

and $f(z, w)$ is a polynomial. ■

Unlike the classical Chow theorem, in o-minimal settings analytic subset $X \subset \mathbb{C}^n$ need not be an analytic subvariety of the projective space. Lemma 9 plays an important role in showing the algebraicity of X. By the tameness of o-minimal structure the definable holomorphic functions are restricted to polynomial functions.

In [PS], Peterzil and Starchenco give another proof of O-mimimal Chow Theorem. First they show;

Theorem 10 (Theorem 4.4. [PS])**.** *Let M be a complex manifold and $E \subseteq M$ a $\mathbb{C}$-analytic subset of M (of arbitrary dimension). If A is a $\mathbb{C}$-analytic subset of $M \setminus E$ which is also subanalytic in M then $Cl(A)$ is a $\mathbb{C}$-analytic subset of M.*

Proof : (O-mimimal Chow Theorem)

Since $\mathbb{C}^n$ is obtained from $\mathbb{P}^n(\mathbb{C})$ by removing a $\mathbb{C}$-analytic set, it follows from Theorem 10 that the closure of X in $\mathbb{P}^n(\mathbb{C})$ is a $\mathbb{C}$-analytic subset of $\mathbb{P}^n(\mathbb{C})$. Now apply the classical Chow's Theorem. ■

Theorem 11 (Theorem ([PS]). *Let S be a quasi-projective complex variety and $Z \subset S$ be a closed analytic subset. If there exists an o-minimal structure expanding $\mathbb{R}_{\text{an}}$ in which Z is definable, then Z is algebraic.*

Corollary 12. *Note that every closed analytic subset of a projective variety is $\mathbb{R}_{\text{an}}$-definable. Therefore the classical Chow theorem is a corollary of this o-minimal version of Chow theorem.*

3 O-minimal GAGA

Before stating the O-minimal GAGA, we need some definitions.

3.1 Definable topological space

Definition 13. $(\mathcal{X}, \{U_i\}, \varphi_i)$ is said to be a definable topological space if we have

- $\mathcal{X}$ is a topological space,
- $\{U_i\}$ is a finite open covering of $\mathcal{X}$,
- each $\varphi_i : U_i \to V_i \subset \mathbb{R}^n$ is a homeomorphisms,
- all V_i and $V_{ij} = \varphi_i(U_i \cap U_j)$ are definable,
- the maps $\varphi_i \circ \varphi_j^{-1} : V_{ij} \to V_{ij}$ are definable.
- (U_i, φ_i) are called charts.
- A morphism between two such topological spaces is a continuous map which is definable on the given charts.

3.2 Definable geometric quotient

Let $\mathcal{X}$ be a definable topological space and $\mathcal{R} \subset \mathcal{X} \times \mathcal{X}$ be a closed definable equivalence relation.

A *definable geometric quotient* $\mathcal{X}/\mathcal{R}$ is a surjective morphism $p : \mathcal{X} \to \mathcal{Y}$ of definable topological spaces such that the fibres of p are the equivalence classes of $\mathcal{R}$ and $\mathcal{Y}$ carries the quotient topology.

If such a quotient exists, then it is unique. We say that $\mathcal{R}$ is definably proper if the preimages by the projections of definable compact subsets of $\mathcal{X}$ are compact subsets of $\mathcal{R}$.

Theorem 14 (van den Dries 1998)**.** If $\mathcal{X}$ is a definable topological space and $\mathcal{R}$ is a closed definably proper equivalence relation, then the geometric quotient $\mathcal{X}/\mathcal{R}$ exists.

3.3 Definable analytic space

Let $\mathcal{X}$ be a definable topological space and $\underline{\mathcal{X}}$ be definable site, objects are definable subsets $U \subset \mathcal{X}$ and admissible coverings are finite coverings.

Definition 15 (Definable analytic space)**.** *A definable analytic space is a pair $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ consisting of a definable topological space $\mathcal{X}$ and a sheaf $\mathcal{O}_{\mathcal{X}}$ on $\underline{\mathcal{X}}$.*

NB: In the above definition, analyticity of the space is hidden in the sheaf $\mathcal{O}_{\mathcal{X}}$.

Theorem 16 ([BBT])**.** *Geometric quotients also exist, even in the category of definable analytic spaces, if $\mathcal{R} \subset \mathcal{X} \times \mathcal{X}$ is an étale closed definable equivalence relation (i.e. the projection maps are open and locally an isomorphism onto their images)*

3.4 Géométrie algébrique et géométrie analytique

In the introduction of the famous GAGA paper, Serre explains the GAGA principle as follows.

Toute variété algébrique X sur le corps des nombres complexes peut être munie, de façon canonique, d'une structure d'espace analytique ; tout faisceau algébrique cohérent sur X détermine un faisceau analytique cohérent. Lorsque X est une variété projective, nous montrons que, réciproquement, tout faisceau analytique cohérent sur X peut être obtenu ainsi, et de façon unique ; de plus, cette correspondance préserve les groupes de cohomologie. Ces résultats contiennent comme cas particuliers des théorémes classiques de Chow et Lefschetz, et permettent d'aborder la comparaison entre espaces fibrés algébriques et espaces fibrés analytiques de base une variété algébrique projective. [1]

In English; for any algebraic complex variety X, we can define canonically an analytic space in such a way that for each coherent algebraic sheaf we can define a coherent analytic sheaf. If X is a projective variety, we can do the contrary: each coherent alalytic sheaf comes from a coherent algebraic sheaf.

The following theorem of Serre generalizes in fact theorems of Chow and Lefschetz;

Theorem 17 (Serre, 1955)**.** *Let X be a proper complex algebraic variety and X^{an} be the associated analytic space. Then the categories of coherent sheaves $\mathrm{Coh}(X)$ and $\mathrm{Coh}(X^{\mathrm{an}})$ are equivalent.*

3.5 O-minimal GAGA

In O-minimal settings, Bakker, Brunebarbe and Tsimermam proved in 2018 an O-minimal version of GAGA type theorem;

Theorem 18 (Thm 1.3 [BBT])**.** *Let X be a separated algebraic space of finite type over $\mathbb{C}$ and X^{def} be the associated definable analytic space. Then the definabilization functor*

$$\mathrm{Def} : \mathrm{Coh}(X) \to \mathrm{Coh}(X^{\mathrm{def}})$$

is fully faithful, exact, and its essential image is closed under subobjects and quotients.

Corollary 19. *O-minimal Chow's theorem also holds for algebraic spaces.*

This corollary will be used repeatedly in the proof of Griffiths conjecture.

4 Hodge Theory

We now discuss basic notions of Hodge Theory.

4.1 Legendre family of ellipitic curves

Consider the Legendre family of elliptic curves $\mathcal{E}_\lambda$ defined by the equation

$$y^2 = x(x-1)(x-\lambda) \quad (\lambda \in \mathbb{C} \setminus \{0,1\})$$

Each $\mathcal{E}_\lambda$ defines a Riemann surface.

For all $\lambda \neq 0, 1$,
$\mathcal{E}_\lambda$ are isomorphic as *differential* manifolds, they are tori.

However

Theorem 20 (Theorem 1.1.1, [CSP], p.4))**.** *Suppose that $\lambda \neq 0, 1$. Then there is an $\epsilon > 0$ such that for all λ' within distance ϵ from λ the Riemann surfaces $\mathcal{E}_\lambda$ and $\mathcal{E}_{\lambda'}$ are not isomorphic as complex manifolds.*

So λ encodes the complex structure of $\mathcal{E}_\lambda$.

4.2 Period maps

We can define structures called Hodge structures in the Legendre family of Riemann surfaces (elliptic curves) $\mathcal{E}_\lambda$. Period maps describe properties of elliptic curves.

Similarly, in general, period maps describe how Hodge structures vary on a family of smooth projective varieties.

[1] Jean-Pierre Serre, Annales de l'Institut Fourier, Vol 6 (1956), p.1-42

4.2.1 Hodge locus

The *Hodge locus of* the variation of Hodge structures is defined as the union $\mathrm{HL}(S,\mathcal{V}) \subset S$ of all preimages which are not the whole S of Mumford-Tate subvarieties under the period map, i.e., $\mathrm{HL}(S,\mathcal{V})$ is a countable union of irreducible closed analytic subvarieties of S.

As soon as one leaves the realm of abelian varieties, these arithmetic quotients are complex analytic spaces which almost never carry an algebraic structure, so the holomorphic, non-algebraic period maps could a priori behave wildly at infinity.

The spaces

$$\Gamma\backslash \mathrm{D} = \Gamma\backslash G(\mathbb{R})/M$$

that are targets of period maps are examples of *arithmetic quotients.*

Theorem 21 ([BKT]). *The real analytic manifold $S_{\Gamma,G,M} = \Gamma\backslash G/M$ can be endowed with a functorial structure of $\mathbb{R}^{\mathrm{alg}}$-definable manifold such that, for each Siegel set $\mathfrak{S} \subset G/M$, the map*

$$\pi|_{\mathfrak{S}} : \mathfrak{S} \to S_{\Gamma,G,M}$$

is $\mathbb{R}^{\mathrm{alg}}$-definable.

4.3 Definability of period maps

Theorem 22 ([BKT]). *Let S be a smooth connected quasi-projective complex variety and $\mathcal{V}$ be a polarised variation of pure Hodge structures of weight k on S.*

Then the associated period map

$$\Phi : S \to S_{\Gamma,G,M}$$

is $\mathbb{R}_{\mathrm{an,exp}}$-definable.

Theorem 23 ([BKT]). *All period maps have tame geometry: they are definable in the o-minimal structure $\mathbb{R}_{\mathrm{an,exp}}$ relatively to a natural semialgebraic structure on $S_{\Gamma,G,M}$.*

the Hodge conjecture

If the variation of Hodge structures $\mathcal{V}$ arises from a family of smooth projective varieties then the Hodge locus $\mathrm{HL}(S,\mathcal{V})$ is a countable union of closed irreducible algebraic subvarieties of S.

Theorem 24 (Cattani, Deligne, and Kaplan (1995)). *The conjecture holds unconditionally and for all variations of Hodge structures, whether they have geometric origin or not.*

As a corollary of the definability of period maps, Bakker, Klingler, and Tsimerman obtain a new proof of this theorem.

Theorem 25 ([BKT]). *Let $\mathcal{V}$ be a polarised variation of pure Hodge structures of weight k on a smooth connected quasi-projective complex variety S. Then the Hodge locus $\mathrm{HL}(S,\mathcal{V}) \subset S$ is a countable union of closed irreducible algebraic subvarieties.*

Proof : Let $\Phi : S \to S_{\Gamma,G,M}$ be the period map associated with the variation of Hodge structures. We show that

$$\mathrm{HL}(S,\mathcal{V}) = \bigcup_{Y:\ \text{Mumford-Tate subvariety of } S_{\Gamma,G,M}} \Phi^{-1}(Y).$$

It suffices to prove that the preimage $W = \Phi^{-1}(Y)$ of such a $Y \subset S_{\Gamma,G,M}$ is algebraic.

- Mumford-Tate subvariety $Y \subset S_{\Gamma,G,M}$ is of the form $S_{\Gamma',G',M'}$ and hence $\mathbb{R}^{\mathrm{alg}}$-definable.
- It follows from the definability of the period map that the subset $W = \Phi^{-1}(Y) \subset S$ is $\mathbb{R}_{\mathrm{an,exp}}$-definable.
- W is also a complex analytic subvariety, so the o-minimal Chow theorem implies that W is algebraic.

■

4.4 Summarly

New proof of Theorem (Cattani,Deligne, and Kaplan (1995))

Definability of period maps (BKT 2018) + o-minimal Chow Thm (PS 2009)

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the algebaicity of Hodge loci (BKT 2018)

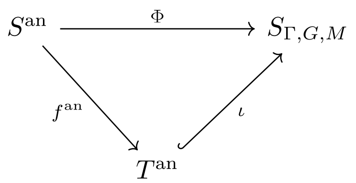
4.5 Proof of Griffiths conjecture

Around fifty years ago, P. A. Griffiths (1970) conjectured that period maps have quasi-projective images and proved it when S is compact. Then Sommese (1978) showed that, up to a proper modification, the image is algebraic.

Equipped with the definability of period maps and o-minimal GAGA theorem, Bakker, Brunebarbe, and Tsimerman (2018) proved a long-standing conjecture of Griffiths that images of period maps are quasi-projective algebraic varieties.

The main result of Bakker, Brunebarbe, and Tsimerman (2018) is the general case of this conjecture.

Theorem 26 ([BBT]). *Let S be a smooth connected quasi-projective complex variety and $\Phi : S \to S_{\Gamma,G,M}$ be a period map. Then there exists a unique dominant morphism of complex algebraic varieties $f : S \to T$ and a closed immersion $\iota : T^{an} \to S_{\Gamma,G,M}$ of analytic spaces such that Φ factors as:*



Moreover, the variety T is quasi-projective.

4.5.1 Proof of Griffiths conjecture BBT 2018

Definability of period maps (BKT 2018) + o-minimal GAGA (BBT 2018)

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Proof of Griffiths conjecture (BBT 2018)

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