Locally definable $C^\infty G$ manifolds

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1 Introduction

Let $\mathcal{M} = (\mathbb{R}, <, +, \cdot, e^x, \dots)$ be an exponential o-minimal expansion of the field $\mathbb{R}$ of real numbers with C^∞ cell decomposition. By [7], there exist uncountably many o-minimal expansions of $\mathbb{R}$.

We refer [1], [2] for definable maps, definable sets. By [8], a generalization of o-minimal structures of $\mathbb{R}$ are studied. There exists an o-minimal expansion of $\mathbb{R}$ such that it does not admit C^∞ cell decomposition ([6]). Everything is considered in $\mathcal{M}$ unless otherwise stated.

We study approximations of locally definable $C^r G$ maps by locally definable $C^\infty G$ maps.

2 Locally definable $C^\infty G$ manifolds

Definition 2.1. *(1) A subset X of $\mathbb{R}^n$ is locally definable set if for each point $x \in X$, there exists an open ball A such that $X \cap A$ is definable.*

Note that every open subset of $\mathbb{R}^n$ is locally definable.

(2) Let $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$ be locally definable sets. A continuous map $f : X \to Y$ is locally definable if the graph of f $(\subset X \times Y \subset \mathbb{R}^n \times \mathbb{R}^m)$ is a locally definable set. A locally definable map $f : X \to Y$ is a locally definable

2010 *Mathematics Subject Classification.* 14P10, 03C64.
Key Words and Phrases. O-minimal, locally definable C^∞ manifolds.

homeomorphism if there exists a locally definable map $f' : Y \to X$ such that $f \circ f' = id_Y, f' \circ f = id_X$.

(3) Assume that X, Y are open. A locally definable map $f : X \to Y$ is a locally definable C^r map if it is a C^r map. A locally definable C^r map $f : X \to Y$ is a locally definable C^r diffeomorphism if there exists a locally definable C^r map $f' : Y \to X$ such that $f \circ f' = id_Y, f' \circ f = id_X$.

Remark that the complement of a locally definable set and the projection image of a locally definable set are not always locally definable ([5]).

Definition 2.2. *(1) A group G is a definable group if G is a definable set and the group operations $G \times G \to G$ and $G \to G$ are definable.*

(2) Let G be a definable group. A pair (X, ϕ) consisting a definable set X and a G action $\phi : G \times X \to X$ is a definable G set if ϕ is definable. We simply write X instead of (X, ϕ).

(3) A definable map $f : X \to Y$ between definable G sets is a definable G map if for any $x \in X, g \in G$, $f(gx) = gf(x)$. A definable G map is a definable G homeomorphism if it is a homeomorphism.

Definition 2.3. *A Hausdorff space X is an n-dimensional locally definable C^r manifold if there exist a countable open cover $\{U_i\}_{i=1}^\infty$ of X, countable open sets $\{V_i\}_{i=1}^\infty$ of $\mathbb{R}^n$, and a countable collection of homeomorphisms $\{\phi_i : U_i \to V_i\}_{i=1}^\infty$ such that for any i, j with $U_i \cap U_j \neq \emptyset$, $\phi_i(U_i \cap U_j)$ is definable and open, and $\phi_j \circ \phi_i^{-1} : \phi_i(U_i \cap U_j) \to \phi_j(U_i \cap U_j)$ is a definable C^r diffeomorphism. This pair $(\{U_i\}_{i=1}^\infty, \{\phi_i : U_i \to V_i\}_{i=1}^\infty)$ of sets and homeomorphisms is called a locally definable C^r coordinate system.*

Definition 2.4. *A pair (X, ϕ) consisting a locally definable C^r manifold and a group action $\phi : G \times X \to X$ such that ϕ is a locally definable C^r map is a locally definable $C^r G$ manifold.*

The following is our result.

Theorem 2.5 ([4])**.** *Let G be a compact definable C^∞ group. Let X, Y be an affine locally definable $C^\infty G$ manifold and r a positive integer. Suppose that $f : X \to Y$ is locally definable $C^r G$ map. Then f is approximated by locally definable $C^\infty G$ map with respect to the Whitney C^r topology*

This is a generalization of the following theorem.

Theorem 2.6 ([3]). *Let G be a compact definable C^∞ group. Let X, Y be an affine definable $C^\infty G$ manifold and r a positive integer. Suppose that $f : X \to Y$ is definable $C^r G$ map. Then f is approximated by definable $C^\infty G$ map with respect to the definable C^r topology*

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