Some Remarks on Automorphisms with a Single Orbit

Hirotaka Kikyo
Graduate School of System Informatics
Kobe University

Abstract

Suppose $l > m \geq 3$. The random m-hypergraphs omitting the complete m-hyper-subgraphs of size l admit automorphisms with a single orbit.

1 Introduction

This is a part of a work in progress with Akito Tsuboi.

W. Henson [3] showed that any random graphs which have no complete subgraphs of size $p \geq 4$ have no automorphisms with a single orbit. We show that the situation will be different in the hypergraph case.

Let $m < l < \omega$, and let R be an m-ary relation symbol. By an m-hypergraph, we mean an R-structure where R is symmetric and irreflexive. For a finite m-hypergraph X, $R(X)$ denotes the set of hyperedges on X, and $\mathrm{e}(X)$ denote the number of hyperedges on X. So, we have $R(X) = \{A \in [X]^m : X \models R(A)\}$ and $\mathrm{e}(X) = |R(X)|$.

Let $\mathcal{H}_l^m$ denote the class of all countable m-hypergraphs which have no complete m-hyper-subgraphs of size l. We have $X \in \mathcal{H}_l^m$ if and only if whenever $A \subseteq X$ with $|A| = l$ then $\mathrm{e}(A) < \binom{l}{m}$.

Let $\mathcal{K}$ be an infinite class of finite R-structures.

M is a *random* structure for $\mathcal{K}$ if

(1) M is countable,

(2) whenever $A \subset_{finite} M$ then $A \in \mathcal{K}$, and

(3) whenever $A \subset_{finite} M$, $A \subset B \in \mathcal{K}$ and B is finite then there is an L-embedding $f : B \to M$ with $f(x) = x$ for $x \in A$.

A random structure for $\mathcal{K}$ is also known as a Fraïssé limit of $\mathcal{K}$ and a generic structure for $\mathcal{K}$.

If $\mathcal{K}$ has HP, JEP, and AP then there is a random structure for $\mathcal{K}$.

Let A, B, C and D be R-structures. We say that D is a *free amalgam* of B and C over A if $D = B \cup C$, $B \cap C = A$ as the sets of domains, and $R(D) = R(B) \cup R(C)$. We say that a class of finite R-structures $\mathcal{K}$ has the *free amalgamation property* (FAP in short) if whenever D is a free amalgam of B and C over A with A, B, $C \in \mathcal{K}$ then $D \in \mathcal{K}$.

If $2 \leq m < l$ then $\mathcal{H}_l^m$ has HP, JEP, and FAP.

Suppose that an R-structure M has an automorphism σ with a single orbit. Then, M must be countable. Moreover, for any $b \in M$, we have $M = \{\sigma^n(b) : n \in \mathbb{Z}\}$. Therefore, we can assume that $\mathrm{dom}(M) = \mathbb{Z}$ and $\sigma(x) = x + 1$ (a shift function). Note that

$$M \models R(a, b) \iff R(a + p, b + p)$$

for any $a, b, p \in \mathbb{Z} = \mathrm{dom}(M)$.

2 Henson's Theorem

The random structures for $\mathcal{H}_l^2$ with $2 < l$ are called Henson graphs, denoted by H_l.

Fact 2.1 (Henson). (1) *The random graph has an automorphism with a single orbit.*

(2) *H_3 has an automorphism with a single orbit.*

(3) *H_l with $l \geq 4$ has no automorphisms with a single orbit.*

We prove (3) with $l = 4$. Suppose H_4 has an automorphism σ with a single orbit. We can assume that $H_4 = (\mathbb{Z}, R)$ and $\sigma(x) = x + 1$.

Let K_4 denote the complete graph with 4 vertices. Consider a new path of length 1 with the ends $0 \in H_4$ and b'. Since this graph is K_4-free, it can be embedded into H_4 over 0. Let b be the isomorphic image of b' in H_4. So, b is adjacent to 0. Note that $-b$ is also adjacent to 0 because we can shift 0 and

b by $-b$. Similarly, we can choose $a \in H_4$ such that a is adjacent to 0, but not adjacent to b and $-b$. Then $(0, a, a+b, b)$ is a 4 cycle with no diagonals. Since 0 and b are adjacent in H_4, a and $a+b$ are adjacent by shifting by a. Similarly, since 0 and a are adjacent in H_4, b and $a+b$ are adjacent by shifting by b.

We can choose c such that c is adjacent to 0, a, b, and $a+b$. Then $\{0, a, c, c-b\}$ forms a tetrahedron K_4. Since c and $a+b$ are adjacent, $c-b$ and a are adjacent by shifting by $-b$. Since 0 and $-b$ are adjacent, c and $c-b$ are adjacent by shifting by c.

3 Random structures for $\mathcal{H}_l^m$

We show that automorphisms with a single orbit exist on random hypergraphs.

Theorem 3.1. *Assume $l > m \geq 3$. Let M be a random structure for $\mathcal{H}_l^m$. Then M has an automorphism with a single orbit.*

Proof. Let $\mathbb{Z}$ be the domain of the structure, and σ the shift function ($\sigma(x) = x+1$).

We are going to construct a relation R on $\mathbb{Z}$ such that $(\mathbb{Z}, R)$ is a random structure for $\mathcal{H}_l^m$ and σ will be an R-automorphism of $(\mathbb{Z}, R)$.

Enumerate all tuples (A, D) as (A_n, D_n), where $A \in \mathcal{H}_l^m$, and D is a finite subset of $\mathbb{Z}$ with $|D| = |A| - 1$. For a finite subset $F \subset \mathbb{Z}$, we define the width of F to be $\mathrm{wd}(F) = \max F - \min F$.

We inductively define R_n, $w_n \in \omega$ and $d_n \in \mathbb{Z}$ satisfying the following:

1. $R_0 \subset R_1 \subset \cdots \subset R_n$; R_n is shift-closed.
2. $2(w_i + \mathrm{wd}(D_{i+1})) < w_{i+1}$ $(i < n)$.
3. Let $B \in R_n - R_{n-1}$ be a new hyperedge and $b = \max(B)$. If $x \in B - \{b\}$ then $w_{n-1} < b - x \leq w_n$.
4. The hypergraph $(\mathbb{Z}, R_n)$ belongs to $\mathcal{H}_l^m$.
5. If $(D_n, R_{n-1}) \cong A'_n$, where A'_n denotes the substructure obtained from A_n by omitting the last point (of the enumeration of A_n), then this isomorphism can be extended to $(D_n \cup \{d_n\}, R_n) \cong A_n$.

We consider the n-th stage.

To clarify our argument, we introduce terminology. For x and y in $\mathbb{Z}$, we say they have a *short distance* at the n-th stage if $|x-y| \leq w_{n-1}$, otherwise, we refer to it as a *long distance*. We also say that x is *close to* y if x and y have a short distance, otherwise we say that x is *far from* y.

We map d to $d_n \in \mathbb{Z}$ so that d_n is far from elements in D_n, and add new hyperedges to form R_n so that $(D_n \cup \{d_n\}, R_n) \cong A_n$.

Suppose, for a contradiction, that $A = \{a_1, a_2, \ldots, a_l\}$ is a substructure of $(\mathbb{Z}, R_n)$ with $a_1 < a_2 < \ldots < a_l$ and

$$\mathrm{e}(A) = \binom{l}{m}.$$

Since a new hyperedge must exist in A, it follows that a_1 and a_l have a long distance. By shifting, we can assume $a_l = d_n$.

Note that If $1 \leq i < j \leq l$ then a_i and a_j are adjacent in $A = \{a_1, \ldots, a_l\}$ in the sense that there is $B \in [A]^m$ with $a_i, a_j \in B$ and $A \models R(B)$.

Claim 1. *a_l and a_i ($i < l$) are adjacent with new hyperedges in $R_n(A) - R_{n-1}(A)$.*

Suppose the claim is false. Then a_l and some a_j are adjacent with an old hyperedge in R_{n-1}. Hence, a_l is close to a_j at the stage n. But a_l is far from a_1. Choose $B \in [A]^m$ with $a_1, a_j, a_l \in B$. Then $B \notin R_{n-1}$ because a_l is far from a_1 but also $B \notin R_n - R_{n-1}$ because a_l is close to a_i. This means that A is not complete m-hypergraph at the stage n. We have the claim.

Since we are assuming $a_l = d_n$, we have $a_i \in D_n$ for $i = 1, \ldots, l-1$. Therefore, $(A, R_n(A))$ is isomorphic to some substructure of $(D_n \cup \{d\}, R_n(D_n \cup \{d\}) \cong A_n \in \mathcal{H}_l^m$. Therefore, A should be a member of $\mathcal{H}_l^m$. But this contradicts the choice of A. □

A more general case will be treated in [5].

Acknowledgments

The work is supported by the Research Institute for Mathematical Sciences, an International Joint Usage/Research Center located in Kyoto University.

References

[1] P. J. Cameron, Cyclic automorphisms of a countable graph and random sum-free sets, Graphs and Combinatorics **1** (1985), 129–135.

[2] R. Diestel, *Graph Theory*, Fourth Edition, Springer, New York (2010).

[3] C. W. Henson, A family of countable homogeneous graphs, Pacific Journal of Mathematics **38**, No. 1 (1971), 69–83.

[4] E. Hrushovski, Simplicity and the Lascar group, preprint, 1998.

[5] H. Kikyo and A. Tsuboi, On automorphisms with a single orbit of some random structures, in preparation.

Graduate School of System Informatics
Kobe University
1-1 Rokkodai, Nada, Kobe 657-8501
JAPAN
kikyo@kobe-u.ac.jp

神戸大学大学院システム情報学研究科　桔梗　宏孝