RELATIVIZED LASCAR, KIM-PILLAY, SHELAH GROUPS

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Abstract. We study relativized Lascar groups, and show that some fundamental facts about the Galois groups of first-order theories can be generalized to the relativized context.

1. Preliminaries

1.1. **Hyperimaginaries and model theoretic Galois groups.** The proofs for basic properties of hyperimaginaries can be found on [C11] and [K14]. Most of the basic definitions and facts on the Lascar group can be found on [KL23], [Lee22] and [K14], which collect and generalize the results in [Z02], [CLPZ01], [LP01] and more papers to the context of hyperimaginaries.

We denote the automorphism group of $\mathfrak{C}$ by $\mathrm{Aut}(\mathfrak{C})$ and for $A \subset \mathfrak{C}$, we denote the set of automorphisms of $\mathfrak{C}$ fixing A pointwise by $\mathrm{Aut}_A(\mathfrak{C})$

Definition 1.1. Let E be an equivalence relation defined on $\mathfrak{C}^\alpha$ and $A \subseteq \mathfrak{C}$. Then E is said to be

(1) *finite* if the number of its equivalence classes is finite,
(2) *bounded* if the number of its equivalence classes is small,
(3) *A-invariant* if for any $f \in \mathrm{Aut}_A(\mathfrak{C})$, $E(a, b)$ if and only if $E(f(a), f(b))$,
(4) *A-definable* if there is a formula $\varphi(x, y)$ over A such that $\models \varphi(a, b)$ if and only if $E(a, b)$ holds, just *definable* if it is definable over some parameters, and
(5) *A-type-definable* if there is a partial type $\Phi(x, y)$ over A such that $\models \Phi(a, b)$ if and only if $E(a, b)$ holds, just *type-definable* if it is type-definable over some parameters.

Definition 1.2. Let E be an $\emptyset$-type-definable equivalence relation on $\mathfrak{C}^\alpha$. An equivalence class of E is called a *hyperimaginary* and it is denoted by a_E for a representative a. A hyperimaginary a_E is *countable* if $|a|$ is countable.

Definition 1.3. For a hyperimaginary e,

$$\mathrm{Aut}_{\boldsymbol{e}}(\mathfrak{C}) := \{f \in \mathrm{Aut}(\mathfrak{C}) : f(\boldsymbol{e}) = \boldsymbol{e} \text{ (setwise)}\}.$$

We say an equivalence relation E is *$\boldsymbol{e}$-invariant* if for any $f \in \mathrm{Aut}_{\boldsymbol{e}}(\mathfrak{C})$, $E(a, b)$ holds if and only if $E(f(a), f(b))$ holds.

Definition 1.4. For a hyperimaginary e, we say two 'objects' (e.g. elements of $\mathfrak{C}$, tuples of equivalence classes, enumerations of sets) b and c are *interdefinable over* $\boldsymbol{e}$ if for any $f \in \mathrm{Aut}_{\boldsymbol{e}}(\mathfrak{C})$, $f(b) = b$ if and only if $f(c) = c$.

Fact 1.5 ([K14, Section 4.1] or [C11, Chapter 15])**.**

(1) Any tuple (of elements) b in $\mathfrak{C}$, any tuple c of imaginaries of $\mathfrak{C}$, and any tuple of hyperimaginaries are interdefinable with a single hyperimaginary.
(2) Any hyperimaginary is interdefinable with a sequence of countable hyperimaginaries.

Until the end of this paper, we will fix some arbitrary $\emptyset$-type-definable equivalence relation E and a hyperimagianry $\boldsymbol{e} := a_E$.

Definition 1.6.

(1) A hyperimaginary $\boldsymbol{e}'$ is *definable over* $\boldsymbol{e}$ if $f(\boldsymbol{e}') = \boldsymbol{e}'$ for any $f \in \mathrm{Aut}_{\boldsymbol{e}}(\mathfrak{C})$.
(2) A hyperimaginary $\boldsymbol{e}'$ is *algebraic over* $\boldsymbol{e}$ if $\{f(\boldsymbol{e}') : f \in \mathrm{Aut}_{\boldsymbol{e}}(\mathfrak{C})\}$ is finite.
(3) A hyperimaginary $\boldsymbol{e}'$ is *bounded over* $\boldsymbol{e}$ if $\{f(\boldsymbol{e}') : f \in \mathrm{Aut}_{\boldsymbol{e}}(\mathfrak{C})\}$ is small.
(4) The *definable closure* of $\boldsymbol{e}$, denoted by $\mathrm{dcl}(\boldsymbol{e})$ is the set of all countable hyperimaginaries $\boldsymbol{e}'$ such that $f(\boldsymbol{e}') = \boldsymbol{e}'$ for any $f \in \mathrm{Aut}_{\boldsymbol{e}}(\mathfrak{C})$.

(5) The *algebraic closure* of $\boldsymbol{e}$, denoted by $\operatorname{acl}(\boldsymbol{e})$ is the set of all countable hyperimaginaries $\boldsymbol{e}'$ such that $\{f(\boldsymbol{e}') : f \in \operatorname{Aut}_{\boldsymbol{e}}(\mathfrak{C})\}$ is finite.
(6) The *bounded closure* of $\boldsymbol{e}$, denoted by $\operatorname{bdd}(\boldsymbol{e})$ is the set of all countable hyperimaginaries $\boldsymbol{e}'$ such that $\{f(\boldsymbol{e}') : f \in \operatorname{Aut}_{\boldsymbol{e}}(\mathfrak{C})\}$ is bounded.

Definition/Remark 1.7.

(1) By Fact 1.5(2), if $f \in \operatorname{Aut}(\mathfrak{C})$ fixes $\operatorname{bdd}(\boldsymbol{e})$, then for any hyperimaginary $\boldsymbol{e}'$ which is bounded over $\boldsymbol{e}$, $f(\boldsymbol{e}') = \boldsymbol{e}'$. Similar statements also hold for $\operatorname{dcl}(\boldsymbol{e})$ and $\operatorname{acl}(\boldsymbol{e})$.
(2) By (1), for a hyperimaginary b_F which is possibly not countable, we write $b_F \in \operatorname{bdd}(\boldsymbol{e})$ if b_F is bounded over $\boldsymbol{e}$. We use notation $b_F \in \operatorname{dcl}(\boldsymbol{e})$, $\operatorname{acl}(\boldsymbol{e})$ in a similar way.
(3) Each of $\operatorname{dcl}(\boldsymbol{e})$, $\operatorname{acl}(\boldsymbol{e})$, and $\operatorname{bdd}(\boldsymbol{e})$ is small and interdefinable with a single hyperimaginary (c.f. [C11, Proposition 15.18]). Thus it makes sense to consider $\operatorname{Aut}_{\operatorname{bdd}(\boldsymbol{e})}(\mathfrak{C})$.

Definition 1.8 ([K14, Section 4.1])**.** Let b_F and c_F be hyperimaginaries.

(1) The *complete type of* b_F *over* $\boldsymbol{e}$, $\operatorname{tp}_x(b_F/\boldsymbol{e})$ is a partial type over a
$$\exists z_1 z_2(\operatorname{tp}_{z_1 z_2}(ba) \wedge F(x, z_1) \wedge E(a, z_2)),$$
whose solution set is the union of automoprhic images of b_F over $\boldsymbol{e}$.
(2) We write $b_F \equiv_e c_F$ if there is an automorphism $f \in \operatorname{Aut}_{\boldsymbol{e}}(\mathfrak{C})$ such that $f(b_F) = c_F$. Then, the equivalence relation $x_F \equiv_e y_F$ in variables xy is a-type-definable, given by the partial type:
$$\exists z_1 z_2 w_1 w_2(E(a, z_1) \wedge E(a, z_2) \wedge \operatorname{tp}(z_1 w_1) = \operatorname{tp}(z_2 w_2) \wedge F(w_1, x) \wedge F(w_2, y)).$$
We also write $\operatorname{tp}_x(b_F/e) = \operatorname{tp}_y(c_F/e)$ for $b_F \equiv_e c_F$.

Now we start to recall the model theoretic Galois groups.

Definition 1.9.

(1) $\operatorname{Autf}_{\mathrm{L}}(\mathfrak{C}, \boldsymbol{e})$ is a normal subgroup of $\operatorname{Aut}_{\boldsymbol{e}}(\mathfrak{C})$ generated by
$$\{f \in \operatorname{Aut}_{\boldsymbol{e}}(\mathfrak{C}) : f \in \operatorname{Aut}_M(\mathfrak{C}) \text{ for some } M \models T \text{ such that } \boldsymbol{e} \in \operatorname{dcl}(M)\}.$$
(2) The quotient group $\operatorname{Gal}_{\mathrm{L}}(T, \boldsymbol{e}) = \operatorname{Aut}_{\boldsymbol{e}}(\mathfrak{C}) / \operatorname{Autf}_{\mathrm{L}}(\mathfrak{C}, \boldsymbol{e})$ is called the *Lascar group* of T over $\boldsymbol{e}$.

Fact 1.10 ([KL23, Section 1])**.**

(1) $\operatorname{Gal}_{\mathrm{L}}(T, \boldsymbol{e})$ *does not depend on the choice of a monster model up to isomorphism, so it is legitimate to write* $\operatorname{Gal}_{\mathrm{L}}(T, \boldsymbol{e})$ *instead of* $\operatorname{Gal}_{\mathrm{L}}(\mathfrak{C}, \boldsymbol{e})$.
(2) $[\operatorname{Aut}_{\boldsymbol{e}}(\mathfrak{C}) : \operatorname{Autf}_{\mathrm{L}}(\mathfrak{C}, \boldsymbol{e})] = |\operatorname{Gal}_{\mathrm{L}}(T, \boldsymbol{e})| \leq 2^{|T|+|a|}$*, which is small.*

Fact 1.11 ([Z02, Lemma 1])**.** *Let* M *be a small model of* T *such that* $\boldsymbol{e} \in \operatorname{dcl}(M)$ *and* $f, g \in \operatorname{Aut}_{\boldsymbol{e}}(\mathfrak{C})$*. If* $\operatorname{tp}(f(M)/M) = \operatorname{tp}(g(M)/M)$*, then* $f \cdot \operatorname{Autf}_{\mathrm{L}}(\mathfrak{C}, \boldsymbol{e}) = g \cdot \operatorname{Autf}_{\mathrm{L}}(\mathfrak{C}, \boldsymbol{e})$ *as elements in* $\operatorname{Gal}_{\mathrm{L}}(T, \boldsymbol{e})$.

Definition 1.12. Let M be a small model of T such that $\boldsymbol{e} \in \operatorname{dcl}(M)$.

(1) $S_M(M) = \{\operatorname{tp}(f(M)/M) : f \in \operatorname{Aut}_{\boldsymbol{e}}(\mathfrak{C})\}$.
(2) $\nu : S_M(M) \to \operatorname{Gal}_{\mathrm{L}}(T, \boldsymbol{e})$ is defined by $\nu(\operatorname{tp}(f(M)/M)) = f \cdot \operatorname{Autf}_{\mathrm{L}}(\mathfrak{C}, \boldsymbol{e}) = [f]$, which is well-defined by Fact 1.11.
(3) $\mu : \operatorname{Aut}_{\boldsymbol{e}}(\mathfrak{C}) \to S_M(M)$ is defined by $\mu(f) = \operatorname{tp}(f(M)/M)$.
(4) $\pi = \nu \circ \mu : \operatorname{Aut}_{\boldsymbol{e}}(\mathfrak{C}) \to \operatorname{Gal}_{\mathrm{L}}(T, \boldsymbol{e})$, so that $\pi(f) = [f]$ in $\operatorname{Gal}_{\mathrm{L}}(T, \boldsymbol{e})$.

Remark 1.13 ([KL23, Remark 1.9])**.** Let $S_x(M) = \{p(x) : |x| = |M| \text{ and } p(x) \text{ is a complete type over } M\}$ be the compact space of complete types. Note that even if $\boldsymbol{e} \in \operatorname{dcl}(M)$, possibly a is not in M, so that $S_M(M)$ is *not* $\{p \in S_x(M) : \operatorname{tp}(M/\boldsymbol{e}) \subseteq p\}$. But for any small model M such that $\boldsymbol{e} \in \operatorname{dcl}(M)$, $S_M(M)$ is a closed (so compact) subspace.

Proof. Let
$$r(x, a) = \operatorname{tp}(M/\boldsymbol{e}) = \exists z_1 z_2(\operatorname{tp}_{z_1 z_2}(Ma) \wedge x = z_1 \wedge E(a, z_2))$$
and $r_0(y, M) = \operatorname{tp}(a/M)$. Then
$$r'(x, M) = \exists y(r(x, y) \wedge r_0(y, M))$$
has the same solution set as $\operatorname{tp}(M/\boldsymbol{e})$. Thus $S_M(M) = \{p \in S_x(M) : r'(x, M) \subseteq p\}$, which is closed. □

Fact 1.14 ([KL23, Corollary 1.20])**.**

(1) We give the quotient topology on $\operatorname{Gal}_{\mathrm{L}}(T, \boldsymbol{e})$ *induced by* $\nu : S_M(M) \to \operatorname{Gal}_{\mathrm{L}}(T, \boldsymbol{e})$*. The quotient topology does not depend on the choice of* M.

(2) $\mathrm{Gal_L}(T,\boldsymbol{e})$ *is a quasi-compact topological group.*

Definition 1.15. Let [id] be the identity in $\mathrm{Gal_L}(T,\boldsymbol{e})$, $\mathrm{Gal^c_L}(T,\boldsymbol{e})$ be the topological closure of the trivial subgroup of $\mathrm{Gal_L}(T,\boldsymbol{e})$, and $\mathrm{Gal^0_L}(T,\boldsymbol{e})$ be the connected component containing the identity in $\mathrm{Gal_L}(T,\boldsymbol{e})$. Put $\mathrm{Autf_{KP}}(\mathfrak{C},\boldsymbol{e})=\pi^{-1}[\mathrm{Gal^c_L}(T,\boldsymbol{e})]$ and $\mathrm{Autf_S}(\mathfrak{C},\boldsymbol{e})=\pi^{-1}[\mathrm{Gal^0_L}(T,\boldsymbol{e})]$.

(1) The *KP(-Galois) group of* T over $\boldsymbol{e}$ is

$$\mathrm{Gal_{KP}}(T,\boldsymbol{e}):=\mathrm{Aut}_{\boldsymbol{e}}(\mathfrak{C})/\,\mathrm{Autf_{KP}}(\mathfrak{C},\boldsymbol{e}).$$

(2) The *Shelah(-Galois) group of* T over $\boldsymbol{e}$ is

$$\mathrm{Gal_S}(T,\boldsymbol{e}):=\mathrm{Aut}_{\boldsymbol{e}}(\mathfrak{C})/\,\mathrm{Autf_S}(\mathfrak{C},\boldsymbol{e}).$$

Note that we have

$$\mathrm{Gal_{KP}}(T,\boldsymbol{e})\cong\mathrm{Gal_L}(T,\boldsymbol{e})/\,\mathrm{Gal^c_L}(T,\boldsymbol{e}),\ \ \mathrm{Gal_S}(T,\boldsymbol{e})\cong\mathrm{Gal_L}(T,\boldsymbol{e})/\,\mathrm{Gal^0_L}(T,\boldsymbol{e}).$$

Definition 1.16. Given hyperimaginaries b_F, c_F, they are said to have the same

(1) *Lascar strong type* over $\boldsymbol{e}$ if there is $f\in\mathrm{Autf_L}(\mathfrak{C},\boldsymbol{e})$ such that $f(b_F)=c_F$, and it is denoted by $b_F\equiv^{\mathrm{L}}_{\boldsymbol{e}}c_F$,

(2) *KP strong type* over $\boldsymbol{e}$ (where KP is an abbreviation for Kim-Pillay) if there is $f\in\mathrm{Autf_{KP}}(\mathfrak{C},\boldsymbol{e})$ such that $f(b_F)=c_F$, and it is denoted by $b_F\equiv^{\mathrm{KP}}_{\boldsymbol{e}}c_F$, and

(3) *Shelah strong type* over $\boldsymbol{e}$ if there is $f\in\mathrm{Autf_S}(\mathfrak{C},\boldsymbol{e})$ such that $f(b_F)=c_F$, and it is denoted by $b_F\equiv^{\mathrm{S}}_{\boldsymbol{e}}c_F$.

Fact 1.17 ([KL23])**.** *Let* b_F *and* c_F *be hyperimaginaries.*

(1) $b_F\equiv^{\mathrm{L}}_{\boldsymbol{e}}c_F$ *if and only if for any* $\boldsymbol{e}$*-invariant bounded equivalence relation* E *which is coarser than* F*,* $E(b,c)$ *holds;* $x_F\equiv^{\mathrm{L}}_{\boldsymbol{e}}y_F$ *is the finest such an equivalence relation among them.*

(2) The following are equivalent.

(a) $b_F\equiv^{\mathrm{KP}}_{\boldsymbol{e}}c_F$.

(b) $b_F\equiv_{\mathrm{bdd}(\boldsymbol{e})}c_F$.

(c) For any $\boldsymbol{e}$*-invariant type-definable bounded equivalence relation* E *which is coarser* F*,* $E(b,c)$ *holds (*$x_F\equiv^{\mathrm{KP}}_{\boldsymbol{e}}y_F$ *is the finest such an equivalence relation among them).*

(3) The following are equivalent.

(a) $b_F\equiv^{\mathrm{S}}_{\boldsymbol{e}}c_F$.

(b) $b_F\equiv_{\mathrm{acl}(\boldsymbol{e})}c_F$.

(c) For any $\boldsymbol{e}$*-invariant type-definable equivalence relation* L *coarser than* F*, if* b_L *has finitely many conjugates over* $\boldsymbol{e}$*, then* $L(b,c)$ *holds.*

If F *is just* $=$ *so that* b_F *and* c_F *are just real tuples, then we can omit "coarser than* F*".*

Fact 1.18. *Let* F *be an* $\boldsymbol{e}$*-invariant type-definable equivalence relation on* $\mathfrak{C}^\alpha$*. Then there is an* $\emptyset$*-type-definable equivalence relation* F' *such that for any* $c\in\mathfrak{C}^\alpha$*,* c_F *and* $(ca)_{F'}$ *are interdefinable over* $\boldsymbol{e}$ *so that we can 'replace' an equivalence class of an* $\boldsymbol{e}$*-invariant type-definable equivalence relation with a hyperimaginary.*

Proof. Since F is $\boldsymbol{e}(=a_E)$-invariant, F is type-definable over a, say by $F(x,y;a)$. Then for $p(x)=\mathrm{tp}(a)$, put

$$F'(xz,yw):=(F(x,y;z)\wedge E(z,w)\wedge p(z)\wedge p(w))\vee xz=yw.$$

Then F' is the desired one. □

1.2. **Relativized model theoretic Galois groups.** From now on, we fix an $\boldsymbol{e}$-invariant partial type $\Sigma(x)$ where x is a possibly infinite tuple of variables, that is, for any $f\in\mathrm{Aut}_{\boldsymbol{e}}(\mathfrak{C})$, $b\models\Sigma(x)$ if and only if $f(b)\models\Sigma(x)$. We write $\Sigma(\mathfrak{C})$ for the set of tuples b of elements in $\mathfrak{C}$ such that $b\models\Sigma(x)$.

Definition 1.19 (Restriction of automoprhism groups to Σ)**.** Let $X\in\{\mathrm{L},\mathrm{KP},\mathrm{S}\}$.

(1) $\mathrm{Aut}_{\boldsymbol{e}}(\Sigma)=\mathrm{Aut}_{\boldsymbol{e}}(\Sigma(\mathfrak{C}))=\{f\restriction\Sigma(\mathfrak{C}):f\in\mathrm{Aut}_{\boldsymbol{e}}(\mathfrak{C})\}$.

(2) For a cardinal λ, $\mathrm{Autf}^{\lambda}_X(\Sigma,\boldsymbol{e})=$

$$\{\sigma\in\mathrm{Aut}_{\boldsymbol{e}}(\Sigma):\ \text{for any of tuple } b=(b_i)_{i<\lambda} \text{ where each } b_i\models\Sigma(x_i),\ b\equiv^X_{\boldsymbol{e}}\sigma(b)\}.$$

(3) $\mathrm{Autf}_X(\Sigma,\boldsymbol{e})=$

$$\{\sigma\in\mathrm{Aut}_{\boldsymbol{e}}(\Sigma):\text{for any cardinal }\lambda\text{ and for any tuple } b=(b_i)_{i<\lambda}\ \text{with each } b_i\models\Sigma(x_i),\ b\equiv^X_{\boldsymbol{e}}\sigma(b)\}.$$

Remark 1.20. For $X \in \{\mathrm{L}, \mathrm{KP}, \mathrm{S}\}$, it is easy to check that $\operatorname{Autf}^{\lambda}_X(\Sigma, \boldsymbol{e})$ and $\operatorname{Autf}_X(\Sigma, \boldsymbol{e})$ are normal subgroups of $\operatorname{Aut}_{\boldsymbol{e}}(\Sigma)$.

Definition 1.21.

(1) For $X \in \{\mathrm{L}, \mathrm{KP}, \mathrm{S}\}$, for any cardinal λ, $\operatorname{Gal}^{\lambda}_X(\Sigma, \boldsymbol{e}) = \operatorname{Aut}_{\boldsymbol{e}}(\Sigma)/\operatorname{Autf}^{\lambda}_X(\Sigma, \boldsymbol{e})$.
(2) $\operatorname{Gal}_{\mathrm{L}}(\Sigma, \boldsymbol{e}) = \operatorname{Aut}_{\boldsymbol{e}}(\Sigma)/\operatorname{Autf}_{\mathrm{L}}(\Sigma, \boldsymbol{e})$ is the *Lascar(-Galois) group over* $\boldsymbol{e}$ *relativized to* Σ.
(3) $\operatorname{Gal}_{\mathrm{KP}}(\Sigma, \boldsymbol{e}) = \operatorname{Aut}_{\boldsymbol{e}}(\Sigma)/\operatorname{Autf}_{\mathrm{KP}}(\Sigma, \boldsymbol{e})$ is the *KP(-Galois) group over* $\boldsymbol{e}$ *relativized to* Σ.
(4) $\operatorname{Gal}_{\mathrm{S}}(\Sigma, \boldsymbol{e}) = \operatorname{Aut}_{\boldsymbol{e}}(\Sigma)/\operatorname{Autf}_{\mathrm{S}}(\Sigma, \boldsymbol{e})$ is the *Shelah(-Galois) group over* $\boldsymbol{e}$ *relativized to* Σ.

Remark 1.22.

(1) For $X \in \{\mathrm{L}, \mathrm{KP}, \mathrm{S}\}$, if $[f] = [\mathrm{id}]$ in $\operatorname{Gal}_X(T, \boldsymbol{e})$, then $[f] = [\mathrm{id}]$ in $\operatorname{Gal}_X(\Sigma, \boldsymbol{e})$.
(2) In general, $\operatorname{Gal}^1_{\mathrm{L}}(\Sigma) \neq \operatorname{Gal}^2_{\mathrm{L}}(\Sigma)$ (c.f. [DKKL21, Example 2.3]).

Fact 1.23.

(1) $\operatorname{Autf}_{\mathrm{L}}(\Sigma, \boldsymbol{e}) = \operatorname{Autf}^{\omega}_{\mathrm{L}}(\Sigma, \boldsymbol{e})$.
(2) $\operatorname{Autf}_{\mathrm{KP}}(\Sigma, \boldsymbol{e}) = \operatorname{Autf}^{\omega}_{\mathrm{KP}}(\Sigma, \boldsymbol{e})$.
(3) $\operatorname{Autf}_{\mathrm{S}}(\Sigma, \boldsymbol{e}) = \operatorname{Autf}^{\omega}_{\mathrm{S}}(\Sigma, \boldsymbol{e})$.

Proof. (1): The proof is the same as [DKL17, Remark 3.3] and [Lee22, Proposition 6.3] over a hyperimaginary, so we omit it.

(2): $\sigma \in \operatorname{Autf}_{\mathrm{KP}}(\Sigma, \boldsymbol{e})$ if and only if for any tuple b of realizations of Σ, $b \equiv^{\mathrm{KP}}_{\boldsymbol{e}} \sigma(b)$. But $\equiv^{\mathrm{KP}}_{\boldsymbol{e}}$ is equivalent to $\equiv_{\mathrm{bdd}(\boldsymbol{e})}$ (which is type-definable) by Fact 1.17, thus if $b' \equiv^{\mathrm{KP}}_{\boldsymbol{e}} \sigma(b)'$ for any corresponding subtuples $b', \sigma(b')$ of $b, \sigma(b')$, which are tuples of finitely many realizations of Σ, then $b \equiv^{\mathrm{KP}}_{\boldsymbol{e}} \sigma(b)$ by compactness. Thus if $\sigma \in \operatorname{Autf}^{\omega}_{\mathrm{L}}(\Sigma, \boldsymbol{e})$, then $\sigma \in \operatorname{Autf}_{\mathrm{L}}(\Sigma, \boldsymbol{e})$. The proof for (3) is the same as (2). □

Fact 1.24 ([DKL17, Proposition 3.6] or [Lee22, Proposition 6.5])**.** *The relativized Lascar group* $\operatorname{Gal}_{\mathrm{L}}(\Sigma, \boldsymbol{e})$ *does not depend on the choice of* $\mathfrak{C}$.

2. Topology on relativized model theoretic Galois groups

In this section, we will generalize Fact 1.14 and Fact 1.17 into the relativized Lascar group. More precisely, we will find a topology $\mathfrak{t}$ on the relativized Lascar group $\operatorname{Gal}_{\mathrm{L}}(\Sigma, \boldsymbol{e})$ such that

- the group $\operatorname{Gal}_{\mathrm{L}}(\Sigma, \boldsymbol{e})$ is a quasi-compact group with respect to $\mathfrak{t}$, and
- $\operatorname{Autf}_{\mathrm{KP}}(\Sigma, \boldsymbol{e}) = \pi'^{-1}[\operatorname{Gal}^0_{\mathrm{L}}(\Sigma, \boldsymbol{e})]$ and $\operatorname{Autf}_{\mathrm{S}}(\Sigma, \boldsymbol{e}) = \pi'^{-1}[\operatorname{Gal}^c_{\mathrm{L}}(\Sigma, \boldsymbol{e})]$,

where $\pi' : \operatorname{Aut}_{\boldsymbol{e}}(\Sigma) \to \operatorname{Gal}_{\mathrm{L}}(\Sigma, \boldsymbol{e})$ is the natural surjective map, and $\operatorname{Gal}^0_{\mathrm{L}}(\Sigma, \boldsymbol{e})$ and $\operatorname{Gal}^c_{\mathrm{L}}(\Sigma, \boldsymbol{e})$ are the identity closure and the connected component of $\operatorname{Gal}_{\mathrm{L}}(\Sigma, \boldsymbol{e})$ in the topology $\mathfrak{t}$.

Definition 2.1. A small tuple b of realizations of Σ is called a *Lascar tuple* (in Σ) if $\operatorname{Autf}_{\mathrm{L}}(\Sigma, \boldsymbol{e}) = \{\sigma \in \operatorname{Aut}_{\boldsymbol{e}}(\Sigma) : b \equiv^{\mathrm{L}}_{\boldsymbol{e}} \sigma(b)\}$.

Lemma 2.2. *For any* $\boldsymbol{e}$*-invariant partial type* $\Sigma(x)$*, there is a Lascar tuple in* Σ.

Proof. By Fact 1.10(2), the set of Lascar equivalence classes

$$C = \{c_{\equiv^{\mathrm{L}}_{\boldsymbol{e}}} : c \text{ is a countable tuple of realizations of } \Sigma\}$$

is small, say its cardinal is κ.

Let $b = (b_i : i < \kappa)$ be a small tuple that collects representatives of Lascar classes in C, only one for each class. Then b is the desired one; by Fact 1.23, it is enough to show that for an automorphism $f \in \operatorname{Aut}_{\boldsymbol{e}}(\mathfrak{C})$, if $b \equiv^{\mathrm{L}}_{\boldsymbol{e}} f(b)$, then for any tuple d of countable realizations of Σ, $d \equiv^{\mathrm{L}}_{\boldsymbol{e}} f(d)$. Note that there is $i < \kappa$ such that $d \equiv^{\mathrm{L}}_{\boldsymbol{e}} b_i$. Then we have

$$d \equiv^{\mathrm{L}}_{\boldsymbol{e}} b_i \equiv^{\mathrm{L}}_{\boldsymbol{e}} f(b_i) \equiv^{\mathrm{L}}_{\boldsymbol{e}} f(d)$$

where the last equivalence follows from the invariance of $\equiv^{\mathrm{L}}_{\boldsymbol{e}}$. □

Remark 2.3.

(1) Any small tuple of realizations of Σ can be extended into a Lascar tuple.
(2) Any concatenation of two Lascar tuples is again a Lascar tuple.

Now we fix a Lascar tuple b in Σ and a small model M with $b \in M$ and $e \in \mathrm{dcl}(M)$. Let

$$\begin{aligned} S_b(b) &:= \{\mathrm{tp}(\sigma(b)/b) : \sigma \in \mathrm{Aut}_{\boldsymbol{e}}(\Sigma)\} \\ &= \{\mathrm{tp}(f(b)/b) : f \in \mathrm{Aut}_{\boldsymbol{e}}(\mathfrak{C})\}. \end{aligned}$$

For the natural restriction map $r : S_M(M) \to S_b(b)$, which is continuous with respect to the logic topology on the type space, we have that $S_b(b)$ is a compact space. Next, Consider a map

$$\nu_b : S_b(b) \to \mathrm{Gal}^{\lambda}_{\mathrm{L}}(\Sigma, \boldsymbol{e}),\ p = \mathrm{tp}(\sigma(b)/b) \mapsto [\sigma].$$

Then, it is not hard to check that ν_b is well-defined because b is a Lascar tuple (the same proof of Fact 1.11 works).

Remark 2.4. We have the following commutative diagram of natural surjective maps:

$$\begin{array}{ccccccccc} \mathrm{Aut}_{\boldsymbol{e}}(\mathfrak{C}) & \xrightarrow{\mu} & S_M(M) & \xrightarrow{\nu} & \mathrm{Gal}_{\mathrm{L}}(T, \boldsymbol{e}) & \xrightarrow{\eta_{\mathrm{KP}}} & \mathrm{Gal}_{\mathrm{KP}}(T, \boldsymbol{e}) & \xrightarrow{\eta_{\mathrm{S}}} & \mathrm{Gal}_{\mathrm{S}}(T, \boldsymbol{e}) \\ \downarrow{\scriptstyle \xi} & & \downarrow{\scriptstyle r} & & \downarrow{\scriptstyle \xi_{\mathrm{L}}} & & \downarrow{\scriptstyle \xi_{\mathrm{KP}}} & & \downarrow{\scriptstyle \xi_{\mathrm{S}}} \\ \mathrm{Aut}_{\boldsymbol{e}}(\Sigma) & \xrightarrow{\mu_b} & S_b(b) & \xrightarrow{\nu_b} & \mathrm{Gal}_{\mathrm{L}}(\Sigma, \boldsymbol{e}) & \xrightarrow{\eta_{\mathrm{KP},\Sigma}} & \mathrm{Gal}_{\mathrm{KP}}(\Sigma, \boldsymbol{e}) & \xrightarrow{\eta_{\mathrm{S},\Sigma}} & \mathrm{Gal}_{\mathrm{S}}(\Sigma, \boldsymbol{e}) \end{array}$$

Put $\pi_b := \nu_b \circ \mu_b$ and $\pi_\Sigma := \pi_b \circ \xi = \xi_{\mathrm{L}} \circ \pi$. Note that $\pi = \nu \circ \mu : \mathrm{Aut}_{\boldsymbol{e}}(\mathfrak{C}) \to \mathrm{Gal}_{\mathrm{L}}(T, \boldsymbol{e})$ and $\pi_b = \pi' : \mathrm{Aut}_{\boldsymbol{e}}(\Sigma) \to \mathrm{Gal}_{\mathrm{L}}(\Sigma, \boldsymbol{e})$.

Remark 2.5 (Relaivized Galois groups are topological groups)**.** Consider a topology $\mathfrak{t}_b$ on $\mathrm{Gal}_{\mathrm{L}}(\Sigma, \boldsymbol{e})$ given by the quotient topology via ν_b. Then, $(\mathrm{Gal}_{\mathrm{L}}(\Sigma, \boldsymbol{e}), \mathfrak{t}_b)$ is a quasi-compact topological group whose topology $\mathfrak{t}_b$ is independent of the choice of a Lascar tuple b.

Proof. Then the restriction map $r : S_M(M) \to S_b(b)$ is a continuous surjective map between compact Hausdorff spaces, hence a quotient map. Thus $\nu_b : S_b(b) \to \mathrm{Gal}_{\mathrm{L}}(\Sigma, \boldsymbol{e})$ and $\nu_b \circ r : S_M(M) \to \mathrm{Gal}_{\mathrm{L}}(\Sigma, \boldsymbol{e})$ induce the same quotient topology on $\mathrm{Gal}_{\mathrm{L}}(\Sigma, \boldsymbol{e})$.

But the quotient topology on $\mathrm{Gal}_{\mathrm{L}}(\Sigma, \boldsymbol{e})$ given by the natural projection map $\xi_{\mathrm{L}} : \mathrm{Gal}_{\mathrm{L}}(T, \boldsymbol{e}) \to \mathrm{Gal}_{\mathrm{L}}(\Sigma, \boldsymbol{e})$ is also the same as the above topology, thus the topology of $\mathrm{Gal}_{\mathrm{L}}(\Sigma, \boldsymbol{e})$ is independent of the choice b (Fact 1.14) and it is a topological group since a quotient group of a topological group with quotient topology is a topological group (it is the same reasoning as [DKL17, Remark 3.4]). □

The following is a relativized analogue of [KL23, Proposition 2.3]. The purpose of Proposition 2.6 is to interpret closed subgroups of $\mathrm{Gal}_{\mathrm{L}}(\Sigma, \boldsymbol{e})$ using bounded hyperimaginaries.

Proposition 2.6. *Let $H \le \mathrm{Aut}_{\boldsymbol{e}}(\mathfrak{C})$. The following are equivalent.*

(1) $\pi_\Sigma(H)$ is closed in $\mathrm{Gal}_{\mathrm{L}}(\Sigma, \boldsymbol{e})$ and $H = \pi_\Sigma^{-1}[\pi_\Sigma(H)]$.

(2) $H = \mathrm{Aut}_{\boldsymbol{e}'\boldsymbol{e}}(\mathfrak{C})$ for some hyperimaginary $\boldsymbol{e}' \in \mathrm{bdd}(\boldsymbol{e})$, and one of the representatives of $\boldsymbol{e}'$ is a tuple of realizations of Σ.

Proof. The method of proof is the same as the proof of [KL23, Proposition 2.3], but we use a Lascar tuple b instead of a model M.

($\Rightarrow$): We have

$$\mathrm{Aut}_b(\mathfrak{C}) = \mathrm{Aut}_{b\boldsymbol{e}}(\mathfrak{C}) \le \xi^{-1}[\mathrm{Autf}_{\mathrm{L}}(\Sigma, \boldsymbol{e})] \le H$$

and since $\pi_\Sigma(H)$ is closed, $\nu_b^{-1}(\pi_\Sigma(H))$ is closed and thus $\{h(b) : h \in H\}$ is type-definable over b. Hence by [KL23, Proposition 2.2], $H = \mathrm{Aut}_{b_F\boldsymbol{e}}(\mathfrak{C})$ for some $\emptyset$-type-definable equivalence relation F.

We have $b_F \in \mathrm{bdd}(\boldsymbol{e})$ because $[\mathrm{Aut}_{\boldsymbol{e}}(\mathfrak{C}) : H] = \kappa$ is small since $\mathrm{Autf}_{\mathrm{L}}(\mathfrak{C}, \boldsymbol{e}) \le H$, and so there is $\{f_i \in \mathrm{Aut}_{\boldsymbol{e}}(\mathfrak{C}) : i < \kappa\}$ such that $\mathrm{Aut}_{\boldsymbol{e}}(\mathfrak{C}) = \bigsqcup_{i<\kappa} f_i \cdot H$. Then for all $g, h \in \mathrm{Aut}_{\boldsymbol{e}}(\mathfrak{C})$, if $g \cdot H = h \cdot H$, then $h^{-1}g \in H$ and hence $g(b_F) = h(b_F)$.

($\Leftarrow$): Say $\boldsymbol{e}' = c_F$ where c is a tuple of realizations of Σ and F is an $\emptyset$-type-definable equivalence relation. By Remark 2.3 and Remark 2.5, we may choose a Lascar tuple b which contains c. For $q(x) = \mathrm{tp}(c/\boldsymbol{e})$, because $\boldsymbol{e}' \in \mathrm{bdd}(\boldsymbol{e})$,

$$F'(z_1, z_2) := (q(z_1) \wedge q(z_2) \wedge F(z_1, z_2)) \vee (\neg q(z_1) \wedge \neg q(z_2))$$

is an $\boldsymbol{e}$-invariant bounded equivalence relation on $\mathfrak{C}^{|c|}$. Since c is a tuple in $\Sigma(\mathfrak{C})$, for any $\sigma \in \mathrm{Autf}_{\mathrm{L}}(\Sigma, \boldsymbol{e})$, $\sigma(c) \equiv^{\mathrm{L}}_{\boldsymbol{e}} c$ and thus $F'(\sigma(c), c)$ by Fact 1.17(1). Note that $c_F = c_{F'}$, so we have $\xi^{-1}[\mathrm{Autf}_{\mathrm{L}}(\Sigma, \boldsymbol{e})] \le H = \mathrm{Aut}_{c_F\boldsymbol{e}}(\mathfrak{C})$, hence $\pi_\Sigma^{-1}[\pi_\Sigma[H]] = H$. Notice that $H = \{f \in \mathrm{Aut}_{\boldsymbol{e}}(\mathfrak{C}) : f(c) \models F(z, c)\}$. Then $\nu_b^{-1}[\pi_\Sigma[H]] = \{p(z') \in S_b(b) : F(z, c) \subseteq p(z')\}$ where $z \subseteq z'$ and $|z'| = |b|$. Thus H is closed. □

Definition 2.7.

(1) For $H \leq \mathrm{Aut}_{\boldsymbol{e}}(\mathfrak{C})$, $\equiv^H$ is an orbit equivalence relation such that for tuples b, c in $\mathfrak{C}$, $b \equiv^H c$ if and only if there is $h \in H$ such that $h(b) = c$.
(2) For $H \leq \mathrm{Aut}_{\boldsymbol{e}}(\Sigma)$, we will use the same notation $\equiv^H$ but it is confined to the tuples of realizations of Σ.

Remark 2.8. Let $H \leq \mathrm{Aut}_{\boldsymbol{e}}(\Sigma)$ and c, d be tuples of realizations of Σ.
(1) If $H = \mathrm{Autf_L}(\Sigma, \boldsymbol{e})$, then $c \equiv^H d$ if and only if $c \equiv^{\mathrm{L}}_{\boldsymbol{e}} d$.
(2) If $H = \mathrm{Autf_{KP}}(\Sigma, \boldsymbol{e})$, then $c \equiv^H d$ if and only if $c \equiv^{\mathrm{KP}}_{\boldsymbol{e}} d$.
(3) If $H = \mathrm{Autf_S}(\Sigma)$, then $c \equiv^H d$ if and only if $c \equiv^{\mathrm{S}}_{\boldsymbol{e}} d$.

Proposition 2.9. *Give the following topology on $\mathrm{Aut}_{\boldsymbol{e}}(\Sigma)$: Its basic open sets are of the form $O_{c,d}$ where c and d are tuples of finitely many realizations of Σ, and $\sigma \in O_{c,d}$ if and only if $\sigma(c) = d$. Then $\pi_b : \mathrm{Aut}_{\boldsymbol{e}}(\Sigma) \to \mathrm{Gal_L}(\Sigma, \boldsymbol{e})$ is continuous.*

Proof. It can be proved in the same way as in [KL23, Proposition 3.1] using a Lascar tuple b. Let U be an open subset of $\mathrm{Gal_L}(\Sigma, \boldsymbol{e})$ and $[\sigma] = \sigma \cdot \mathrm{Autf_L}(\Sigma, \boldsymbol{e}) \in U$. Then $\nu_b^{-1}[U]$ is open and so there is a basic open set $V_{\varphi(x)} = \{p \in S_b(b) : \varphi(x) \in p\} \subseteq \nu_b^{-1}[U]$ such that $\mathrm{tp}(\sigma(b)/b) \in V_{\varphi(x)}$. Let b_0 be subtuple of b, which is a tuple of finitely many realizations of Σ and contains the finite tuple of b which corresponds to $\varphi(x)$ in $V_{\varphi(x)}$. Then $\mu_b^{-1}[V_{\varphi(x)}] = \{\tau \in \mathrm{Aut}_{\boldsymbol{e}}(\Sigma) : \tau(b_0) \models \varphi(x)\}$ contains σ. Note that the basic open set $O_{b_0,\sigma(b_0)} = \{\tau \in \mathrm{Aut}_{\boldsymbol{e}}(\Sigma) : \tau(b_0) = \sigma(b_0)\}$ contains σ and is contained in $\mu_b^{-1}[V_{\varphi(x)}]$. Thus $\pi_b^{-1}[U]$ is open. □

Proposition 2.10. *Let H be a subgroup of $\mathrm{Aut}_{\boldsymbol{e}}(\Sigma)$ containing $\mathrm{Autf_L}(\Sigma, \boldsymbol{e})$ such that $\pi_b(H)$ is a closed subgroup of $\mathrm{Gal_L}(\Sigma, \boldsymbol{e})$. Then*
(1) For any tuples c and d of realizations of Σ, $c \equiv^H d$ if and only if for all corresponding subtuples c' and d' of c and d, which are tuples of finitely many realizations of Σ, $c' \equiv^H d'$.
(2) $H = \{\sigma \in \mathrm{Aut}_{\boldsymbol{e}}(\Sigma) : \sigma$ fixes all the $\equiv^H$ -classes of any tuples of realizations of $\Sigma\}$

$$= \{\sigma \in \mathrm{Aut}_{\boldsymbol{e}}(\Sigma) : \sigma \text{ fixes all the } \equiv^H \text{-classes of any tuples of finitely many realizations of } \Sigma\}.$$

Proof. (1): Suppose that for all corresponding subtuples c' and d' of c and d, which are tuples of finitely many realizations of Σ, $c' \equiv^H d'$. Note that, by Proposition 2.6, $H' := \pi_\Sigma^{-1}[\pi_b[H]] = \mathrm{Aut}_{\boldsymbol{e}'\boldsymbol{e}}(\mathfrak{C})$ for some hyperimaginary $\boldsymbol{e}' \in \mathrm{bdd}(\boldsymbol{e})$ such that one of its representatives is a tuple of realizations of Σ. Then by commutativity of diagram in Remark 2.4, $H = \xi[H'] = \mathrm{Aut}_{\boldsymbol{e}'\boldsymbol{e}}(\Sigma)$ and so $c \equiv^H d$ if and only if $\mathrm{tp}(c/\boldsymbol{e}'\boldsymbol{e}) = \mathrm{tp}(d/\boldsymbol{e}'\boldsymbol{e})$. Since $\mathrm{tp}(c'/\boldsymbol{e}'\boldsymbol{e}) = \mathrm{tp}(d'/\boldsymbol{e}'\boldsymbol{e})$ for all corresponding subtuples c' and d' of c and d, which are tuples of finitely many realizations of Σ, by compactness, we have that $\mathrm{tp}(c/\boldsymbol{e}'\boldsymbol{e}) = \mathrm{tp}(d/\boldsymbol{e}'\boldsymbol{e})$ and so $c \equiv^H d$.

(2): We show that

$$H \supseteq \{\sigma \in \mathrm{Aut}_{\boldsymbol{e}}(\Sigma) : \sigma \text{ fixes all the } \equiv^H \text{-classes of any tuples of finitely many realizations of } \Sigma\}.$$

Let σ be an element on the right. For any subtuple b' of b, which is a tuple of finitely many realizations of Σ, $b' \equiv^H \sigma(b')$, and so by (1), $b \equiv^H \sigma(b)$. Thus there is $\tau \in H$ such that $\tau(b) = \sigma(b)$ and $\tau^{-1}\sigma(b) = b$. We have

$$\tau^{-1}\sigma \in \mathrm{Aut}_b(\Sigma) \leq \mathrm{Autf_L}(\Sigma, \boldsymbol{e}) \leq H,$$

and conclude that $\sigma \in \tau H = H$. □

Definition 2.11.
(1) $\mathrm{Gal}^c_{\mathrm{L}}(\Sigma, \boldsymbol{e})$ is the topological closure of the trivial subgroup of $\mathrm{Gal_L}(\Sigma, \boldsymbol{e})$.
(2) $\mathrm{Gal}^0_{\mathrm{L}}(\Sigma, \boldsymbol{e})$ is the connected component containing the identity in $\mathrm{Gal_L}(\Sigma, \boldsymbol{e})$.

Note that $\mathrm{Gal}^c_{\mathrm{L}}(\Sigma, \boldsymbol{e})$ and $\mathrm{Gal}^0_{\mathrm{L}}(\Sigma, \boldsymbol{e})$ are closed normal subgroups.

Lemma 2.12. $\mathrm{Autf_{KP}}(\Sigma, \boldsymbol{e})/\mathrm{Autf_L}(\Sigma, \boldsymbol{e})$ *and* $\mathrm{Autf_S}(\Sigma, \boldsymbol{e})/\mathrm{Autf_L}(\Sigma, \boldsymbol{e})$ *are closed in* $\mathrm{Gal_L}(\Sigma, \boldsymbol{e})$.

Proof. We have

$$\nu_b^{-1}\eta^{-1}_{\mathrm{KP},\Sigma}[\mathrm{Autf_{KP}}(\Sigma, \boldsymbol{e})/\mathrm{Autf_L}(\Sigma, \boldsymbol{e})] = \{\mathrm{tp}(f(b)/b) : b \equiv^{\mathrm{KP}}_{\boldsymbol{e}} f(b)\}.$$

By Fact 1.17(2) and the proof of Remark 1.13, $\equiv^{\mathrm{KP}}_{\boldsymbol{e}}$ is type-definable and there is an $\boldsymbol{e}$-invariant partial type $\Gamma(x, y)$ over b such that $\{\mathrm{tp}(f(b)/b) : b \equiv^{\mathrm{KP}}_{\boldsymbol{e}} f(b)\} = \{p(x) \in S_b(b) :\models \Gamma(x, b)\}$, implying that $\mathrm{Autf_{KP}}(\Sigma, \boldsymbol{e})/\mathrm{Autf_L}(\Sigma, \boldsymbol{e})$ is closed. The proof for $\mathrm{Autf_S}(\Sigma, \boldsymbol{e})/\mathrm{Autf_L}(\Sigma, \boldsymbol{e})$ is exactly the same; replace KP by S. □

Proposition 2.13.

(1) Let $H = \pi_b^{-1}(\mathrm{Gal}^c_{\mathrm{L}}(\Sigma, \boldsymbol{e})) \leq \mathrm{Aut}_{\boldsymbol{e}}(\Sigma)$. *Then* $c \equiv^H d$ *if and only if* $c \equiv^{\mathrm{KP}}_{\boldsymbol{e}} d$.

(2) Let $H = \pi_b^{-1}(\mathrm{Gal}^0_{\mathrm{L}}(\Sigma)) \leq \mathrm{Aut}_{\boldsymbol{e}}(\Sigma)$. *Then* $c \equiv^H d$ *if and only if* $c \equiv^{\mathrm{S}}_{\boldsymbol{e}} d$.

Proof. (1): By Proposition 2.6, we know that $\pi_\Sigma^{-1}[\mathrm{Gal}^c_{\mathrm{L}}(\Sigma, \boldsymbol{e})] = \mathrm{Aut}_{\boldsymbol{e}'\boldsymbol{e}}(\mathfrak{C})$ for $\boldsymbol{e}' \in \mathrm{bdd}(\boldsymbol{e})$. Thus by commutativity of diagram in Remark 2.4, $H = \xi(\mathrm{Aut}_{\boldsymbol{e}'\boldsymbol{e}}(\mathfrak{C})) = \mathrm{Aut}_{\boldsymbol{e}'\boldsymbol{e}}(\Sigma)$ and so $c \equiv^H d$ if and only if $c \equiv_{\boldsymbol{e}'\boldsymbol{e}} d$. By Fact 1.17(2), $c \equiv^{\mathrm{KP}}_{\boldsymbol{e}} d$ if and only if $c \equiv_{\mathrm{bdd}(\boldsymbol{e})} d$, and $\boldsymbol{e}' \in \mathrm{bdd}(\boldsymbol{e})$, thus $c \equiv^{\mathrm{KP}}_{\boldsymbol{e}} d$ implies $c \equiv^H d$.

By Lemma 2.12, $\mathrm{Gal}^c_{\mathrm{L}}(\Sigma, \boldsymbol{e}) \leq \mathrm{Autf}_{\mathrm{KP}}(\Sigma, \boldsymbol{e}) / \mathrm{Autf}_{\mathrm{L}}(\Sigma, \boldsymbol{e})$. We already have proved that $c \equiv^{\mathrm{KP}}_{\boldsymbol{e}} d$ implies $c \equiv^H d$, and $c \equiv^{\mathrm{Autf}_{\mathrm{KP}}(\Sigma, \boldsymbol{e})} d$ if and only if $c \equiv^{\mathrm{KP}}_{\boldsymbol{e}} d$ by Remark 2.8, so it follows that $c \equiv^H d$ if and only if $c \equiv^{\mathrm{KP}}_{\boldsymbol{e}} d$.

(2): It can be proved using Fact 1.17(3) and the fact that in a topological group, the connected component containing the identity is the intersection of all closed normal subgroups of finite indices. □

Theorem 2.14.

(1) $\mathrm{Gal}^c_{\mathrm{L}}(\Sigma, \boldsymbol{e}) = \mathrm{Autf}_{\mathrm{KP}}(\Sigma, \boldsymbol{e}) / \mathrm{Autf}_{\mathrm{L}}(\Sigma, \boldsymbol{e})$.

(2) $\mathrm{Gal}^0_{\mathrm{L}}(\Sigma, \boldsymbol{e}) = \mathrm{Autf}_{\mathrm{S}}(\Sigma, \boldsymbol{e}) / \mathrm{Autf}_{\mathrm{L}}(\Sigma, \boldsymbol{e})$.

Proof. (1): Let $H_1 = \pi_b^{-1}[\mathrm{Gal}^c_{\mathrm{L}}(\Sigma, \boldsymbol{e})]$ and $H_2 = \mathrm{Autf}_{\mathrm{KP}}(\Sigma, \boldsymbol{e})$. By Remark 2.8 and Proposition 2.13, $\equiv^{H_1}$ and $\equiv^{H_2}$ are the same equivalence relations on the tuples of realizations of $\Sigma(\mathfrak{C})$. Then by Corollary 2.10, $H_1 = H_2$.

(2): By exactly the same proof as (1), by letting $H_1 = \pi_b^{-1}[\mathrm{Gal}^0_{\mathrm{L}}(\Sigma, \boldsymbol{e})]$ and $H_2 = \mathrm{Autf}_{\mathrm{S}}(\Sigma, \boldsymbol{e})$. □

References

[C11] Enrique Casanovas, Simple theories and hyperimaginaries (Lecture Notes in Logic), Cambridge University Press, 2011.

[CLPZ01] Enrique Casanovas, Daniel Lascar, Anand Pillay and Martin Ziegler, Galois groups of first order theories, Journal of Mathematical Logic, **1**, (2001), 305–319.

[DKKL21] Jan Dobrowolski, Byunghan Kim, Alexei Kolesnikov and Junguk Lee, The relativized Lascar groups, type-amalgamation, and algebraicity, Journal of Symbolic Logic, **86**, (2021), 531–557.

[DKL17] J. Dobrowolski, B. Kim, and J. Lee, The Lascar groups and the first homology groups in model theory, *Ann. Pure Appl. Logic*, **168**, (2017), 2129–2151.

[K14] Byunghan Kim, Simplicity theory, Oxford University Press, 2014.

[KL23] B. Kim, and H. Lee, Automorphism groups over a hyperimaginary, J. Math. Soc. Japan, **75**, (2023), 21–49,

[L82] Daniel Lascar, On the category of models of a complete theory, Journal of Symbolic Logic, **47**, (1982), 249–266.

[LP01] Daniel Lascar and Anand Pillay, Hyperimaginaries and automorphism groups, Journal of Symbolic Logic, **66**, (2001), 127–143.

[Lee22] H. Lee, Quotient groups of the Lascar group and strong types in the context of hyperimaginaries, Ph.D thesis, Yonsei University, 2022.

[Z02] Martin Ziegler, Introduction to the Lascar group, Tits buildings and the model theory of groups, London Math. Lecture Note Series, **291**, Cambridge University Press, (2002), 279–298.

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