EXISTENCE AXIOM OF PRE-INDEPENDENCE RELATIONS IN NSOP_1 THEORIES

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1. Introduction

This note is based on the author's talk in RIMS Symposia, "Model theoretic aspects of the notion of independence and dimension", held on December 4 − 6, 2023.

Kaplan and Ramsey introduced Kim-independence in [8], which partially inherits what non-forking independence satisfies in simple theories, assuming the theory is NSOP_1, and it successfully gives a characterization of NSOP_1 theories. However, it is still difficult to say that Kim-independence is a complete generalization of non-forking independence since it is still not known whether Kim-independence is equivalent to non-forking independence in simple theories over an arbitrary set. The goal of our work is finding a pre-independence relation over sets which generalizes feature of non-forking independence in simple theories over sets and Kim-independence in NSOP_1 theories over models. In other words, we want to find a pre-independence relation $\mathop{\smile\!\!\!\!\!\!|}\,$ such that:

(i) $\mathop{\smile\!\!\!\!\!\!|}\, = \mathop{\smile\!\!\!\!\!\!|}\,^f$ in simple theories over an arbitrary set,
(ii) $\mathop{\smile\!\!\!\!\!\!|}\, = \mathop{\smile\!\!\!\!\!\!|}\,^K$ in NSOP_1 theories over models,

and satisfies properties what $\mathop{\smile\!\!\!\!\!\!|}\,^K$ satisfies in NSOP_1 theories over models (such as existence, symmetry, independence theorem, etc.), in NSOP_1 theories over an arbitrary set.

As a partial achievement of this goal, we prove that $\mathop{\smile\!\!\!\!\!\!|}\,^{Kf}$ satisfies (i), (ii) above, and existence in NSOP_1 theories over sets. We leave a sketch of proof of the main theorem (Theorem 3.12) below.

2. Preliminary and notation

We quote the following notions of pre-independence relations from [1], [2], and [3].

Definition 2.1. [1][2][3, Definition 2.4] A *pre-independence relation* is an invariant ternary relation $\mathop{\smile\!\!\!\!\!\!|}\,$ on sets. If a triple of sets (a, b, c) is in the pre-independence relation $\mathop{\smile\!\!\!\!\!\!|}\,$, then we write it $a \mathop{\smile\!\!\!\!\!\!|}\,_c b$ and say *"a is $\mathop{\smile\!\!\!\!\!\!|}\,$-independent from b over c"*. Throughout this paper we will consider the following properties for a pre-independence relation. (If it is clear in the context, then we omit the words in the parenthesis.)

(i) Monotonicity (over d): If $aa' \mathop{\smile\!\!\!\!\!\!|}\,_d bb'$, then $a \mathop{\smile\!\!\!\!\!\!|}\,_d b$.
(ii) Base monotonicity (over d): If $a \mathop{\smile\!\!\!\!\!\!|}\,_d bb'$, then $a \mathop{\smile\!\!\!\!\!\!|}\,_{db} b'$.
(iii) Transitivity (over d): If $a \mathop{\smile\!\!\!\!\!\!|}\,_{db} c$ and $b \mathop{\smile\!\!\!\!\!\!|}\,_d c$, then $ab \mathop{\smile\!\!\!\!\!\!|}\,_d c$.

(iv) Right extension (over d): If $a \perp_d b$, then for all c, there exists $c' \equiv_{db} c$ such that $a \perp_d bc'$.
(v) Existence (over d): $a \perp_d d$ for all a. We say a set d is an *extension base for* $\perp$ if $\perp$ satisfies existence over d.
(vi) Finite character (over d): If $a \not\perp_d b$, then there exist finite $a' \subseteq a$ and $b' \subseteq b$ such that $a' \not\perp_d b'$.
(vii) Strong finite character (over d): If $a \not\perp_d b$, then there exist finite subtuple $b' \subseteq b$, finite tuples x', y' of variables with $|x'| \leq |a|$, $|y'| = |b'|$, and a formula $\varphi(x', y') \in \mathcal{L}(d)$ such that $\varphi(x', b') \in \mathrm{tp}(a/db)$ and $a' \not\perp_d b'$ for all $a' \models \varphi(x', b')$.

Definition 2.2. We say a formula with parameters $\varphi(x, a)$ divides over a set B, if there exists an indiscernible sequence $(a_i)_{i<\omega}$ over B with $a_0 = a$ such that $\{\varphi(x, a_i)\}_{i<\omega}$ is inconsistent. We say $\varphi(x, a)$ forks over B if there are dividing formulas $\psi_0(x, b_0), ..., \psi_{n-1}(x, b_{n-1})$ over B such that $\varphi(x, a) \vdash \bigvee_{i<n} \psi_i(x, b_i)$.

We write $a \perp^f_C b$ if $\mathrm{tp}(a/bC)$ has no forking formula over C. We write $a \perp^i_C b$ if there exists a global invariant type over C containing $\mathrm{tp}(a/bC)$. We write $a \perp^u_C b$ if $\mathrm{tp}(a/bC)$ is finitely satisfiable in C.

The definitions of Kim-independence was introdoced by Kaplan and Ramsey in [8]. We import a generalized version of it introduced by Mutchnik in [17].

Definition 2.3. Let $\perp$ be a pre-independence relation, κ an infinite cardinal, A a set of parameters. We call a sequence $(b_i)_{i<\kappa}$ an $\perp$*-Morley sequence over* A if it is indiscernible over A and $b_i \perp_A b_{<i}$ for all $i < \kappa$. Let $\varphi(x, y)$ be a formula. We say $\varphi(x, b)$ $\perp$*-Kim-divides over* A if there exists an $\perp$-Morley sequence $(b_i)_{i<\kappa}$ over A with $b_0 = b$ such that $\{\varphi(x, b_i)\}_{i<\kappa}$ is inconsistent. We say $\varphi(x, b)$ $\perp$*-Kim-forks over* A if there exist $\psi_0(x, b_0), ..., \psi_n(x, b_n)$ such that $\varphi(x, b) \vdash \bigvee_{i\leq n} \psi_i(x, b_i)$ and each $\psi_i(x, b_i)$ $\perp$-divides over A. We say a partial type $\Sigma(x)$ $\perp$*-Kim-forks over* A if it implies a formula that $\perp$-Kim-forks over A.

For a given pre-independence relation $\perp$, we say *a is $\perp$-Kim-independent from b over C* and write $a \perp^{K^{\perp}}_C b$ if $\mathrm{tp}(a/bC)$ does not $\perp$-forks over C. We will write $\perp^{K^f}$, $\perp^{K^i}$ for $\perp^{K^{\perp^f}}$, $\perp^{K^{\perp^i}}$ respectively, to simplify notation.

By $a \perp^{Kd^i}_C b$, we mean $\mathrm{tp}(a/bC)$ has no $\perp^i$-Kim-dividing formula.

We will frequently use a notion of ill-founded tree $\mathcal{T}^{\delta}_{\alpha}$ which is originally introduced in [8] by Kaplan and Ramsey. In this note we import a generalized version which appears in [13], with some additional notations.

Definition 2.4. [8][13] Suppose α and δ are ordinals. We define $\mathcal{T}^{\delta}_{\alpha}$ to be the set of functions η so that

(i) $\mathrm{dom}(\eta)$ is an end-segment of α of the form $[\beta, \alpha)$ for β equal to 0 or a successor ordinal. If α is a successor or 0, we allow $\beta = \alpha$, *i.e.* $\mathrm{dom}(\eta) = \emptyset$. Note that $\mathcal{T}^{\delta}_0 = \{\emptyset\}$.
(ii) $\mathrm{ran}(\eta) \subseteq \delta$.
(iii) Finite support: the set $\{\gamma \in \mathrm{dom}(\eta) : \eta(\gamma) \neq 0\}$ is finite.

If $\delta = \omega$, then we just write $\mathcal{T}_{\alpha}$.

Let $\mathcal{L}_{s,\alpha} = \{\lhd, <_{lex}, \wedge, \{P_\beta\}_{\beta<\alpha}\}$ for each ordinal α, where $\lhd, <_{lex}$ are binary relation symbols, $\wedge$ is a binary function symbol, and P_β is an unary relation symbol for each β. If it is clear in the context, we will omit 'α' in $\mathcal{L}_{s,\alpha}$ and just write it $\mathcal{L}_s$. We interpret $\mathcal{T}_\alpha^\delta$ as an $\mathcal{L}_s$-structure by defining each symbol as below.

(iv) $\eta \lhd \nu$ if and only if $\eta \subseteq \nu$. Write $\eta \perp \nu$ if $\neg(\eta \lhd \nu)$ and $\neg(\nu \lhd \eta)$.
(v) $\eta \wedge \nu = \eta|_{[\beta,\alpha)} = \nu|_{[\beta,\alpha)}$ where $\beta = \min\{\gamma : \eta|_{[\gamma,\alpha)} = \nu|_{[\gamma,\alpha)}\}$, if non-empty (note that β will not be a limit, by finite support). Define $\eta \wedge \nu$ to be the empty function if this set is empty (note that this cannot occur if α is a limit).
(vi) $\eta <_{lex} \nu$ if and only if $\eta \lhd \nu$ or, $\eta \perp \nu$ with $\mathrm{dom}(\eta \wedge \nu)=[\gamma+1, \alpha)$ and $\eta(\gamma) < \nu(\gamma)$.
(vii) For each ordinal $\beta < \alpha$, let $P_\beta^{\alpha,\delta} = \{\eta \in \mathcal{T}_\alpha^\delta : \mathrm{dom}(\eta) = [\beta, \alpha)\}$ (the β-th floor in $\mathcal{T}_\alpha^\delta$). If it is clear in the context, we omit α and δ, just write P_β. Note that if β is limit then P_β is empty.
(viii) Canonical inclusion: For $\alpha < \alpha'$, $\mathcal{T}_\alpha^2$ can be embedded in $\mathcal{T}_{\alpha'}^2$ with respect to $\mathcal{L}_{s,\alpha'}$ by a map $f_{\alpha,\alpha'} : \mathcal{T}_\alpha^2 \to \mathcal{T}_{\alpha'}^2 : \eta \mapsto \eta \cup \{(\beta, 0) : \beta \in \alpha' \setminus \alpha\}$. Unless otherwise stated, we regard $\mathcal{T}_\alpha^2$ as $f_{\alpha,\alpha'}(\mathcal{T}_\alpha^2)$ in $\mathcal{T}_{\alpha'}^2$. Note that by finite support, $\mathcal{T}_\alpha^2$ can be regarded as $\bigcup_{\beta<\alpha} \mathcal{T}_\beta^2$ with respect to canonical inclusions, for each limit ordinal α.
(ix) $\eta \perp_{lex} \nu$ if and only if $\eta <_{lex} \nu$ and $\eta \not\lhd \nu$. For an indexed set $\{a_\eta\}_{\eta \in \mathcal{T}_\alpha^2}$ and $\eta \in \mathcal{T}_\alpha^2$, by $a_{\perp_{lex}\eta}$ we mean the set $\{a_\nu : \nu \perp_{lex} \eta\}$.
(x) For each $\eta \in \mathcal{T}_\alpha^\delta$, let $t(\eta)$ be an ordinal such that $\mathrm{dom}(\eta) = [t(\eta), \alpha)$. If α is not limit, then $t(\emptyset) = \alpha$.
(xi) For each $\eta \in \mathcal{T}_\alpha^\delta$, $i < \delta$, let $(i)^\frown\eta = \eta \cup \{(\alpha, i)\} \in \mathcal{T}_{\alpha+1}^\delta$.
(xii) $(i)^\frown\mathcal{T}_\alpha^\delta = \{(i)^\frown\eta : \eta \in \mathcal{T}_\alpha^\delta\} \subseteq_{\alpha+1}^\delta$ for each α, δ and $i < \delta$.
(xiii) We let $\eta^\frown(i) := \eta \cup \{(t(\eta)-1, i)\}$, for each $i < \delta$ and $\eta \in \mathcal{T}_\alpha^\delta$ with $\mathrm{dom}(\eta) \neq [0, \alpha)$.
(xiv) $\mathrm{dom}^*(\eta)$ is $\{i \in \mathrm{dom}(\eta) : i \text{ is not a limit ordinal}\}$ for each $\eta \in \mathcal{T}_\alpha^\delta$.
(xv) For $\beta < \alpha$, $\zeta_\beta \in \mathcal{T}_\alpha^\delta$ is a function from $[\beta, \alpha)$ to δ such that $\zeta(i) = 0$ for all $i \in [\beta, \alpha)$.
(xvi) For all $v \subseteq \alpha$, $\mathcal{T}_{\alpha|v}^\delta$ is the set of all tuples $\eta \in \mathcal{T}_\alpha^\delta$ such that $t(\eta) \in v$ and $\eta(i) = 0$ for all $i \in \mathrm{dom}(\eta) \setminus v$.

Now we recall the notions of indiscernibility of tree and modeling property (cf. [11], [12], and [18]). Let M be a structure in a language $\mathcal{L}$. For a tuple $\bar{a}$ of elements in M and a subset A of M, by $\mathrm{qftp}_\mathcal{L}(\bar{a}/A)$ and $\mathrm{tp}_\mathcal{L}(\bar{a}/A)$, we mean the set of quantifier-free $\mathcal{L}_A$-formulas and the set of $\mathcal{L}_A$-formulas realized by $\bar{a}$ in M respectively. If there is no confusion, we may omit the subscript $\mathcal{L}$.

Let $\mathcal{I}$ be a structure in a language $\mathcal{L}_\mathcal{I}$. For a set $\{b_i : i \in \mathcal{I}\}$ and a finite tuple $\bar{i} = (i_0, \ldots, i_n)$ in $\mathcal{I}$, we write $\bar{b}_{\bar{i}}$ for the tuple $(b_{i_0}, \ldots, b_{i_n})$. Let $\mathbb{M}$ be a monster model of a complete theory T in a language $\mathcal{L}$. By $\bar{a}_{\bar{\eta}} \equiv_{\Delta,A} \bar{b}_{\bar{\nu}}$ (or $\mathrm{tp}_\Delta(\bar{a}_{\bar{\eta}}/A) = \mathrm{tp}_\Delta(\bar{b}_{\bar{\nu}}/A)$), we mean that for any $\mathcal{L}_A$-formula $\varphi(\bar{x}) \in \Delta$ where $\bar{x} = x_0 \cdots x_n$, $\bar{a}_{\bar{\eta}} \models \varphi(\bar{x})$ if and only if $\bar{b}_{\bar{\nu}} \models \varphi(\bar{x})$.

Definition 2.5. Let $\mathbb{M}$ be a monster model in a language $\mathcal{L}$ and $\mathcal{I}$ be an index structure in a language $\mathcal{L}_\mathcal{I}$.

(i) For $(b_i)_{i\in\mathcal{I}}$ of tuples of elements in $\mathbb{M}$ and a subset A of $\mathbb{M}$, we say that $(b_i)_{i\in\mathcal{I}}$ is *$\mathcal{I}$-indiscernible over A* if for any finite tuples $\bar{i}$ and $\bar{j}$ in $\mathcal{I}$,

$$\mathrm{qftp}_{\mathcal{L}_\mathcal{I}}(\bar{i}) = \mathrm{qftp}_{\mathcal{L}_\mathcal{I}}(\bar{j}) \text{ implies } \mathrm{tp}_\mathcal{L}(\bar{b}_{\bar{i}}/A) = \mathrm{tp}_\mathcal{L}(\bar{b}_{\bar{j}}/A).$$

(ii) We say $(b_i)_{i\in\mathcal{I}}$ is *$\mathcal{I}$-locally based* on $(a_i)_{i\in\mathcal{I}}$ over A if for all $\bar{i}$ and a finite set of $\mathcal{L}_A$-formulas Δ, there is $\bar{j}$ such that $\mathrm{qftp}_{\mathcal{L}_\mathcal{I}}(\bar{i}) = \mathrm{qftp}_{\mathcal{L}_\mathcal{I}}(\bar{j})$ and $\bar{b}_{\bar{i}} \equiv_{\Delta,A} \bar{a}_{\bar{j}}$.

We say that $\mathcal{I}$-indexed sets have the *modeling property* if for any $\mathcal{I}$-indexed set $(a_i)_{i\in\mathcal{I}}$, there is an $\mathcal{I}$-indiscernible set $(b_i)_{i\in\mathcal{I}}$, which is $\mathcal{I}$-locally based on $(a_i)_{i\in\mathcal{I}}$.

Definition 2.6. For ordinals α and δ, let $\mathcal{T}_\alpha^\delta$ be a tree in the language $\mathcal{L}_s$ with interpretations in Definition 2.4. We refer to a $\mathcal{T}_\alpha^\delta$-indexed indiscernible set as a *s-indiscernible tree*. We say that $(b_\eta)_{\eta\in\mathcal{T}_\alpha^\delta}$ is *s-locally based* on $(a_\eta)_{\eta\in\mathcal{T}_\alpha^\delta}$ over A if $(b_\eta)_{\eta\in\mathcal{T}_\alpha^\delta}$ is $\mathcal{T}_\alpha^\delta$-locally based on $(a_\eta)_{\eta\in\mathcal{T}_\alpha^\delta}$ over A.

Notation 2.7. For a tree $\mathcal{T}_\alpha^\delta$ in the language $\mathcal{L}_s$, by $\bar{\eta} \sim_s \bar{\nu}$, we mean $\mathrm{qftp}_{\mathcal{L}_s}(\bar{\eta}) = \mathrm{qftp}_{\mathcal{L}_s}(\bar{\nu})$ and say they are *s-isomorphic*.

Fact 2.8. *If δ is infinite, then for any ordinal α, a set A, and a tree of parameters $(a_\eta)_{\eta\in\mathcal{T}_\alpha^\delta}$, there exists an s-indiscernible $(b_\eta)_{\eta\in\mathcal{T}_\alpha^\delta}$ over A which is s-locally based on $(a_\eta)_{\eta\in\mathcal{T}_\alpha^\delta}$ over A.*

The proof of the above fact can be found in [11], [12], and [18]. We call Fact 2.8 the *s-modeling property*.

The following is easy to show but important when we want to construct a sequence of s-indiscernible trees whose members are pairwise distinct. It will be implicitly used when we apply Lemma 3.9 and Lemma 3.10 in the proof of the main theorem.

Remark 2.9. Let δ and δ' be infinite cardinal with $\delta < \delta'$. If $(a_\eta)_{\eta\in\mathcal{T}_\alpha^\delta}$ is s-indiscernible over A, then there exists an s-indiscernible tree $(b_\eta)_{\eta\in\mathcal{T}_\alpha^{\delta'}}$ over A such that $a_\eta = b_\eta$ for all $\eta \in \mathcal{T}_\alpha^\delta$ by compactness.

The following two techniques are useful when we want to construct an indiscernible sequence or tree whose information is based on a priorly given sequence or tree. Note that Fact 2.11 can be proved by using the same argument in [8, Lemma 5.10].

Fact 2.10. [10, Lemma 1.5] *Let A be a set, δ a cardinal, and $\kappa = \beth_{\lambda^+}(\lambda)$ where $\lambda = 2^{|A|+|T|+\delta}$. For any sequence of parameters $(a_i)_{i<\kappa}$ with $|a_i| = |a_j| \le \delta$ for all $i, j < \kappa$, there exists $(b_i)_{i<\omega}$ such that*

(i) *for all $n < \omega$, there exists $i_0 < \cdots < i_{n-1} < \kappa$ such that $b_0...b_{n-1} \equiv_A a_{i_0}...a_{i_{n-1}}$,*
(ii) *$(b_i)_{i<\omega}$ is indiscernible over A.*

Fact 2.11. [8, Lemma 5.10] *Let A be a set, δ a cardinal, δ' an infinite cardinal, and $\kappa = \beth_{\lambda^+}(\lambda)$ where $\lambda = 2^{|A|+|T|+\delta+\delta'}$. For any tree of parameters $(a_\eta)_{\eta\in\mathcal{T}_\kappa^{\delta'}}$ with $|a_\eta| = |a_\nu| \le \delta$ for all $\eta,\nu \in \mathcal{T}_\alpha^{\delta'}$, there exists $(b_\eta)_{\eta\in\mathcal{T}_\omega^{\delta'}}$ such that*

(i) *for all $n < \omega$, there exists $u \subseteq \kappa$ with $|u| = n$ such that $(b_\eta)_{\eta\in\mathcal{T}_n^{\delta'}} \equiv_A (a_\eta)_{\eta\in\mathcal{T}_{\kappa|v}^{\delta'}}$,*
(ii) *for all finite $u, v \subseteq \omega$ with $|u| = |v|$, $(b_\eta)_{\eta\in\mathcal{T}_{\omega|u}^{\delta'}} \equiv_A (b_\eta)_{\eta\in\mathcal{T}_{\omega|v}^{\delta'}}$.*

3. Existence of $\mathop{\smile\!\!\!\!\!\!\mid}$-Kim-independence

Lemma 3.1. *Let T be any complete theory and $\mathbb{M}$ its monster model. Let $A, B \subseteq \mathbb{M}$ be sets and κ an infinite cardinal. Suppose that we have a sequence $(b_i)_{i<\kappa} \in \mathbb{M}$ such* *that $b_i \mathop{\smile\!\!\!\!\!\!|}^f_A b_{<i}B$ for all $i < \kappa$. Then for any set $C \subseteq \mathbb{M}$, there exists $(b'_i)_{i<\kappa} \equiv_{AB} (b_i)_{i<\kappa}$ such that $b'_i \mathop{\smile\!\!\!\!\!\!|}^f_A b'_{<i}BC$ for all $i < \kappa$.*

Lemma 3.2. *Let T be any complete theory and $\mathbb{M}$ its monster model. Let A, B be small sets in the monster model and κ a sufficiently large infinite cardinal. Suppose that we have an indiscernible sequence $(b_i)_{i<\kappa}$ over AB such that $b_i \mathop{\smile\!\!\!\!\!\!|}^f_A b_{<i}B$ for all $i < \kappa$. Then for any small set C, there exists $(b'_i)_{i<\kappa} \equiv_{AB} (b_i)_{i<\kappa}$ such that*

- (i) *$b'_i \mathop{\smile\!\!\!\!\!\!|}^f_A b'_{<i}BC$ for all $i < \kappa$,*
- (ii) *$(b'_i)_{i<\kappa}$ is indiscernible over ABC.*

If $\mathop{\smile\!\!\!\!\!\!|}^f$ satisfies base monotonicity additionally, then $(b'_i)_{i<\kappa}$ is a $\mathop{\smile\!\!\!\!\!\!|}^f$-Morley sequence over D for any $A \subseteq D \subseteq ABC$.

Lemma 3.3. *Let T be any complete theory, α an ordinal, λ an infinite cardinal, B a set, and $(a_\eta)_{\eta\in\mathcal{T}_\alpha^\lambda}$ an s-indiscernible tree over B. If $(a'_\eta)_{\eta\in\mathcal{T}_\alpha^\lambda}$ is s-locally based on $(a_\eta)_{\eta\in\mathcal{T}_\alpha^\lambda}$ over B, then $(a'_\eta)_{\eta\in\mathcal{T}_\alpha^\lambda} \equiv_B (a_\eta)_{\eta\in\mathcal{T}_\alpha^\lambda}$.*

Lemma 3.4. *Let T be any complete theory, α a successor ordinal, λ an infinite cardinal, and B a set. If $(a_\eta)_{\eta\in\mathcal{T}_\alpha^\lambda}$ is an s-indiscernible tree over B, then it is s-indiscernible over $Ba_\emptyset$.*

Lemma 3.5. *Let T be any complete theory, α an ordinal, κ an infinite cardinal, and A, B, C sets. Let $(a_\eta)_{\eta\in\mathcal{T}_\alpha^\kappa}$ be an s-indiscernible tree over BC such that $A \mathop{\smile\!\!\!\!\!\!|}^f_C B(a_\eta)_{\eta\in\mathcal{T}_\alpha^\kappa}$. If $(a'_\eta)_{\eta\in\mathcal{T}_\alpha^\kappa}$ is s-locally based on $(a_\eta)_{\eta\in\mathcal{T}_\alpha^\kappa}$ over ABC, then $A \mathop{\smile\!\!\!\!\!\!|}^f_C B(a'_\eta)_{\eta\in\mathcal{T}_\alpha^\kappa}$.*

Lemma 3.6. *Let T be any complete theory, α a successor ordinal, λ an infinite cardinal, and κ a cardinal. Let B be a set, $(a_\eta)_{\eta\in\mathcal{T}_\alpha^\lambda}$ an s-indiscernible tree over B, and $(b_i)_{i<\kappa}$ a $\mathop{\smile\!\!\!\!\!\!|}^f$-Morley sequence over B with $b_0 := a_\emptyset$. Then there exists $(b'_i)_{i<\kappa} \equiv_{Bb_0} (b_i)_{i<\kappa}$ such that*

- (i) *$(a_\eta)_{\eta\in\mathcal{T}_\alpha^\lambda}$ is s-indiscernible over $Bb'_{<\kappa}$,*
- (ii) *$b'_i \mathop{\smile\!\!\!\!\!\!|}^f_B (a_\eta)_{\eta\in\mathcal{T}_\alpha^\lambda} b'_{<i}$ for each $0 < i < \kappa$.*

Definition 3.7. [4] A sequence of trees $(A_i)_{i<\beta}$ is said to be *mutually s-indiscernible over a set D* if A_i is s-indiscernible over $DA_{\neq i}$ for each $i < \beta$.

Remark 3.8. Let α be a successor ordinal. If a sequence of trees $((a^i_\eta)_{\eta\in\mathcal{T}_\alpha^\lambda})_{i<\beta}$ is mutually s-indiscernible over D, then it is mutually s-indiscernible over $Da_\emptyset^{<\beta}$ by Lemma 3.4.

Lemma 3.9. *Let T be any complete theory, α an ordinal, λ an infinite cardinal, and D a set. Assume that a mutually s-indiscernible sequence $((a^i_\eta)_{\eta\in\mathcal{T}_\alpha^\lambda})_{i<\beta}$ over D is given. If there exists $((\hat{a}^i_\eta)_{\eta\in\mathcal{T}_\alpha^\lambda})_{i\le\beta}$ such that*

- (i) *$(\hat{a}^\beta_\eta)_{\eta\in\mathcal{T}_\alpha^\lambda}$ is s-indiscernible over $D(a^{<\beta}_\eta)_{\eta\in\mathcal{T}_\alpha^\lambda}$,*
- (ii) *$(\hat{a}^i_\eta)_{\eta\in\mathcal{T}_\alpha^\lambda}$ is s-locally based on $(a^i_\eta)_{\eta\in\mathcal{T}_\alpha^\lambda}$ over $D(\hat{a}^{<i}_\eta)_{\eta\in\mathcal{T}_\alpha^\lambda}(a^{>i}_\eta)_{\eta\in\mathcal{T}_\alpha^\lambda}(\hat{a}^\beta_\eta)_{\eta\in\mathcal{T}_\alpha^\lambda}$ for each $i < \beta$,*
- (iii) *$(\hat{a}^i_\eta)_{\eta\in\mathcal{T}_\alpha^\lambda}$ is s-indiscernible over $D(\hat{a}^{<i}_\eta)_{\eta\in\mathcal{T}_\alpha^\lambda}(a^{>i}_\eta)_{\eta\in\mathcal{T}_\alpha^\lambda}(\hat{a}^\beta_\eta)_{\eta\in\mathcal{T}_\alpha^\lambda}$ for each $i < \beta$,*

then $((\hat{a}^i_\eta)_{\eta\in\mathcal{T}_\alpha^\lambda})_{i<\beta} \equiv_D ((a^i_\eta)_{\eta\in\mathcal{T}_\alpha^\lambda})_{i<\beta}$ and $((\hat{a}^i_\eta)_{\eta\in\mathcal{T}_\alpha^\lambda})_{i\le\beta}$ is mutually s-indiscernible over D.

Lemma 3.10. *Let T be any complete theory, α a successor ordinal, λ an infinite cardinal, κ a sufficiently large cardinal with $\kappa > \lambda$ and $cf(\kappa) = \kappa$, D a set, $(b_i)_{i<\kappa}$ a $\mathop{\smile\!\!\!\!\!\!|}^f$-Morley sequence over D, and $(a_\eta)_{\eta\in\mathcal{T}_\alpha^\lambda}$ an s-indiscernible tree over $Db_{<\kappa}$ with $a_\emptyset = b_0$. If $b_i \mathop{\smile\!\!\!\!\!\!|}^f_D (a_\eta)_{\eta\in\mathcal{T}_\alpha^\lambda}(b_k)_{0<k<i}$ for each $0 < i < \kappa$, then there exist $(A_i := (a^i_\eta)_{\eta\in\mathcal{T}_\alpha^\lambda})_{i<\kappa}$ and $(b'_i)_{i<\kappa} \equiv_{Db_0} (b_i)_{i<\kappa}$ such that $a^0_\eta := a_\eta$ for each $\eta \in \mathcal{T}_\alpha^\lambda$, $b'_i = a^i_\emptyset$ for each $i < \kappa$, and satisfy*

(i) *$(A_i)_{i<\kappa}$ is indiscernible over D,*
(ii) *$(A_i)_{i<\kappa}$ is mutually s-indiscernible over D,*
(iii) *$b'_j \mathop{\smile\!\!\!\!\!\!|}^f_D A_i(b'_k)_{i<k<j}$ for all $i < j < \kappa$.*

Lemma 3.11. *Let T be any complete theory, α an ordinal, λ an infinite cardinal, $n < \omega$, $\phi(x), \varphi_0(x, y_0), ..., \varphi_{n-1}(x, y_{n-1}) \in \mathcal{L}$, and $b_0, ..., b_{n-1}$ tuples of parameters. Suppose that*

(i) *$\vdash \forall x(\phi(x) \leftrightarrow \varphi_0(x, b_0) \vee \cdots \vee \varphi_{n-1}(x, b_{n-1}))$,*

and for a sufficient large cardinal κ with $\kappa > \lambda$, there exist $(A_k := (a^k_\eta)_{\eta\in\mathcal{T}_\alpha^\lambda})_{k<\kappa}$, $f : \alpha^* \to n$, and $D := (b'_0, ..., b'_{n-1}) \equiv (b_0, ..., b_{n-1})$ satisfying

(ii) *$\{\varphi_{f(t(\nu))}(x, a^k_\nu) \mid \nu \trianglelefteq \eta\}$ is consistent for each $k < \kappa$ and $\eta \in \mathcal{T}_\alpha^\lambda$,*
(iii) *$a^k_\eta \equiv b_{f(t(\eta))}$ for all $k < \kappa$ and $\eta \in \mathcal{T}_\alpha^\lambda$,*
(iv) *$(A_k)_{k<\kappa}$ is indiscernible over D,*
(v) *A_0 is s-indiscernible over D,*

where $\alpha^* := \alpha + 1$ if α is 0 or a successor, $\alpha^* := \alpha$ if α is limit. Then there exists $j < n$ and such that

$$\{\varphi_{f(t(\nu))}(x, a^k_\nu) \mid \nu \trianglelefteq \eta\} \cup \{\varphi_j(x, b'_j)\}$$

is consistent for each $\eta \in \mathcal{T}_\alpha^\lambda$ and $k < \kappa$.

Now we are ready to prove the main theorem.

Theorem 3.12. *Suppose T is NSOP$_1$. Then $\mathop{\smile\!\!\!\!\!\!|}^{Kf}$ satisfies existence over an arbitrary set.*

Proof. It is enough to show that $a \mathop{\smile\!\!\!\!\!\!|}^{Kf}_E \emptyset$ for any tuple of parameters a and a set E. Without loss of generality, we may assume $E = \emptyset$. So we show $a \mathop{\smile\!\!\!\!\!\!|}^{Kf} \emptyset$. Suppose not. Then there exists $\phi(x) \in \text{tp}(a)$ such that $\phi(x)$ $\mathop{\smile\!\!\!\!\!\!|}^f$-Kim-forks over $\emptyset$. So there exist $\varphi_0(x, b_0), ..., \varphi_{n-1}(x, b_{n-1})$ such that $\phi(x) \models \varphi_0(x, b_0) \vee \cdots \vee \varphi_{n-1}(x, b_{n-1})$ and $\varphi_i(x, b_i)$ $\mathop{\smile\!\!\!\!\!\!|}^f$-Kim-divides over $\emptyset$ for all $i < n$.

Let $\mu' < \lambda < \mu$ be sufficiently large infinite cardinals with self-cofinality such that $\mu' > \beth_{(2^{|T|+\omega})^+}(2^{|T|+\omega})$, $\lambda > \beth_{(2^{\mu'})^+}(2^{\mu'})$, and $\mu > \beth_{(2^\lambda)^+}(2^\lambda)$. Since $\varphi_i(x, b_i)$ $\mathop{\smile\!\!\!\!\!\!|}^f$-Kim-divides over $\emptyset$, there exists a $\mathop{\smile\!\!\!\!\!\!|}^f$-Morley sequence $(b^l_i)_{l<\lambda}$ over $\emptyset$ with $b^0_i = b_i$ such that $\{\varphi_i(x, b^l_i)\}_{l<\lambda}$ is inconsistent, for each $i < n$. By replacing $\varphi_i(x, b_i)$ with $\phi(x) \wedge \varphi_i(x, b_i)$, we may assume that $\vdash \forall x(\phi(x) \leftrightarrow \varphi_0(x, b_0) \vee \cdots \vee \varphi_{n-1}(x, b_{n-1}))$.

For each ordinal α, let $\alpha^* := \alpha + 1$ if α is 0 or a successor, and $\alpha^* := \alpha$ if α is limit.

Claim 3.12.1. There exist a sequence of s-indiscernible trees $((a^\alpha_\eta)_{\eta\in\mathcal{T}_\alpha^\lambda})_{\alpha<\mu}$ over $\emptyset$ and a sequence of functions $(f_\alpha : \alpha^* \to n)_{\alpha<\mu}$ such that

(i) $a^\alpha_\eta \equiv b_{f_\alpha(t(\eta))}$ for each $\eta \in \mathcal{T}_\alpha^\lambda$,
(ii) $(a^\alpha_{\eta^\frown\langle l\rangle})_{l<\lambda} \equiv (b^l_{f_\alpha(t(\eta)-1)})_{l<\lambda}$ for each $\eta \in \mathcal{T}_\alpha^\lambda$ with $t(\eta) \neq 0$,

(iii) $\{\varphi_{f_\alpha(t(\nu))}(x, a^\alpha_\nu) \mid \nu \trianglelefteq \eta\}$ is consistent for each $\eta \in \mathcal{T}^\lambda_\alpha$,
(iv) $a^\alpha_{\eta^\frown(l)} \mathop{\smile\!\!\!\!\!\!|}^f (a^\alpha_\nu)_{\nu \trianglerighteq \eta^\frown(0)}(a^\alpha_{\eta^\frown(l')})_{l'<l}$ for each $0 < l < \lambda$ and $\eta \in \mathcal{T}^\lambda_\alpha$ with $t(\eta) \neq 0$,

for each $\alpha < \mu$, and

(v) $(a^{\alpha_0}_\eta)_{\eta\in\mathcal{T}^\lambda_{\alpha_0}} \subseteq (a^{\alpha_1}_\eta)_{\eta\in\mathcal{T}^\lambda_{\alpha_1}}$ by canonical inclusion and $f_{\alpha_0} \subseteq f_{\alpha_1}$ for each $\alpha_0 < \alpha_1 < \mu$.

Proof of Claim 3.12.1. For $(a^0_\eta)_{\eta\in\mathcal{T}^\lambda_0}$ and $f_0 : 1 \to n$, choose any $j < n$ and let $a^0_\emptyset := b_j$ and $f_0(0) = j$. Then they satisfy all conditions we want.

Suppose that we have constructed $((a^\beta_\eta)_{\eta\in\mathcal{T}^\lambda_\beta})_{\beta<\alpha}$ and $(f_\beta : \beta^* \to n)_{\beta<\alpha}$ satisfying all conditions, for some $\alpha < \mu$. First we assume that α is a successor. Then there exists $\beta < \mu$ such that $\alpha = \beta + 1$. Suppose β is also a successor. Choose any sufficiently large cardinal κ with $\kappa > \lambda$ and $\mathrm{cf}(\kappa) = \kappa$. Then there exists a $\mathop{\smile\!\!\!\!\!\!|}^f$-Morley sequence $(b^k)_{k<\kappa}$ over $\emptyset$ such that $(b^l)_{l<\lambda} \equiv (b^l_{f_\beta(\beta)})_{l<\lambda}$ and $b^0 = a^\beta_\emptyset$.

By Lemma 3.2, there exists $D := (b^*_0, ..., b^*_{n-1}) \equiv (b_0, ..., b_{n-1})$ such that $(b^k)_{k<\kappa}$ is a $\mathop{\smile\!\!\!\!\!\!|}^f$-Morley sequence over D.

By applying s-modeling property, Lemma 3.3, Lemma 3.4, and automorphism, we may assume that $(a^\beta_\eta)_{\eta\in\mathcal{T}^\lambda_\beta}$ is s-indiscernible over D.

By Lemma 3.6, there exists $(b'_k)_{k<\kappa} \equiv_{Db^0} (b^k)_{k<\kappa}$ such that $(a^\beta_\eta)_{\eta\in\mathcal{T}^\lambda_\beta}$ is s-indiscernible over $Db'_{<\kappa}$ and $b'_k \mathop{\smile\!\!\!\!\!\!|}^f_D (a^\beta_\eta)_{\eta\in\mathcal{T}^\lambda_\beta} b'_{<k}$ for all $0 < k < \kappa$.

By Lemma 3.10, there exist $(A_k := (a^{\beta,k}_\eta)_{\eta\in\mathcal{T}^\lambda_\beta})_{k<\kappa}$ and $(b''_k)_{k<\kappa} \equiv_{Db'_0} (b'_k)_{k<\kappa}$ such that $a^{\beta,0}_\eta := a^\beta_\eta$ for each $\eta \in \mathcal{T}^\lambda_\beta$, $b''_k = a^{\beta,k}_\emptyset$ for each $k < \kappa$, and satisfy

(I) $(A_k)_{k<\kappa}$ is indiscernible over D,
(II) $(A_k)_{k<\kappa}$ is mutually s-indiscernible over D,
(III) $b''_j \mathop{\smile\!\!\!\!\!\!|}^f_D A_i(b''_k)_{i<k<j}$ for all $0 < j < \kappa$.

By Lemma 3.11, there exists $j < n$ such that

$$\{\varphi_{f_\beta(t(\nu))}(x, a^{\beta,k}_\nu) \mid \nu \trianglelefteq \eta\} \cup \{\varphi_j(x, b^*_j)\}$$

is consistent for each $\eta \in \mathcal{T}^\lambda_\beta$ and $k < \kappa$. For each $\eta \in \mathcal{T}^\lambda_\alpha$, let

$$\hat{a}^\alpha_\eta = \begin{cases} a^{\beta,l}_\nu & \text{if } \eta = (l)^\frown\nu \\ b^*_j & \text{if } \eta = \emptyset, \end{cases}$$

and $f_\alpha = f_\beta \cup \{(\alpha, j)\}$. Let $(\hat{\hat{a}}^\alpha_\eta)_{\eta\in\mathcal{T}^\lambda_\alpha}$ be an s-indiscernible tree over $\emptyset$ which is locally based on $(\hat{a}^\alpha_\eta)_{\eta\in\mathcal{T}^\lambda_\alpha}$ over $\emptyset$. Then by Lemma 3.3, $(\hat{\hat{a}}^\alpha_\eta)_{\eta\in\mathcal{T}^\lambda_\beta} \equiv (a^\beta_\eta)_{\eta\in\mathcal{T}^\lambda_\beta}$. By automorphism, we can find an s-indiscernible tree $(a^\alpha_\eta)_{\eta\in\mathcal{T}^\lambda_\alpha}$ containing $(a^\beta_\eta)_{\eta\in\mathcal{T}^\lambda_\beta}$ with respect to canonical inclusion, satisfying (i), (ii), (iii), (iv), and (v) with f_α.

Now we suppose that β is limit. Again, we choose a sufficiently large cardinal κ with $\kappa > \lambda$ and $\mathrm{cf}(\kappa) = \kappa$. By Lemma 3.3 and s-modeling property, we can find $D := (b^*_0, ..., b^*_{n-1}) \equiv (b_0, ..., b_{n-1})$ such that $(a^\beta_\eta)_{\eta\in\mathcal{T}^\lambda_\beta}$ is s-indiscernible over D. Since κ is sufficiently large, we can find $(A_k := (a^{\beta,k}_\eta)_{\eta\in\mathcal{T}^\lambda_\beta})_{k<\kappa}$ such that $(A_k)_{k<\kappa}$ is indiscernible and mutually s-indiscernible over D, by applying Lemma 3.9 and s-modeling property repeatedly. By Lemma 3.11, there exists $j < n$ such that

$$\{\varphi_{f_\beta(t(\nu))}(x, a^{\beta,k}_\nu) \mid \nu \trianglelefteq \eta\} \cup \{\varphi_j(x, b^*_j)\}$$

is consistent for each $\eta \in \mathcal{T}^\lambda_\beta$ and $k < \kappa$. By the same construction above, we can find $(a^\alpha_\eta)_{\eta \in \mathcal{T}^\lambda_\alpha}$ and $f_\alpha : \alpha^* \to n$ satisfying (i), (ii), (iii), (iv), and (v).

If α is limit, then we just take $(a^\alpha_\eta)_{\eta \in \mathcal{T}^\lambda_\alpha} := \bigcup_{\beta<\alpha} (a^\beta_\eta)_{\eta \in \mathcal{T}^\lambda_\beta}$ and $f_\alpha := \bigcup_{\beta<\alpha} f_\beta$. By finite support, they satisfy all conditions we want. This completes proof of Claim 3.12.1. $\square$

Claim 3.12.2. There exists an s-indiscernible tree $(a_\eta)_{\eta \in \mathcal{T}^\lambda_\mu}$ over $\emptyset$ and $\varphi(x,y) \in \mathcal{L}$, and a $\mathop{\smile\hskip -0.9em ^| \ }^f$-Morley sequence $(b^l)_{l<\lambda}$ such that

- (†) $\{\varphi(x,b^l)\}_{l<\lambda}$ is inconsistent,
- (‡) $(a_{\eta^\frown(l)})_{l<\lambda} \equiv (b^l)_{l<\lambda}$ for each $\eta \in \mathcal{T}^\lambda_\mu$ with $t(\eta) \neq 0$,
- (‡†) $\{\varphi(x,a_\nu) \mid \nu \trianglelefteq \eta\}$ is consistent for each $\eta \in \mathcal{T}^\lambda_\mu$,
- (‡‡) $a_{\eta^\frown(l)} \mathop{\smile\hskip -0.9em ^| \ }^f (a_\nu)_{\nu \trianglerighteq \eta^\frown(0)} (a_{\eta^\frown(l')})_{l'<l}$ for each $0 < l < \lambda$ and $\eta \in \mathcal{T}^\lambda_\mu$ with $t(\eta) \neq 0$.

Proof of Claim 3.12.2. Let $((a^\alpha_\eta)_{\eta \in \mathcal{T}^\lambda_\alpha})_{\alpha<\mu}$ and $(f_\alpha : \alpha^* \to n)_{\alpha<\mu}$ be sequences given by Claim 3.12.1. Take unions $(a'_\eta)_{\eta \in \mathcal{T}^\lambda_\mu} := \bigcup_{\alpha<\mu}((a^\alpha_\eta)_{\alpha<\mu})$ and $f := \bigcup_{\alpha<\mu} f_\alpha$. By pigeonhole principle, we may assume that f is monochromatic. $\square$

Recall the choice of μ' in the begining of the proof.

Claim 3.12.3. There exists a mutually indiscernible $(I_m = (c^m_l)_{l<\lambda})_{m<\mu'}$ such that

- $(*)$ $c^m_l \mathop{\smile\hskip -0.9em ^| \ }^f I_{<m} c^m_{<l}$ for all $m < \mu'$, $l < \lambda$,
- $(\substack{*\\ *})$ $I_{m_0} \ldots I_{m_{n-1}} \equiv I_0 \ldots I_{n-1}$ for all $m_0 < \cdots < m_{n-1} < \mu'$,
- $(\substack{**\\ *})$ $\{\varphi(x, c^m_{g(m)}) \mid m < \mu'\}$ is consistent for each $g : \mu' \to \lambda$,
- $(\substack{**\\ **})$ $\{\varphi(x, c^m_l) \mid l < \lambda\}$ is inconsistent for each $m < \mu'$,

Proof of Claim 3.12.3. Let $(a_\eta)_{\eta \in \mathcal{T}^\lambda_\mu}$, $\varphi(x,y)$, and $(b^l)_{l<\lambda}$ be given by Claim 3.12.2. By Fact 2.11, there exists $(a'_\eta)_{\eta \in \mathcal{T}^\lambda_\omega}$ such that

- (♯) for all $n < \omega$, there exists $u \subseteq \kappa$ with $|u| = n$ such that $(a'_\eta)_{\eta \in \mathcal{T}^\lambda_n} \equiv (a_\eta)_{\eta \in \mathcal{T}^\lambda_{\kappa|v}}$,
- (♯♯) for all finite $u, v \subseteq \omega$ with $|u| = |v|$, $(a'_\eta)_{\eta \in \mathcal{T}^\lambda_{\omega|u}} \equiv (a'_\eta)_{\eta \in \mathcal{T}^\lambda_{\omega|v}}$.

For each $m < \omega$, $l < \lambda$, let $c^m_l = a_{\zeta_{m+1}{}^\frown(l+1)}$ and $I_m := (c^m_l)_{l<\lambda}$. Then $(I_m)_{m<\omega}$ is mutually indiscernible and satisfies $(*)$, $(\substack{*\\ *})$, and $(\substack{**\\ **})$. $(\substack{**\\ *})$ is by the assumption that T is NSOP$_1$. By using EM-type of $(I_m)_{m<\omega}$ over $\emptyset$, we can extend $(I_m)_{m<\omega}$ to $(I_m)_{m<\mu'}$. $\square$

From now we work on $(I_m)_{m<\mu'}$ obtained in Claim 3.12.3. Choose any small model M.

Claim 3.12.4. There exists $(J_m := (d^m_i)_{i<\omega})_{m<\mu'}$ such that

- $(\substack{***\\ **})$ $\varphi(x, d^m_i)$ $\mathop{\smile\hskip -0.9em ^| \ }^f$-Kim-divides over M for each $m < \mu'$ and $i < \omega$.
- $(\substack{***\\ ***})$ J_m is a subsequence of I_m for each $m < \mu'$,
- $(\substack{****\\ ***})$ $d^m_0 \mathop{\smile\hskip -0.9em ^| \ }^{K^i}_M J_{<m}$ for each $m < \mu'$.

Proof of Claim 3.12.4. I_0 is $\mathop{\smile\hskip -0.9em ^| \ }^f$-Morley sequence over $\emptyset$ and M is small, there exists $I'_0 \equiv I_0$ such that I'_0 is a $\mathop{\smile\hskip -0.9em ^| \ }^f$-Morley sequence over M by Lemma 3.2. By replacing automorphic image, we may assume $I'_0 = I_0$. Let $d^0_i := c^0_i$ for each $i < \omega$ and put $J_0 := (d^0_i)_{i<\omega}$. Note that $e \mathop{\smile\hskip -0.9em ^| \ }^{K^i}_M \emptyset$ for all e since M is a model. Thus $d^0_0 \mathop{\smile\hskip -0.9em ^| \ }^{K^i}_M \emptyset$. $(\substack{***\\ **})$ is clear.

Now suppose that we have constructed $(J_m)_{m<\mu_0}$ satisfying $(\substack{***\\**})$, $(\substack{***\\***})$, and $(\substack{****\\***})$ for some $\mu_0 < \mu'$. Note that I_{μ_0} is indiscernible over $J_{<\mu_0}$ and $c_l^{\mu_0} \mathop{\smile\!\!\!\!\!\!\mid}^f J_{<\mu_0} c_{<l}^{\mu_0}$ for all $l < \lambda$ by (*). By the choice of μ' and λ, we can apply Lemma 3.2 to obtain $(I'_{\mu_0}) \equiv_{J_{<\mu_0}} I_{\mu_0}$ which is a $\mathop{\smile\!\!\!\!\!\!\mid}^f$-Morley sequence over M. By replacing automorphic image again, we may assume $I'_{\mu_0} = I_{\mu_0}$. Let $d_i^{\mu_0} := c_i^{\mu_0}$ for each $i < \omega$ and put $J_{\mu_0} := (d_i^{\mu_0})_{i<\omega}$.

Since $\mathop{\smile\!\!\!\!\!\!\mid}^f$ is stronger than $\mathop{\smile\!\!\!\!\!\!\mid}^{K^i}$ over models, J_{μ_0} is a $\mathop{\smile\!\!\!\!\!\!\mid}^{K^i}$-Morley sequence over M. Thus we have $J_{<\mu_0} \mathop{\smile\!\!\!\!\!\!\mid}_M^{Kd^i} d_0^{\mu_0}$ by Kim's lemma over models in NSOP$_1$. Since $\mathop{\smile\!\!\!\!\!\!\mid}^i$-Kim-dividing and $\mathop{\smile\!\!\!\!\!\!\mid}^i$-Kim-forking are equivalent over models in NSOP$_1$, we have $J_{<\mu_0} \mathop{\smile\!\!\!\!\!\!\mid}_M^{K^i} d_0^{\mu_0}$. By symmetry of $\mathop{\smile\!\!\!\!\!\!\mid}^{K^i}$ over models, we have $d_0^{\mu_0} \mathop{\smile\!\!\!\!\!\!\mid}_M^{K^i} J_{<\mu_0}$. Thus J_{μ_0} satisfies $(\substack{***\\***})$ and $(\substack{****\\***})$. $(\substack{***\\**})$ is clear.

By continuing this we can find $(J_m)_{m<\mu'}$ satisfying all conditions we want. □

In particular, $d_0^m \mathop{\smile\!\!\!\!\!\!\mid}_M^{K^i} d_0^{<m}$ for all $m < \mu'$. By Fact 2.10, we may assume $(d_0^m)_{m<\mu'}$ is indiscernible over M. Thus $(d_0^m)_{m<\mu'}$ is a $\mathop{\smile\!\!\!\!\!\!\mid}^{K^i}$-Morley sequence over M. But $\{\varphi(x, d_0^m)\}_{m<\mu'}$ is consistent, thus $\varphi(x, d_0^0)$ does not $\mathop{\smile\!\!\!\!\!\!\mid}^{K^i}$-Kim-divides over M by Kim's lemma. It yields a contradiction with $(\substack{***\\**})$. □

Corollary 3.13. *Suppose T is NSOP$_1$. Then for all C and $p \in S(C)$, there exists a $\mathop{\smile\!\!\!\!\!\!\mid}^{K^f}$-Morley sequence in p over C.*

References

[1] Hans Adler, A Geometric Introduction to Forking and Thorn-Forking, *Journal of Mathematical Logic*, **9**, (2009), 1–20.

[2] Hans Adler, An introduction to theories without the independence property, *Archive for Mathematical Logic*, *accepted*.

[3] Artem Chernikov and Itay Kaplan, Forking and Dividing in NTP$_2$ theories, *The Journal of Symbolic Logic*, **77**, (2012), 1–20.

[4] Artem Chernikov, Byunghan Kim, and Nicholas Ramsey, Transitivity, Lowness, and Ranks in NSOP$_1$ Theories, (2020).

[5] A. Chernikov and N. Ramsey, On model-theoretic tree properties, *J. of Math. Logic* (2016).

[6] J. Dobrowolski, B. Kim, and N. Ramsey, Independence over Arbitrary Sets in NSOP$_1$ Theories, *preprint*, 2022.

[7] James Hanson, Kim-forking implies quasi-dividing via fracturing, unpublished note, 2023

[8] I. Kaplan and N. Ramsey, 'On Kim-independence', to appear in *J. of EMS.*

[9] I. Kaplan, N. Ramsey, and S. Shelah, Local character of Kim-independence, https://arxiv.org/abs/1707.02902v2

[10] Byunghan Kim, Simplicity theory, Oxford University Press, 2013

[11] Byunghan Kim and HyeungJoon Kim, Notions around tree property 1, *Ann. Pure Appl. Logic*, **162**, (2011), 698–709.

[12] Byunghan Kim, HyeungJoon Kim and Lynn Scow, Tree indiscernibilities, revisited, *Arch. Math. Logic* **53** (2014), 211–232.

[13] J. Kim and H. Lee, Some Remarks on Kim-dividing in NATP Theories, submitted.

[14] A. Kruckman and N. Ramsey, A New Kim's Lemma, https://arxiv.org/abs/2309.02718, 2023

[15] David Marker, Model Theory: An Introduction, Spinger, 2002.

[16] Scott Mutchnik, On NSOP$_2$ Theories, *preprint* (2022).

[17] Scott Mutchnik, Conant-Independence and Generalized Free Amalgamation, *preprint* (2022).

[18] Kota Takeuchi and Akito Tsuboi, On the existence of indiscernible trees, *Ann. Pure Appl. Logic*, **163**, (2012), 1891–1902.