A note on ω-categorical stable theories

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Hrushovski constructed ω-categorical strictly stable structures to refute Lachlan's conjecture.

Theorem 0.1 ([1], 1988) If $\alpha \in (1/2, 2/3)$ and index$(\alpha) = \infty$, then the $\mathbf{H}_{f_\alpha}$-generic graph is ω-categorical and strictly stable.

$\mathbf{H}_{f_\alpha}$ is a class of finite graphs controlled by a function $f_\alpha : \omega \to \mathbf{R}$, which is called Hrushovski's class. However, Hrushovski's class $\mathbf{H}_{f_\alpha}$ contains something unnecessary to construct a counterexample of Lachlan's conjecture. In this short note, we introduce $\mathbf{K}_\alpha (\subset \mathbf{H}_{f_\alpha})$ as the class of finite graphs generated by an edge graph, and prove the following proposition.

Proposition 0.2 (I.) If $\alpha \in (0, 1)$ and index$(\alpha) = \infty$, then the $\mathbf{K}_\alpha$-generic graph is ω-categorical and strictly stable.

1 Preliminaries

In this note, graphs means undirected simple graphs. Let $A, B, C, \cdots$ be finite graphs. For α with $0 < \alpha < 1$, we define a predimension of A by $\delta_\alpha(A) = |A| - \alpha|R^A|$. We often abbreviate $\delta_\alpha(*)$ to $\delta(*)$. Let denote $\delta(B/A) = \delta(B \cup A) - \delta(A)$. A is said to be closed in B (in symbol, $A \leq B$), if $\delta(X/A) \geq 0$ for any $X \subset B - A$. For $A \subset B$, let denote the closure of A in B as $\mathrm{cl}_B(A) = \bigcap\{C : A \subset C \leq B\}$.

Let $A, B,$ be finite graphs with $A = B \cap C$. Then the free amalgam B and C over A is defined by $B \oplus_A C = (B \cup C, R^B \cup R^C)$. Let $\mathbf{K}$ be a class

*The author is supported by Grants-in-Aid for Scientific Research (No.20K03725). This work was supported by the Research Institute for Mathematical Sciences, an International Joint Usage/Research Center located in Kyoto University.

of finite graphs closed under substructures. Then $\mathbf{K}$ is said to have the free amalgamation property (for short, FAP), if whenever $A \leq B, C \in \mathbf{K}$ then $B \oplus_A C \in \mathbf{K}$. A countable graph M is said to be $\mathbf{K}$-generic, if it satisfies the following conditions.

1. $A \subset_{\text{fin}} M$ implies $A \in \mathbf{K}$,
2. if $A \leq B \in \mathbf{K}$ and $A \leq M$, then there exists some $B' \cong_A B$ with $B' \leq M$,
3. $A \subset_{\text{fin}} M$ implies that $\text{cl}_M(A)$ is finite.

If $\mathbf{K}$ has the amalgamation property, there exists a $\mathbf{K}$-generic graph M. Moreover, by the back-and-forth argument, any $\mathbf{K}$-generic graph is isomorphic to M.

2 Proposition

An edge graph means the graph ab with $R(a, b)$. Let $\mathbf{K}_\alpha$ be the class of finite graphs which is generated from an edge graph by FAP. For example, any finite tree is in $\mathbf{K}_\alpha$, and a triangle is not in $\mathbf{K}_\alpha$. Moreover, it can be seen that if $1/2 < \alpha \leq 2/3$ then a square and a pentagon are not in $\mathbf{K}_\alpha$, but a hexagon is in $\mathbf{K}_\alpha$.

For each $i \in \omega - \{0\}$, let $s_\alpha(i) = \min\{(n - m\alpha)/n : n \leq i, n - m\alpha \geq 0\}$ and $s_\alpha(0) = 1$. For instance, when $\alpha = 4/7$, it can be seen that $s_\alpha(1) = 3/7, s_\alpha(2) = 2/7, s_\alpha(3) = 1/7, s_\alpha(4) = 0$. Then we can define Hrushovski's control function $f_\alpha : \omega \to \mathbf{R}$ by $f_\alpha(0) = 0$ and $f_\alpha(n + 1) = \sum_{i \leq n} s_\alpha(i)$ for each $n \in \omega$. Let $\text{index}(\alpha) = \sum_{i \in \omega} s_\alpha(i)$.

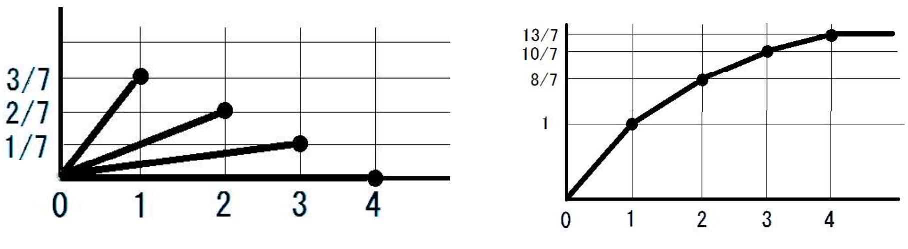

Figure 1: $s_\alpha(i)$ and $f_\alpha(n)$ in the case of $\alpha = 4/7$

Remark 2.1 1. If α is rational, then index(α) is finite.
2. $\{\alpha \in (0,1) : \text{index}(\alpha) = \infty\}$ is dense.

Proof. For 1, if α is rational, then it is clear that there is some i with $s_\alpha(j) = 0$ for any $j \geq i$. Hence index(α) is finite. 2 is proved by the Baire category theorem. For $n \in \omega$, let $O_n = \{\alpha \in (0,1) : \text{index}(\alpha) > n\}$.
Claim 1: O_n is open.
Proof. Take any $\alpha \in O_n$. Since index(α) $> n$, we can take $k \in \omega$ with $\sum_{i=1}^k s_\alpha(i) > n$. For any β sufficiently close to α, $s_\beta(i)$ is nearly equal to $s_\alpha(i)$ for each $i \leq k$. Then $\sum_{i=1}^k s_\beta(i) > n$, and then $\beta \in O_n$.
Claim 2: O_n is dense.
Proof. Take any rational $p/q \in (0,1)$. Take $k \in \omega$ with $\sum_{i=1}^k (1/i) > 2nq$. Let $\alpha = p/q + 1/2kq^2$. By the definition of $s_\alpha(i)$, we can take $m_i, l_i \in \omega$ with $s_\alpha(i) = 1 - (l_i/m_i)\alpha$ where $m_i \leq i$. Note that $l_i \leq m_i q/p \leq kq/p \leq kq$. Then $s_\alpha(i) = 1 - (p/q + 1/2kq^2)(l_i/m_i) = \{(qm_i - pl_i)/q - l_i/2kq^2\}/m_i \geq (1/q - l_i/2kq^2)/i \geq (1/q - 1/2q)/i = 1/2qi$. So $\sum_{i=1}^k s_\alpha(i) \geq \sum_{i=1}^k (1/2qi) > 2nq/2q = n$. Hence $\alpha \in O_n$.

By Claim 1 and 2, $\{\alpha \in (0,1) : \text{index}(\alpha) = \infty\}$ is dense.

Let $\mathbf{H}_{f_\alpha} = \{A : \delta(A') \geq f_\alpha(|A'|) \text{ for any } A' \subset A\}$ be Hrushovski's class.

Lemma 2.2 $\mathbf{H}_{f_\alpha}$ has FAP.

Proof. Take A, B, C with $A \leq B, C \in H_{f_\alpha}$. We can assume that $\delta(B/A) \leq \delta(C/A)$. By the definition of $s_\alpha(i)$, we have $(\delta(B) - \delta(A))/|B - A| \geq s_\alpha(|B|)$. So we have $\delta(B \oplus_A C) \geq f_\alpha(|B \oplus_A C|)$. By the similar argument, it can be checked that if $X \subset B \oplus_A C$ then $\delta(X) \geq f_\alpha(|X|)$. Hence $B \oplus_A C \in \mathbf{H}_{f_\alpha}$.

Since $\mathbf{K}_\alpha$ has FAP, there exists the $\mathbf{K}_\alpha$-generic graph M_α.

Lemma 2.3 If index(α) $= \infty$, then M_α is ω-categorical.

Proof. Since $\mathbf{H}_{f_\alpha}$ has FAP, we have $\mathbf{K}_\alpha \subset \mathbf{H}_{f_\alpha}$. Since index($\alpha$) $= \infty$, f_α is unbounded. Then the statement that a finite set A is closed can be expressed by the first order formula. So, if $M (\equiv M_\alpha)$ is countable then M is generic. By the back-forth method, we have $M \cong M_\alpha$.

Let M be a generic structure and $\mathcal{M}$ a big model of Th(M). For finite $A, B \subset \mathcal{M}$, let $d_\mathcal{M}(A) = \delta(\text{cl}_\mathcal{M}(A))$ and let $d_\mathcal{M}(B/A) = d_\mathcal{M}(BA) - d_\mathcal{M}(A)$.

We sometimes abbreviate $\mathrm{cl}_{\mathcal{M}}(*)$ and $d_{\mathcal{M}}(*)$ to $\mathrm{cl}(*)$ and $d(*)$ respectively. For infinite C, let $d(B/C) = \inf\{d(B/C_0) : C_0 \subset_{\mathrm{fin}} C\}$.

Fact 2.4 Let M be a saturated **K**-generic structure. Then

1. $\mathrm{Th}(M)$ is stable;
2. For $A \leq B \leq M$, $\mathrm{tp}(e/B)$ does not fork over A if and only if $d(e/B) = d(e/A)$ and $\mathrm{cl}(eA) \cap B = A$.

Theorem 2.5 ([2], 2005) Suppose that $\alpha \in (0,1)$ is irrational, any finite tree is in **K** and **K**-generic M is saturated. Then M is strictly stable.

Outline of the proof. By Fact 2.4.1, if M is saturated then $\mathrm{Th}(M)$ is stable. So, it is enough to show that $\mathrm{Th}(M)$ is not superstable. For finite A, B with $A \cap B = \emptyset$, (B, A) is said to be biminimal, if $\delta(B/A) < 0$, $\delta(X/A) \geq 0$ for any nonempty proper $X \subset B$, and $\delta(B/Y) \geq 0$ for any proper $Y \subset A$. Take any $\epsilon \in (0, \alpha)$. Since α is irrational, it can be seen that there are $a, b \in \omega$ with $0 > a - b\alpha > -\epsilon$. Using this, we can construct a finite tree eBC satisfying (i) (C, eB) is biminimal; (ii) $0 > \delta(C/eB) > -\epsilon$; (iii) eB has no relations. Take a sequence $(\epsilon_i)_{i\in\omega}$ with $\alpha > \epsilon_0 > \epsilon_1 > \cdots > 0$ and $\sum_{i=1}^{\infty} \epsilon_i \leq 1$. For each ϵ_i, let eB_iC_i be a finite tree. Amalgamating eB_iC_i's, we can construct a countable tree $D = \bigcup_{i\in\omega} eB_iC_i$ satisfying that $B_i^* \leq eB_i^*C_i^* \leq D$ and there are no relations between B_iC_i and B_jC_j for $i \neq j$, where $B_i^* = \bigcup_{j\leq i} B_j$ and $C_i^* = \bigcup_{j\leq i} C_j$. Since any finite tree belongs to **K**, we can assume that

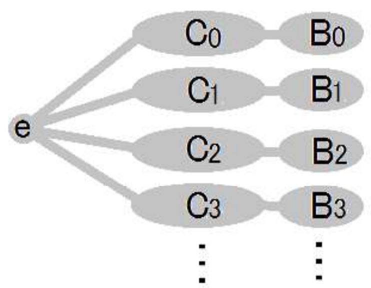

Figure 2: $D = \bigcup_{i\in\omega} eB_iC_i$

$D \leq M$. Then it can be seen that we have $d(e/B_{i+1}^*) = d(e/B_i^*) - \epsilon_{i+1}$ for each $i \in \omega$. Especially, $d(e/B^*_{i+1}) < d(e/B^*_i)$. By Fact 2.4.2, $\mathrm{tp}(e/B^*_{i+1})$ is a forking extension of $\mathrm{tp}(e/B^*_i)$. It follows that $\mathrm{Th}(M)$ is not superstable.

Lemma 2.6 If $\mathrm{index}(\alpha) = \infty$, then M_α is strictly stable.

Proof. Since $\mathrm{index}(\alpha) = \infty$, α is irrational. It is clear that any finite tree is in $\mathbf{K}_\alpha$. By Lemma 2.3, M_α is ω-categorical, so in particular, it is saturated. By Theorem 2.5, M_α is strictly stable.

By Lemma 2.3 and Lemma 2.6, we have Proposition 0.2.

References

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