On rationality of Poincaré series in expansions of the p-adic fields

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1 Introduction

The aim of this article is to describe how the rationality of Poincaré series of expansions of the p-adic fields is related to their stability-theoretic properties. Especially, we are going to introduce the result by the author that there exist NIP expansions of the p-adic fields without the rationality of Poincaré series.

Notation 1.1. For a p-adic integer $a = \sum_{i=0}^{\infty} a_i p^i$ and a number $n \geq 0$, we let $a \bmod p^n$ denote $\sum_{i=0}^{n-1} a_i p^i$, which is in $\mathbb{Z}/p^n\mathbb{Z}$.

Definition 1.2. Let $S \subseteq \mathbb{Z}_p^m$. We define the *Poincaré series* $P_S(T)$ *of* S by the following:

$$N_n = \left| \{(x_1 \bmod p^n, \ldots, x_m \bmod p^n) \in (\mathbb{Z}/p^n\mathbb{Z})^m \mid \overline{x} \in S\} \right|,$$
$$P_S(T) = \sum_{n=0}^{\infty} N_n T^n.$$

In 1984, Denef showed the following prominent result.

Theorem 1.3 ([2])**.** *For any $S \subseteq \mathbb{Z}_p^m$ definable in the p-adic field structure $(\mathbb{Q}_p; +, \cdot)$, $P_S(T)$ is a rational function of T.*

We call this property the *rationality of Poincaré series.* If we expand $(\mathbb{Q}_p; +, \cdot)$ by adding some relation or function, then the expanded structure may have more definable sets. While some expansions still have the rationality of the Poincaré series of their definable sets, there also exist expansions without the rationality of Poincaré series. Our interest is how we can characterize the rationality by stability-theoretic properties. Previous researches tell us the following facts:

- All of the dp-minimal expansions of the p-adic fields, such as the structure expanded by all the analytic functions on a compact set, have the rationality of Poincaré series. In fact, Simon and Walsberg[10] shows that they are all P-minimal, and Kovacsics and Leenknegt[4] shows that all of the P-minimal expansions have the rationality of Poincaré series.

- There is an NIP but not strongly dependent expansion of the p-adic fields having the rationality of Poincaré series. Indeed, Denef[3] proves the rationality of Poincaré series of $(\mathbb{Q}_p; +, \cdot, p^{\mathbb{Z}})$, where $p^{\mathbb{Z}} = \{p^z \mid z \in \mathbb{Z}\}$. the NIP of this structure was shown by Mariaule[6]. For the lack of strong dependence, see my master thesis[7, Section 3.6].
- If an expansion of the p-adic fields defines $\mathbb{Z}$, then it does not have the rationality of Poincaré series, because it defines $p^{2^{\mathbb{N}}} = \{p^r \mid r \in 2^{\mathbb{N}}\}$, whose Poincaré series is not a rational function.

Despite our original expectation that all of the NIP expansions had the rationality of Poincaré series, we found a counterexample.

Theorem 1.4 (O.). *There is an NIP (but not strongly dependent) expansion of the p-adic fields without the rationality of Poincaré series.*

This is our main result. In the rest of this article, we will sketch the proof of this theorem.

2 Presburger Arithmetic augmented by an almost sparse sequence

Let R be an infinite subset of the positive integers. We suppose that $1 \in R$ though it is not an essential assumption at all. We enumerate all the elements of R by the increasing sequence $(r_n)_{n\in\mathbb{N}}$. We define two functions S, S^{-1} on R in the following way: $S(r_n) = r_{n+1}$, $S^{-1}(r_{n+1}) = r_n$, and $S^{-1}(r_0) = r_0$.

Definition 2.1. We call a term of the form $A(x) = \sum_{i=-n}^{m} z_i S^i(x)$, where $z_i \in \mathbb{Z}$, an *operator* on R. If $n = 0$, then we call it an *operator without negative indices.*

Notation 2.2. We write $A(x) >_{ae} 0$ if all but finite number of elements from R meets $A(r) > 0$. We also use similar notations $A(x) <_{ae} 0$ and $A(x) =_{ae} 0$. We believe that the meaning of those notations is clear.

Definition 2.3. We say that R is *almost sparse* when the following two conditions hold:

- For any operator without negative indices $A(x)$, either $A(x) >_{ae} 0$, $A(x) <_{ae} 0$, or $A(x) =_{ae} 0$ holds.
- For any operator without negative indices $A(x)$ with $A(x) >_{ae} 0$, there exists a number $\Delta \geq 0$ such that $A(S^{\Delta}(x)) - x >_{ae} 0$.

Almost sparse sequences include $(2^n)_{n\in\mathbb{N}}$, $(n!)_{n\in\mathbb{N}}$, and the Fibonacci sequence. However, $(n)_{n\in\mathbb{N}}$ and $(2^n+n)_{n\in\mathbb{N}}$ are not almost sparse.

The following theorem by Lambotte is one of the main ingredients for the proof of our main theorem.

Theorem 2.4 ([5])**.** *If R is almost sparse, then the structure $(\mathbb{Z};+,<,R)$ has NIP.*

To sketch the proof, we will only mention some crucial lemmas for this theorem.

Notation 2.5. We define three functions λ_R, S, S^{-1} of $\mathbb{Z}$ in the following manner:

$$\lambda_R(x) = \begin{cases} \max\{r\in R \mid r\le x\} & (x>0) \\ x & (x\le 0)\end{cases};$$

$$S(x) = \begin{cases} S(x) & (x\in R) \\ x & (x\notin R)\end{cases};$$

$$S^{-1}(x) = \begin{cases} S^{-1}(x) & (x\in R) \\ x & (x\notin R)\end{cases}.$$

The next lemma is a modified version of Point's quantifier elimination result ([8, Section 3]). Point gave quantifier elimination imposing an additional condition (eventual periodicity modulo n) on R. The statement of this lemma is weaker than quantifier elimination, but does not need such an extra condition.

Lemma 2.6. *Let $\mathcal{U}$ be a monster model extending $(\mathbb{Z};+,<,R)$. Define $\mathcal{I}$ as the set of partial functions $f:\mathcal{U}\to\mathcal{U}$ which satisfies the following:*

- $\mathrm{Dom}(f)$ *is small and closed under* $0,1,+,-,\lambda_R,S,S^{-1}$.
- *f is a $\{+,<,R\}$-partial embedding.*
- *f is a $\{+,<\}$-partial elementary embedding.*
- *$f|_{\mathrm{Dom}(f)\cap R^{\mathcal{U}}}$ is a partial elementary embedding of the structure $(R^{\mathcal{U}};<,(\cdot\equiv_n i)_{n\ge 2, 0\le i<n})$, where $\cdot\equiv_n i$ is the set $\{r\in R^{\mathcal{U}} \mid \mathcal{U}\models \exists x(nx=r-i)\}$.*

Then, $\mathcal{I}$ forms a back-and-forth system. In particular, any $f\in\mathcal{I}$ is a $\{+,<,R\}$-partial elementary embedding.

In addition to this lemma, we need to prepare the "term separation" lemma below. It is basically the same as one given by Lambotte[5, Lemma 5.6.3].

Lemma 2.7. *Let $\mathcal{U}$ be a monster model extending $(\mathbb{Z};+,<,R)$. Suppose that:*

- *$(a_i)_{i\in I}\subseteq\mathcal{U}$ is indiscernible over $\bar{c}\in\mathcal{U}$.*
- *$\bar{b}\in\mathcal{U}$.*

- $t(x, \overline{y})$ *is a* $\{+, -, \lambda_R, S, S^{-1}\}$*-term.*

Then, there exist a final segment I_f *of* I, $\overline{c}' \in \mathcal{U}$, $e \in \mathcal{U}$, *and a* $\emptyset$*-definable function* $u(x, \overline{zw})$ *such that:*

- $(a_i)_{i \in I_f}$ *is indiscernible over* $\overline{cc}'$*.*
- $t(a_i, \overline{b}) = u(a_i, \overline{cc}') + e$ *for all* $i \in I_f$*.*

We could take advantage of these two lemmas to show that the NIP of Presburger arithmetic ([1]) and that of colored orders ([9, Proposition A.2]) implies that of $(\mathbb{Z}; +, <, R)$, the desired result.

3 The main result

Again, let R be an almost sparse sequence. We consider two subsets of $\mathbb{Q}_p$:

- $p^{\mathbb{Z}} = \{p^z \mid z \in \mathbb{Z}\}$.
- $p^R = \{p^r \mid r \in R\}$.

We can easily show the claim below.

Lemma 3.1. *The Poincaré series of* p^R *is not a rational function.*

Hence, it is sufficient to prove the following theorem to show the main result, i.e., Theorem 1.4.

Theorem 3.2. $(\mathbb{Q}_p; +, \cdot, p^{\mathbb{Z}}, p^R)$ *has NIP.*

The outline of the proof is similar to that of Theorem 2.4 as we are going to see.

Notation 3.3. We let $\pi : \mathbb{Q}_p \to p^{\mathbb{Z}}$ denote the function given by $\pi(x) = p^{v_p(x)}$ for $x \neq 0$ and $\pi(0) = 0$.

Next, note that $(p^{\mathbb{Z}}; \cdot, v(\cdot) < v(\cdot)) \cong (\mathbb{Z}; +, <)$, where $v(\cdot) < v(\cdot)$ is the set of pairs $(x, y) \in (p^{\mathbb{Z}})^2$ such that $v_p(x) < v_p(y)$.

Notation 3.4. We define three functions λ_R, S, S^{-1} on $\mathbb{Q}_p$ so that $(p^{\mathbb{Z}}; \cdot, v(\cdot) < v(\cdot), \lambda_R, S, S^{-1}) \cong (\mathbb{Z}; +, <, \lambda_R, S, S^{-1})$ and that $\lambda_R(x) = S(x) = S^{-1}(x) = x$ for $x \notin p^{\mathbb{Z}}$.

We need to construct a back-and-forth system and prepare the "term separation" result again.

Lemma 3.5. *Let* $\mathcal{U}$ *be a monster model extending* $(\mathbb{Q}_p; +, \cdot, p^{\mathbb{Z}}, p^R)$*. Define* $\mathcal{I}$ *as the set of partial functions* $f : \mathcal{U} \to \mathcal{U}$ *which satisfies the following:*

- $\mathrm{Dom}(f)$ *is small and closed under* $0, 1, +, -, \cdot, ^{-1}, \pi, \lambda_R, S, S^{-1}$*.*
- f *is a* $\{+, \cdot, p^{\mathbb{Z}}\}$*-partial elementary embedding.*

- $f|_{\mathrm{Dom}(f)\cap(p^{\mathbb{Z}})^{\mathcal{U}}}$ *is a partial elementary embedding of the structure* $((p^{\mathbb{Z}})^{\mathcal{U}}; \cdot, v(\cdot) < v(\cdot), (p^R)^{\mathcal{U}})$.

Then, $\mathcal{I}$ *forms a back-and-forth system. In particular, any* $f \in \mathcal{I}$ *is a* $\{+, \cdot, p^{\mathbb{Z}}, p^R\}$*-partial elementary embedding.*

Lemma 3.6. *Let* $\mathcal{U}$ *be a monster model extending* $(\mathbb{Q}_p; +, \cdot, p^{\mathbb{Z}}, p^R)$*. Suppose that:*

- $(a_i)_{i\in I} \subseteq \mathcal{U}$ *is indiscernible over* $\bar{c} \in \mathcal{U}$ *with* $\mathrm{cf}(I) > \aleph_0$*, where* $\mathrm{cf}(I)$ *is the minimal cardinal among those of cofinal subsets of* I*.*
- $\bar{b} \in \mathcal{U}$*.*
- $t(x, \bar{y})$ *is a* $\{+, -, \cdot, ^{-1}, \pi, \lambda_R, S, S^{-1}\}$*-term.*

Then, there exist a final segment I_f *of* I*,* r_l *and* s_l *for* $1 \le l \le n$*,* $\emptyset$*-definable functions* $\alpha_l(x, \overline{zw})$ *and* $\beta_l(x, \overline{zw})$ *for* $1 \le l \le n$*, and* $\bar{c}' \in \mathcal{U}$ *such that:*

- $(a_i)_{i\in I_f}$ *is indiscernible over* $\overline{cc}'$*.*
- $t(a_i, \bar{b}) = \dfrac{\sum_l r_l \alpha_l(a_i, \overline{cc}')}{\sum_l s_l \beta_l(a_i, \overline{cc}')}$ *for all* $i \in I_f$*.*

These two lemmas lead to the fact that the NIP of $(\mathbb{Z}; +, <, R)$ and that of $(\mathbb{Q}_p; +, \cdot, p^{\mathbb{Z}})$ implies that of $(\mathbb{Q}_p; +, \cdot, p^{\mathbb{Z}}, p^R)$. Recall that the former is the result by Lambotte (Theorem 2.4). The latter has also been proved by Mariaule[6]. Thus, Theorem 3.2 follows, and so does the main result of this note.

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