On ultraproducts of o-minimal structures

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概要

abstract We consider about ultraproducts of o-minimal structures. Such structures are definably complete locally o-minimal. We try to characterize them.

1. Introduction

At first we recall some definitions and fundamental facts.

Definition 1. Let M be a densely linearly ordered structure without endpoints.

M is *o-minimal* if every definable subset of M^1 is a finite union of points and intervals.

M is *locally o-minimal* if for any element $a \in M$ and any definable subset $X \subset M^1$, there is an open interval $I \subset M$ such that $I \ni a$ and $I \cap X$ is a finite union of points and intervals.

M is $uniformly\ locally\ o-minimal$ if for any formula $\varphi(x, \bar{y})$ over $\emptyset$ and any $a \in M$, there is an open interval $I \ni a$ such that $I \cap \varphi(M, \bar{b})$ is a finite union of points and intervals for any $\bar{b} \in M^n$ where $\varphi(M, \bar{b})$ is the realization set of $\varphi(x, \bar{b})$ in M.

M is $definably\ complete$ if any definable subset X of M^1 has the supremum and infimum in $M \cup \{\pm\infty\}$.

Example 2. [1], [2]

$(\mathbb{R}, +, <, \mathbb{Z})$ where $\mathbb{Z}$ is the interpretation of a unary predicate, and $(\mathbb{R}, +, <, \sin)$ are definably complete locally o-minimal structures.

Fact 3. [1] *Definably complete local o-minimality is preserved under elementary equivalence.*

Thus we argue in a sufficiently large saturated model $\mathcal{M}$.

We recall the definition of ultraproduct. In this note, we consider ultraproducts of structures only and ultrafilters are always nonprincipal.

Definition 4. Let I be an infinite set and $\mathcal{U}$ be a nonprincipal ultrafilter over I.

And let M_i $(i \in I)$ be structures of some fixed language L.

Consider the equivalence relation $\equiv_{\mathcal{U}}$ on the Cartesian product $C = \prod_{i\in I} M_i$ such that for $f,\ g \in C,\ f \equiv_{\mathcal{U}} g$ if and only if $\{i \in I : f(i) = g(i)\} \in \mathcal{U}$.

We define the *ultraproduct of* M_i *modulo* $\mathcal{U}$ be the set of all equivalence classes of $\equiv_{\mathcal{U}}$, that is,

$\prod_{i\in I} M_i/\mathcal{U} = \{f_{\mathcal{U}} : f \in \prod_{i\in I} M_i\}$.

And we recall a fundamental theorem i.e. Loś' theorem.

Theorem 5. *Let N be an ultraproduct $\prod_{i\in I} M_i/\mathcal{U}$ and let I be the index set. Then ;*

(i) For any term $t(x_1, \cdots, x_n)$ of a language L and elements $f^1_{\mathcal{U}}, \cdots, f^n_{\mathcal{U}} \in N$, we have $t_N[f^1_{\mathcal{U}}, \cdots, f^n_{\mathcal{U}}] = \langle\, t_{M_i}[f^1(i), \cdots, f^n(i)] : i \in I\,\rangle_{\mathcal{U}}$.

(ii) Given any formula $\varphi(x_1, \cdots, x_n)$ of L and $f^1_{\mathcal{U}}, \cdots, f^n_{\mathcal{U}} \in N$, we have $N \models \varphi[f^1_{\mathcal{U}}, \cdots, f^n_{\mathcal{U}}]$ if and only if $\{i \in I : M_i \models \varphi[f^1(i), \cdots, f^n(i)]\} \in \mathcal{U}$.

(iii) For any sentence φ of L, $N \models \varphi$ if and only if $\{i \in I : M_i \models \varphi\} \in \mathcal{U}$.

We try to characterize ultraproducts of locally o-minimal structures. At first we consider about ultraproducts of o-minimal structures.

We verify some elementary facts.

Fact 6. *There are ultraproducts of o-minimal structures which are not o-minimal.*

However ultraproducts of o-minimal structures are locally o-minimal, and definably complete, and infinite 1-types are complete by order formulas, that is,

Let M be a model and $\phi(x, \bar{m}) \in L(M)$ be a one-variable formula. Then there is $b \in M$ such that either for any $c > b$, $M \models \phi(c, \bar{m})$ or for any $c > b$, $M \models \neg\phi(c, \bar{m})$ (it is also true for the lower side).

Some people call these property DCTC, that is, definably complete and type complete (they contain local o-minimality in TC).

I show a poor example.

Example 7. *Let L be the language of ordered fields together with a unary predicate $P(x)$.*

Each $L(P)$-structure $M_n = (\mathbb{R}, \{0, 1, , \cdots, n\})$ is o-minimal, but their ultraproducts $M_{\mathcal{U}} = (\mathbb{R}_{\mathcal{U}}, \mathbb{N}_{\mathcal{U}})$ is not o-minimal.

We recall some notations and facts.

Definition 8. Let L be a language containing a binary predicate $<$ to be interpreted as a dense linear order.

Some people call ultraproducts of o-minimal structures *ultra-o-minimal* structure.

And they call an L-structure M *pseudo-o-minimal* if it is a model of $T^{omin}(L)$, that is, the collection of L-sentences that hold true in every o-minimal L-structure.

A.Rennet showed that T^{omin} is strictly strong than DCTC in [4].

2. Structural complexity in ultraproducts of o-minimal structures

At first we recall characterization by H.Schoutens. Before that, we recall some definitions from [15].

Definition 9. Let M be an o-minimal structure.

We define the *dimension* of a nonempty definable set $X \subset M^m$ by

$dim\, X = max\{i_1 + \cdots + i_m : X$ contains an $(i_1, \cdots, i_m)$-cell$\}$.

We assign to each cell C of dimension d the integer $E(C) = (-1)^d$, and given a finite partition $\mathcal{P}$ of a definable set $S \subset M^m$ into cells, we put

$E_{\mathcal{P}}(S) = \Sigma_{C \in \mathcal{P}} E(C) = k_0 - k_1 + \cdots + (-1)^d k_d + \cdots + (-1)^m k_m$

where k_d is the number of d-dimensional cells in $\mathcal{P}$.

We call $E_{\mathcal{P}}(S)$ ($E(S)$) *Euler characteristic of S*.

Theorem 10. [3] *For an ultra-o-minimal strucutre $\prod_{i \in I} M_i/\mathcal{U}$ given as the ultraproduct of o-minimal structures M_i, a necessary and sufficient condition to be o-minimal is that for each formula φ without parameters, there exists an $N_\varphi \in \mathbb{N}$ such that $|E_{M_i}(\varphi)| \leq N_\varphi$ for almost all i.*
(In the above, we take absolute values for cells).

Next example by A.Fornasiero is known as that of locally o-minimal structure which has the independence property. This structure is an ultraproducts of o-minimal fields expanded by a binary relation.

Example 11. [8]

There is an ultraproduct of o-minimal structures which has the independence property.

In the summer meeting of model theory this year, I refered to an example of ultraproduct of o-minimal structures that has the tree property of the second kind. After that, A.Tsuboi suggested a more applicable example. We verify his proof here.

Definition 12. A formula $\phi(\bar{x}, \bar{y})$ has TP_2, that is, *the tree property of the second kind* if there is an array $(\bar{a}_{t,i})_{t,i<\omega}$ such that ;

$\{\phi(\bar{x}, \bar{a}_{t,i})\}_{i<\omega}$ is k-inconsistent for every $t < \omega$ and,

$\{\phi(\bar{x}, \bar{a}_{t,f(t)}\}_{t<\omega}$ is consistent for any $f : \omega \longrightarrow \omega$.

Lemma 13. *Let $\mathcal{M} = (M, <, \cdots)$ be o-minimal and let $X \subset M^n$ be a finite set.*
Then $\mathcal{M}_X = (M, <, \cdots, X)$ is o-minimal where X is an interpretation of a predicate symbol.

Proof ;

For $X \subset M^n$*, choose* $a_1 < \cdots < a_k$ *such that* $X \subset \{a_1, \cdots, a_k\}^n$*. Then* X *is* $\{a_1, \cdots, a_k\}$*-definable. The structure* $(M, <, \cdots, a_1, \cdots, a_k)$ *is clearly o-minimal. So* $\mathcal{M}_X$ *is also o-minimal.* ∎

Proposition 14. *There is a structure* M *which is an ultraproduct of o-minimal structures that has the* TP_2*.*

Proof ;

We prepare a predicate symbol $E(x, y, z)$ *and constant symbols* $\{c_n\}\,(\,n \in \omega\,)$*. Let* $R = (\,R, <, \cdots)$ *be an o-minimal structure where the language* $L = \{<, \cdots\}$*. For each* $n \in \omega$*, we define the expansion* R_n *of* R *whose language is the* $L_n = (\,L \cup \{\,E(x, y, z), c_0, c_1, \cdots\}\,)$*.*

Construct R_n *in the following :*

(1) *First, let* $D = \{\,d_\eta \,|\, \eta : n \longrightarrow n\,\} \subset R$ *where* $d_\eta \neq d_\nu$ *if* $\eta \neq \nu$*.*

(2) $c_i^{R_n} = i\ (\,i \leq n\,)$*,* $c_i^{R_n} = n + 1\ (\,i > n\,)$ *(for some enumeration of a subset in* R_n *).*

(3) $E(x, y, c_i)^{R_n}$ *is an equivalence relation on* D *such that* $\models E(d_\eta, d_\nu, c_i)$ *if and only if* $\eta(i) = \nu(i)$*.*

(4) $R_n := (\,R, E, c_0, c_1, \cdots)$*.*

By the proposition above, each R_n *is o-minimal, since* E *is satisfied by finite elements.*

Now let $R^* = \prod_{n\in\omega} R_n \,/\, \mathcal{U}$ *where* $\mathcal{U}$ *is a non-principal ultrafilter.*

Then

(A) R^* *is a definably complete locally o-minimal structure.*

(B) $E^{R^*}(x, y, c_i)\,(\,i < \omega\,)$ *are cross-cutting equivalence relations.*

In R_n*, we can take an array of formulas* $\{\,E(x, d_{\eta(i,j)}, c_i) : i, j < n\,\}$ *satisfying that ;*

η *is* $\eta : n \longrightarrow n$ *and,*

in the i-th row, $\eta(i, j)(i) = j$ *for* $j < n$*.*

Thus in every row, $\{\,E(x, d_{\eta(i,j)}, c_i) : j < n\,\}$ *is 2-inconsistent and,*

for any $\nu : n \longrightarrow n$*,* $\{\,E(x, d_{\eta(i,\nu(i))}, c_i) : i < n\,\}$ *is consistent.*

(C) *Since the equivalence relation are uniformly defined,* R^* *has the* TP_2*.* ∎

The proof above suggests that we can construct ultraproducts of o-minimal structures having other properties, in particular, properties which have finite approximation.

3. Independence in ultraproducts of o-minimal structures

We recall some definitions at first.

Definition 15. Let M be a densely linearly ordered structure and $p(x) \in S_1(M)$.

We say that $p(x)$ is *cut* (*irrational*) *over* M if for any $a \in M$, if $a < x \in p(x)$, then there is $b \in M$ such that $a < b < x \in p(x)$, and similarly, if $x < a \in p(x)$, then there is $c \in M$ such that $x < c < a \in p(x)$.

We say that $q(x) \in S_1(M)$ is *noncut* (*rational*) *over* M if $q(x)$ is not a cut type.

Here we consider nonisolated types only.

Definition 16. Let M be locally o-minimal and $p(x) \in S_1(M)$ be noncut.

There are four kinds of noncut types ;

$p(x) \supset \{m < x < a \,:\, m < a \in M\}$ or $\{a < x < m \,:\, a < m \in M\}$ for some fixed $a \in M$.

Here we call these types *bounded* noncut types of a over M.

$p(x) \supset \{m < x \,:\, m \in M\}$ or $\{x < m \,:\, m \in M\}$.

We call these types *unbounded* noncut types.

Definition 17. A formula $\varphi(\bar{x}, \bar{a})$ *divides* over a set A if there is a sequence $\{\bar{a}_i : i \in \omega\}$ with $tp(\bar{a}_i/A) = tp(\bar{a}/A)$ such that $\{\varphi(\bar{x}, \bar{a}_i) : i \in \omega\}$ is k-inconsistent for some $k \in \omega$.

A formula $\varphi(\bar{x}, \bar{a})$ *forks* over A if $\varphi(\bar{x}, \bar{a}) \vdash \bigvee_{i<n} \psi_i(\bar{x}, \bar{b}_i)$ and each $i < n$, $\psi_i(\bar{x}, \bar{b}_i)$ divides over A.

We argue about forking of 1-variable types in ultraproducts of o-minimal structures.

Fact 18. *Let M_i ($i \in I$) be an o-minimal structures and let $M_U = \prod_{i \in I} M_i/\mathcal{U}$ be an ultraproduct and $A \subset M_U$.*

And assume that M_U is $|\, A\,|^+$-saturated.

Then any unbounded noncut type over M_U and bounded noncut type of a over M_U for some $a \in A$, does not fork over A.

Fact 19. *Let M_i ($i \in I$) be an o-minimal structures and let $M_U = \prod_{i \in I} M_i/\mathcal{U}$ be an ultraproduct and $A \subset M_U$.*

Assume that for almost all $i \in I$, there are c_i, $d_i \in M_i \setminus A_i$ such that $c_i \equiv_{A_i} d_i$.

Then $c_{\mathcal{U}} < x < d_{\mathcal{U}}$ (or $d_{\mathcal{U}} < x < c_{\mathcal{U}}$) divides over A.

And we can show the next Lemma.

Lemma 20. *Let $M_U = \prod_{i \in I} M_i/\mathcal{U}$ where M_i ($i \in I$) is o-minimal and let $A \subset B \subset M_U$. And assume that M_U is $|\, A\,|^+$-saturated.*

If a B-definable set $X \subset M_U$ is cofinal in an A-definable set $Y \subset M_U$,

then X does not fork over A.

I can show a few results about forking in ultraproducts of o-minimal structures at present.

Problem 21. *There are many results about forking in o-minimal structures, and more generally, in dp-minimal or VC-minimal structures.*

Under what conditions, and what extent do these properties reflect to forking in ultraproducts of o-minimal structures ?

4. Ultraproducts of expanded fields

We recalled a result by H.Schoutens about the necessary and sufficient condition for ultraproducts to be o-minimal. However, it is difficult to confirm that each structure satisfies that condition.

Some people investigated ultraproducts of expanded real closed fields.

Definition 22. [5]

Let $R \models RCF$ and $(f_i)_{i\in I}$ be an I-indexed sequence of polynomials $f_i \in R[x]$ (x could be a tuple).

We consider the $L_{RCF}(f)$-structure consisting of $\mathcal{R} = (R^{\mathcal{U}}, (f_n)_{n\in\mathbb{N}}/\mathcal{U})$ where $R^{\mathcal{U}} = \prod_{n\in\mathbb{N}} R_n/\mathcal{U}$ and R_n is expanded R by f_n.

We denote $\tilde{f}(x) = (f_n)_{n\in\mathbb{N}}/\mathcal{U}$ and call $\tilde{f}(x)$ a *pseudopolynomial over* $R^{\mathcal{U}}$.

And let $f(x)$ be a function.

We say that $\tilde{f}(x)$ is a *pseudopolynomial approximation of* f if for all $x \in dom(f) \subset R$, we have $\tilde{f}(x) = f(x) + \epsilon_f$ for some infinitesimal function ϵ_f.

For example, let $f(x)$ be a real analytic function with Taylor polynomials $(T_n)_{n\in\mathbb{N}}$ defined on the set $dom(f) \subset \mathbb{R}$. Then $\tilde{f}(x) = (T_n)_{n\in\mathbb{N}}/\mathcal{U}$.

In these argument, Pfaffian functions and Khovanskii's theorem about them are available.

Definition 23. Let R be a definably complete expansion of an ordered field.

And let $f_i : R^n \longrightarrow R$ $(i = 1, \cdots, s)$ be definable function and C^1.

We say that $(f_1, \cdots, f_s)$ is a *Pfaffian chain in R of length s* if $\dfrac{\partial f_i}{\partial x_j} \in R[\bar{x}, f_1, \cdots, f_i]$ for $i = 1, \cdots, s$ and $j = 1, \cdots, n$.

A definable function $F = (F_1, \cdots, F_m) : R^n \longrightarrow R^m$ is a *Pfaffian function in R* if $F_l \in R[\bar{x}, f_1, \cdots, f_s]$ $(l = 1, \cdots, m)$ for some Pfaffian chain $(f_1, \cdots, f_s)$ in R.

For example, (i) e^x, (ii) e^x, e^{e^x}, (iii) $(x^2+1)^{-1}$, $arctan\, x$ are Pfaffian chains.

All Pfaffian functions are analytic.

There is a theorem by A.Fornasiero and T.Servi [7].

Theorem 24. *Let R be a definably complete locally o-minimal expansion (Baire expansion)* *of a field by a family of Pfaffian functions.*

Then R is o-minimal.

And there are results by A.Rennet [5].

Theorem 25. *Let $\mathcal{R} = \prod_{n\in\mathbb{N}_{>0}} R_n / \mathcal{U}$ where $R_n = \bar{\mathbb{R}}$ and let $\tilde{e^x}$ be a Taylor polynomial approximation.*

Then $(\mathcal{R}, \tilde{e^x})$ is o-minimal.

Remark 26. *He argued by means of another approximation of e^x.*

By the approximation ; $\lim_{n\to\infty}(1 + x/n)^n = e^x, i.e. f_n(x) = (1 + x/n)^n$ and he proved that $(\mathcal{R}, \tilde{f})$ is o-minimal.

He also considered ultraproducts of expanded fields by iterated functions. And he put a question.

Question 27. *If P is a pseudopolynomial approximation of any Pfaffian function, then P is also Pfaffian ? and is $(\mathcal{R}, P)$ o-minimal ?*

They paid attention to the fact whether constructed ultraproducts are o-minimal or not.

Problem 28. *Can we characterize them ?*

Can we characterize ultraproducts of groups, for example, that of expanded $(\mathbb{R}, +, <)$?

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