Definable $\mathcal{C}^r$ imbedding theorem

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概要

We introduce the zero set property of a definably complete expansion of an ordered field $\mathcal{F} = (F, <, +, \cdot, 0, 1, \ldots)$. A definable $\mathcal{C}^r$ manifold X is definably $\mathcal{C}^r$ imbeddable into some F^n if and only if X is definably normal when the definably complete expansion of an ordered field has the zero set property.

1 Introduction

We introduce the results of an ongoing work by the authors in this paper. Let $0 \leqq r < \infty$. We abbreviate 'definable $\mathcal{C}^r$' to '$\mathcal{D}^r$'. We discuss about $\mathcal{D}^r$ imbeddablity of $\mathcal{D}^r$ manifolds.

We briefly review the previous works on $\mathcal{D}^r$ imbeddablity of $\mathcal{D}^r$ manifolds. In the o-minimal setting, it was proven that every $\mathcal{D}^r$ manifold is $\mathcal{D}^r$ imbeddable into some F^n in [1], [2] and [5]. The reference [1] only treats the case in which the manifold is definably compact. The other two treat more general case, but there are gaps in their proofs[*1]. The authors treated the definably complete locally o-minimal case in [4] and proved a $\mathcal{D}^r$ imbedding theorem in it. The early versions of [4] also have a gap of the same kind though it was fixed in the current version.

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[*1] In the proof of [5, Proposition 2.2], it was proven that the closure of V_i in X_{k-1} is contained in U_i, where X_{k-1} is a subset of the manifold X. However, it needs to be proven that the closure of V_i in X is contained in U_i in that part of the proof. As to [2, Lemma 4.6], the proof that ψ_i is of class $\mathcal{C}^r$ at points in $\partial(\phi_i^{-1}(V_i)) \cap \partial(\phi_i^{-1}(W_i))$ is missing.

2 Our results

Recall some definitions.

Definition 2.1. (1) The structure $\mathcal{F}$ is *definably complete* if for any non-empty definable subset A of F, $\sup A$ and $\inf A$ exist in $\{F \cup \pm\infty\}$ [6].

(2) A definably complete structure $\mathcal{F} = (F, <, \ldots)$ is *d-minimal* if for every m and definable subset A of F^{m+1}, there exists an $N \in \mathbb{N}$ such that for every $x \in F^m$, the set $\{y \in F | (x, y) \in A\}$ has non-empty interior or a union of at most N discrete sets [3, 7].

Let $\mathcal{F} = (F, <, +, \cdot, 0, 1, \ldots)$ be a definably complete expansion of an ordered field and $1 \leqq r < \infty$.

Definition 2.2. (1) A pair $(M, \{\phi_i : U_i \to V_i\}_{i \in I})$ of a topological space and finite family of homeomorphims is a *definable C^r manifold* or a *$\mathcal{D}^r$ manifold* if

- $\{U_i\}_{i \in I}$ is a finite open cover of M,
- U_i' is a $\mathcal{D}^r$ submanifold of F^{m_i} for any $i \in I$ and,
- the composition $(\varphi_j|_{U_i \cap U_j}) \circ (\varphi_i|_{U_i \cap U_j})^{-1} : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$ is a $\mathcal{D}^r$ diffeomorphism whenever $U_i \cap U_j \neq \emptyset$.

Here, the notation $\varphi_i|_{U_i \cap U_j}$ denotes the restriction of φ_i to $U_i \cap U_j$. The family $\{\varphi_i : U_i \to U_i'\}_{i \in I}$ is called a *$\mathcal{D}^r$ atlas* on M. We often write M instead of $(M, \{\varphi_i : U_i \to U_i'\}_{i \in I})$ for short. Note that a $\mathcal{D}^r$ submanifold is naturally a $\mathcal{D}^r$ manifold.

(2) A definable subset Z of X is a *k-dimensional $\mathcal{D}^r$ submanifold* of X if each point $x \in Z$ there exist an open box U_x of x in X and a $\mathcal{D}^r$ diffeomorphism ϕ_x from U_x to some open box V_x of F^d such that $\phi_x(x) = 0$ and $U_x \cap Y = \phi_x^{-1}(F^k \cap V_x)$, where F^k denotes $\{(x_1, \ldots, x_k, 0, \ldots, 0) | x_1, \ldots, x_k \in F\}$.

(3) Let X and Y be $\mathcal{D}^r$ manifolds with $\mathcal{D}^r$ charts $\{\phi_i : U_i \to V_i\}_{i \in A}$ and $\{\psi_j : U_j' \to V_j'\}_{j \in B}$, respectively. A continuous map $f : X \to Y$ is a *$\mathcal{D}^r$ map* if for any $i \in A$ and $j \in B$, the image $\phi_i(f^{-1}(V_j') \cap U_i)$ is definable and open in F^n and the map $\psi_j \circ f \circ \phi_i^{-1} : \phi_i(f^{-1}(V_j) \cap U_i) \to F^m$ is a $\mathcal{D}^r$ map.

(4) Let X and Y be $\mathcal{D}^r$ manifolds. We say that X is *$\mathcal{D}^r$ diffeomorphic to* Y if there

exist $\mathcal{D}^r$ maps $f : X \to Y$ and $h : Y \to X$ such that $f \circ h = \mathrm{id}$ and $h \circ f = \mathrm{id}$.

(5) A $\mathcal{D}^r$ manifold M is *definably normal* if for any definable closed subset C and definably open subset U of M with $C \subseteq U$, there exists a definable open subset V of M such that $C \subseteq V \subseteq cl_M(V) \subseteq U$.

Definition 2.3. A definably complete expansion of an ordered field $\mathcal{F} = (F, <, +, \cdot, 0, 1, \dots)$ has *the zero set property* if for any positive $r > 0$, and any definable closed subset A of F^n, there exists a $\mathcal{D}^r$ function $f : F^n \to F$ such that $f^{-1}(0) = A$.

Theorem 2.4 ([8])**.** *A d-minimal expansion of an ordered field has the zero set property.*

By Theorem 2.4, d-minimal structures are examples of having the zero set property. The following are our results.

Theorem 2.5. *Let $\mathcal{F} = (F, <, +, \cdot, 0, 1, \dots)$ be a definably complete expansion of an ordered field having the zero set property. Every definably normal $\mathcal{D}^r$ manifold is definably imbeddable into some F^n, and its image is a $\mathcal{D}^r$ submanifold of F^n.*

A $\mathcal{D}^r$ submanifold of F^n is definably normal, we have the following theorem:

Theorem 2.6. *Let $\mathcal{F} = (F, <, +, \cdot, 0, 1, \dots)$ be a definably complete expansion of an ordered field having the zero set property. Every definably $\mathcal{D}^r$ manifold X is definably imbeddable into some F^n, and its image is a $\mathcal{D}^r$ submanifold of F^n if and only if X is definably normal.*

We briefly sketch an outline of the proof of Theorem 2.5. The proof is almost the same as that of [2, Theorem 1.3]. In [2], the structure is assumed to be o-minimal, but the proof in [2] is almost valid even when the structure is a definably complete expansion of an ordered field having the zero set property. We show a 'partition of unity' lemma in the course of the proof. Let $\{\varphi_i : U_i \to U'_i\}_{1 \leq i \leq k}$ be $\mathcal{D}^r$ atlases of a definably normal $\mathcal{D}^r$ manifold M. We construct $\mathcal{D}^r$ functions $\psi_i : M \to F$ so that $\mathrm{supp}(\psi_i) \subseteq U_i$ and $M = \bigcup_{i=1}^k \{\psi_i > 0\}$ by induction on i. Set

$$V_i = \bigcup_{j=1}^{i-1} \psi_j^{-1}((0, \infty)) \cup \bigcup_{j=i+1}^{k} U_j.$$

Since M is definably normal, we can take a definable open subset B_i of M such that $M \setminus V_i \subseteq B_i \subseteq \mathrm{cl}_M(B_i) \subseteq U_i$. We construct ψ_i so that $\psi_i > 0$ on $M \setminus V_i$ and $\mathrm{supp}(\psi_i) \subseteq \mathrm{cl}_M(B_i)$ in a standard way. The inclusion $\mathrm{supp}(\psi_i) \subseteq \mathrm{cl}_M(B_i) \subseteq U_i$ guarantees that the function ψ_i is of class $\mathcal{C}^r$ at the boundary of U_i because it is zero at the boundary.

参考文献

[1] A. Berarducci and M. Otero, *Intersection theory for o-minimal manifolds*, Ann. Pure Appl. Logic, **107** (2001), 87-119.

[2] A. Fischer, *Smooth functions in o-minimal structures*, Adv. Math. **218**, (2008), 496–514.

[3] A. Fornasiero, *D-minimal structures version 20*, preprint (2021), arXiv:2107.04293.

[4] M. Fujita and T. Kawakami, *Approximation and zero set of definable functions in a definably complete locally o-minimal structure*, arXiv:2301.02464, (2023).

[5] T. Kawakami, *Every definable C^r manifold is affine*, Bull. Korean Math. Soc. **42**, (2005), 165-167.

[6] C. Miller, *Expansions of dense linear orders with the intermediate value property*, J. Symbolic Logic, **66**, (2001), 1783-1790.

[7] C. Miller, *Tameness in expansions of the real field*, Logic Colloquium '01, 281–316, Lect. Notes Log., **20**, Assoc. Symbol. Logic, Urbana, IL, (2005).

[8] C. Miller and A. Thamrongthanyalak, *D-minimal expansions of the real field have the zero set property*, Proc. Amer. Math. Soc. **146**, (2018), 5169-–5179.