A summary on dimension theory and decomposition into special submanifolds in definably complete locally o-minimal structures

海上保安大学校 * 藤田雅人
Masato Fujita
Liberal Arts, Japan Coast Guard Academy

概要

We summarize recent results on dimension theory and decomposition into (quasi-)special submanifolds in definably complete locally o-minimal structures.

1 Introduction

Locally o-minimal structures are defined by localizing the definition of o-minimal structures. An expansion of a dense linear order without endpoints $\mathcal{M} = (M, <, \ldots)$ is *locally o-minimal* if, for every definable subset X of M and for every point $a \in M$, there exists an open interval I containing the point a such that $X \cap I$ is the union of finitely many points and open intervals. Locally o-minimal structures are investigated in, for instance, [22, 3, 11]. The expansion $\mathcal{M}$ is *definably complete* if any definable subset X of M has the supremum and infimum in $M \cup \{\pm\infty\}$ [14]. This paper treats dimension theory and decomposition into special submanifolds in definably complete locally o-minimal structures.

We first discuss on dimension theory in Section 2. We review the following four definitions of dimension in o-minimal structures. We do not rephrase the definition

* 〒737-8512 広島県呉市若葉町 5-1

and basic properties of o-minimal structures here. They are found in van den Dires's book [2].

(1) van den Dries proposed an alternative definition of dimension by giving axioms to be satisfied by a dimension function [1]. For a definable set X of M^n, we set $\dim X \geq d$ if and only if $\pi(X)$ has a nonempty interior for some coordinate projection $\pi : M^n \to M^d$, where M is the universe of the structure. This dimension map satisfies van den Dries's requirements.
(2) Pillay proposed the notion of first-order topological structures and defined an ordinal-valued dimension in [16]. An o-minimal structure is a first-order topological structure.
(3) The universe of an o-minimal structure together with the definable closure operation forms a pregeometry. It is demonstrated in [19]. The notion of rank on a pregeometry [17, 12] gives an alternative definition of dimension of a set definable in an o-minimal structure.
(4) When a structure admits a definable cell decomposition such as an o-minimal structure, an alternative definition of dimension is proposed such as in [2].

The above four definitions of dimension coincide in the o-minimal case [13].

We return to the definably complete locally o-minimal case. The equivalence of (1) through (4) above holds also in definably complete locally o-minimal structures by slightly changing their statements. It has been proven in the author and his collaborators' works [4, 5, 7, 9, 10]. Section 2 summarizes the results on dimension in these works.

We first introduce the often-used definition of dimension, called 'topological dimension', and introduce that it satisfies van den Dries's requirements in Section 2.1. It follows from the results in [7, 10]. In [7], the author proved many dimension formulas under the assumption that definably complete locally o-minimal structures satisfy the technical property called property (a). In addition, he demonstrated that models of DCTC [20] and DCULOAS [5] satisfy property (a). Finally, Komine proved that property (a) holds for any definably complete locally o-minimal structure in [10]. Other definitions of dimension are introduced in Section 2.2. The equivalence of (1) through (3) is demonstrated in [9]. They are discussed in Section 2.2.1 and Section 2.2.2. In the author's early works on locally o-minimal structures [4, 5], he employed other definition of dimension, which is also topological. The equivalence of (1) with (4) is demonstrated in the series of papers [4, 7, 10] and discussed in Section 2.2.3. Finally, Section 2.3 summarizes the equivalence of these definitions.

We discuss on decomposition into (quasi-)special submanifolds in Section 3. Decomposition into quasi-special submanifold was proven in [7].[*1] Decomposition into special submanifold was given in [8] under the extra assumption that the structure is an expansion of an ordered group. Miller and Fornasiero proved decomposition theorem similar to ours in [3, 15].[*2] Miller's special manifold assumes that the underlying space is the set of reals, and Fornasiero proved decompositions into multi-cells under the assumption that the structure is an expansion of an ordered field. The author demonstrated that these notions coincide with our definition of special manifolds. We compare these definitions in Section 3.1 and introduce the decomposition theorem in Section 3.2.

2 Dimension theory

2.1 van den Dries's requirements and topological dimension

We first recall van den Dries's requirements on dimension function.

Definition 2.1. Consider a structure $\mathcal{M} = (M, \ldots)$. Let $\mathcal{D}$ be the set of all definable sets and $\mathbb{Z}_{\geq 0} := \{n \in \mathbb{Z} \mid n \geq 0\}$. A map $\dim : \mathcal{D} \to \mathbb{Z}_{\geq 0} \cup \{-\infty\}$ *satisfies van den Dries's requirements* if the following conditions are satisfied [1]:

(1) $\dim(S) = -\infty \Leftrightarrow S = \emptyset$; $\dim(\{x\}) = 0$ for all $x \in M$ and $\dim M = 1$.
(2) $\dim(S_1 \cup S_2) = \max\{\dim S_1, \dim S_2\}$ for any definable subsets of M^n.
(3) $\dim S^\sigma = \dim S$ for any definable set $S \subseteq M^n$ and any permutation σ of $\{1, \ldots, n\}$. Here, $S^\sigma = \{(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) \in M^n \mid (x_1, \ldots, x_n) \in S\}$.
(4) Let T be a definable subset of M^{n+1} and $T_x = \{y \in M \mid (x, y) \in T\}$ for any $x \in M^n$. Set $T(i) = \{x \in M^n \mid \dim(T_x) = i\}$ for $i = 0, 1$. Then, $T(i)$ are

[*1] It was proven under the assumption that the structure enjoys property (a). But the assumption is not necessary by Komine [10].

[*2] Miller considered more general structures called d-minimal structures [15]. Miller's proof in [15] has a gap, and the gap was filled in [21] when there exists a definable bijection between bounded and unbounded intervals.

definable and $\dim(\{(x, y) \in T \mid x \in T(i)\}) = \dim T(i) + i$ for $i = 0, 1$.

In [7], we employ the following definition of dimension.

Definition 2.2 (Dimension)**.** Consider an expansion of a densely linearly order without endpoints $\mathcal{M} = (M, <, \ldots)$. Let X be a nonempty definable subset of M^n. We consider that M^0 is a singleton with the trivial topology. The dimension of X is the maximal nonnegative integer d such that $\pi(X)$ has a nonempty interior for some coordinate projection $\pi : M^n \to M^d$. We set $\dim(X) = -\infty$ when X is an empty set.

Theorem 2.3. *When the structure is definably complete and locally o-minimal, the dimension function defined in Definition 2.2 satisfies van den Dries's requirements.*

Proof. It follows from [10, Proposition 2.8]. □

When the structure satisfies van den Dries's requirements, the following assertions hold [1, Corollary 1.5]:

(1) Let $f : X \to M^n$ be a definable map. We have $\dim(f(X)) \leq \dim X$.
(2) Let $\varphi : X \to Y$ be a definable surjective map whose fibers are equi-dimensional; that is, the dimensions of the fibers $\varphi^{-1}(y)$ are constant. We have $\dim X = \dim Y + \dim \varphi^{-1}(y)$ for all $y \in Y$.

The following properties of dimension does not necessarily follow from van den Dries's requirements.

Proposition 2.4. *Consider a definably complete locally o-minimal structure $\mathcal{M} = (M, <, \ldots)$. The following assertions hold:*

(1) Let $f : X \to M^n$ be a definable map. The notation $\mathcal{D}(f)$ denotes the set of points at which the map f is discontinuous. The inequality $\dim(\mathcal{D}(f)) < \dim X$ holds true.
(2) Let X be a definable set. The notation ∂X denotes the frontier of X defined by $\partial X = \overline{X} \setminus X$. We have $\dim(\partial X) < \dim X$.

Proof. See [10, Proposition 2.8(7), (8)]. □

2.2 Other definitions of dimension

We recalled a topological definition of dimension in Definition 2.2. We introduce other definitions of dimension.

2.2.1 Pillay's first-order topological structure

We first recall Pillay's definition. Pillay defined a first-order topological structure and the dimension rank for definable sets.

Definition 2.5. Let $\mathcal{L}$ be a language and $\mathcal{M} = (M, \ldots)$ be an $\mathcal{L}$-structure. The structure $\mathcal{M}$ is called a *first-order topological structure* if there exists an $\mathcal{L}$-formula $\phi(x, \overline{y})$ such that the family $\{\phi(x, \overline{a}) \mid \overline{a} \subseteq M\}$ is a basis for a topology on M. When $\mathcal{M}$ is an expansion of a dense linear order, then $\mathcal{M}$ is a first-order topological structure.

Recall that a set is *constructible* if it is a finite boolean combination of open sets. We consider the case in which any definable set is constructible such as the case in which the structure is a definably complete locally o-minimal structure [7, Corollary 3.10]. For a definable set X, an ordinary valued dimension rank $\mathrm{D}(X)$ is defined as follows:

(1) If X is nonempty, then $\mathrm{D}(X) \geq 0$. Otherwise, set $\mathrm{D}(X) = -\infty$.
(2) If $\mathrm{D}(X) \geq \alpha$ for all $\alpha < \delta$, where δ is limit, then $\mathrm{D}(X) \geq \delta$.
(3) $\mathrm{D}(X) \geq \alpha + 1$ if there exists a definable closed subset Y of X such that Y has an empty interior in X and $\mathrm{D}(Y) \geq \alpha$.

We put $\mathrm{D}(X) = \alpha$ if $\mathrm{D}(X) \geq \alpha$ and $\mathrm{D}(X) \not\geq \alpha+1$. We set $\mathrm{D}(X) = \infty$ when $\mathrm{D}(X) \geq \alpha$ for all α.

2.2.2 Pregeometry

The notion of a pregeometry is a central notion of geometric stability theory [18]. It is found in [18, Chapter 2] and [23, Appendix C].

Definition 2.6 (Pregeometry)**.** Consider a set S and a map $\mathrm{cl} : \mathcal{P}(S) \to \mathcal{P}(S)$, where

$\mathcal{P}(S)$ denotes the power set of S. The pair (S, cl) is a *(combinatorial) pregeometry* if the following conditions are satisfied for any subset A of S:

(i) $A \subseteq \mathrm{cl}(A)$;
(ii) $\mathrm{cl}(\mathrm{cl}(A)) = \mathrm{cl}(A)$;
(iii) For any $a, b \in S$, if $a \in \mathrm{cl}(A \cup \{b\}) \setminus \mathrm{cl}(A)$, then $b \in \mathrm{cl}(A \cup \{a\})$;
(iv) For any $a \in \mathrm{cl}(A)$, there exists a finite subset Y of A such that $a \in \mathrm{cl}(Y)$.

The condition (iii) is called the *exchange property.*

Consider a pregeometry (S, cl). Let A and B be subsets of S. The set A is cl-*independent* over B if, for any $a \in A$, we have $a \notin \mathrm{cl}((A \setminus \{a\}) \cup B)$. A subset A_0 of A is a cl-*basis* for A over B if A is contained in $\mathrm{cl}(A_0 \cup B)$ and A_0 is cl-independent over B. Each basis for A over B has the same cardinality (See Lemma 2.7(1)) and it is denoted by $\mathrm{rk}^{\mathrm{cl}}(A/B)$. We simply denote it by $\mathrm{rk}(A/B)$ when the closure operation cl is clear from the context.

The following result is well-known:

Lemma 2.7. *Consider a pregeometry* (S, cl)*. The symbols A, B and C denote arbitrary subsets of S. The following assertions hold true.*

(i) Two cl-*bases for A over B have the same cardinality.*
(ii) We have $\mathrm{rk}(A/B) \leq \mathrm{rk}(A/C)$ *when C is a subset of B.*
(iii) When $B \subseteq C$ and $\mathrm{rk}(A/B) = \mathrm{rk}(A/C)$*, a* cl-*basis for A over B is a* cl-*basis for A over C.*

Definition 2.8. Let $\mathcal{L}$ be a language and $\mathcal{M} = (M, \ldots)$ be an $\mathcal{L}$-structure. Let $\mathrm{cl} : \mathcal{P}(M) \to \mathcal{P}(M)$ be a closure operation such that the pair (M, cl) is a pregeometry. Consider subsets A and S of M and M^n, respectively. We define

$$\mathrm{rk}^{\mathrm{cl}}_{\mathcal{M}}(S/A) = \max\{\mathrm{rk}^{\mathrm{cl}}(\{a_1, \ldots, a_n\}/A) \mid (a_1, \ldots, a_n) \in S\}.$$

Let $\mathcal{L}$ be a language and T be its theory. Consider a model $\mathcal{M} = (M, \ldots)$ of T and a monster model $\mathbb{M}$ of T. The universe of $\mathbb{M}$ is denoted by the same symbol $\mathbb{M}$ for simplicity. Assume that there exists a closure operation $\mathrm{cl} : \mathcal{P}(\mathbb{M}) \to \mathcal{P}(\mathbb{M})$ such that the pair $(\mathbb{M}, \mathrm{cl})$ is a pregeometry. Let A be a subset of M and S be a definable set.

We set

$$\mathrm{rk}^{\mathrm{cl}}_T(S/A) = \mathrm{rk}^{\mathrm{cl}}_{\mathbb{M}}(S^{\mathbb{M}}/A).$$

We simply denote it by $\mathrm{rk}(S/A)$ when T and cl are clear from the context.

It is known that the universe of an o-minimal structure together with the definable closure operation forms a pregeometry in [19]. The definable operation does not necessarily yield a pregeometry when the structure is not o-minimal. We need to develop an alternative closure operation for definably complete locally o-minimal structures.

Definition 2.9 (Discrete closure)**.** Let $\mathcal{L}$ be a language containing a binary predicate $<$. Consider a definably complete locally o-minimal $\mathcal{L}$-structure $\mathcal{M} = (M, <, \dots)$. The *discrete closure* $\mathrm{discl}_{\mathcal{M}}(A)$ of a subset A of M is the set of points x in M having an $\mathcal{L}(A)$-formula $\phi(t)$ such that

(a) the set $\phi(\mathcal{M}) := \{t \in M \mid \mathcal{M} \models \phi(t)\}$ contains the point x and
(b) it is discrete and closed.

Theorem 2.10. *Let $\mathcal{L}$ be a language containing the binary predicate $<$. Consider a definably complete locally o-minimal $\mathcal{L}$-structure $\mathcal{M} = (M, <, \dots)$. The pair $(M, \mathrm{discl}_{\mathcal{M}})$ is a pregeometry.*

Proof. See [9, Theorem 3.2]. □

2.2.3 When locally definable cell decomposition is admitted

We recall the definition of cells.

Definition 2.11 (Definable cell decomposition)**.** Consider a densely linearly ordered structure $\mathcal{M} = (M, <, \dots)$. Let $(i_1, \dots, i_n)$ be a sequence of zeros and ones of length n. $(i_1, \dots, i_n)$*-cells* are definable subsets of M^n defined inductively as follows:

- A (0)-cell is a point in M and a (1)-cell is an open interval in M.
- An $(i_1, \dots, i_n, 0)$-cell is the graph of a continuous definable function defined on an $(i_1, \dots, i_n)$-cell. An $(i_1, \dots, i_n, 1)$-cell is a definable set of the form $\{(x, y) \in C \times M \mid f(x) < y < g(x)\}$, where C is an $(i_1, \dots, i_n)$-cell and f and g are definable continuous functions defined on C with $f < g$.

A *cell* is an $(i_1, \dots, i_n)$-cell for some sequence $(i_1, \dots, i_n)$ of zeros and ones. An *open cell* is a $(1, 1, \dots, 1)$-cell.

We inductively define a *definable cell decomposition* of an open box $B \subset M^n$. For $n = 1$, a definable cell decomposition of B is a partition $B = \bigcup_{i=1}^m C_i$ into finite cells. For $n > 1$, a definable cell decomposition of B is a partition $B = \bigcup_{i=1}^m C_i$ into finite cells such that $\pi(B) = \bigcup_{i=1}^m \pi(C_i)$ is a definable cell decomposition of $\pi(B)$, where $\pi : M^n \to M^{n-1}$ is the projection forgetting the last coordinate. Consider a finite family $\{A_\lambda\}_{\lambda \in \Lambda}$ of definable subsets of B. A *definable cell decomposition of B partitioning* $\{A_\lambda\}_{\lambda \in \Lambda}$ is a definable cell decomposition of B such that the definable sets A_λ are unions of cells for all $\lambda \in \Lambda$.

We call the structure $\mathcal{M}$ *admits local definable cell decomposition* if, for any positive integer n, any finite family $\{A_\lambda\}_{\lambda \in \Lambda}$ of definable subsets of M^n and any point $a \in M^n$, there exists an open box B containing the point a such that there exists a definable cell decomposition of B partitioning $\{B \cap A_\lambda\}_{\lambda \in \Lambda}$.

A definably complete uniformly locally o-minimal structure of the second kind admits local definable cell decomposition.

Definition 2.12. A locally o-minimal structure $\mathcal{M} = (M, <, \dots)$ is a *uniformly locally o-minimal structure of the second kind* if, for any positive integer n, any definable set $X \subset M^{n+1}$, $a \in M$ and $b \in M^n$, there exist an open interval I containing the point a and an open box B containing b such that the definable sets $X_y \cap I$ are finite unions of points and open intervals for all $y \in B$.

Theorem 2.13. *A definably complete locally o-minimal structure admits local definable cell decomposition if and only if it is a uniformly locally o-minimal structure of the second kind.*

Proof. See [4, Corollary 4.1]. □

In [4], the author proposed two definitions of dimension. These definitions are also employed in [5].

Definition 2.14. [4, Definition 5.1] Consider a densely linearly ordered structure $\mathcal{M} = (M, <, \dots)$. A definable set $X \subseteq M^n$ is of $\dim'(X) \geq m$ if there exists an open box $B \subset M^m$ and a definable continuous injective map $f : B \to X$ which is homeomorphic onto its image. A definable set $X \subset M^n$ is of $\dim'(X) = m$ if it is of $\dim'(X) \geq m$ and it is not of $\dim'(X) \geq m+1$. The empty set is defined to be of dimension $-\infty$.

Definition 2.15. [4, Definition 5.3] Consider a locally o-minimal structure $\mathcal{M} = (M, <, \dots)$ which admits local definable cell decomposition. Let $X \subseteq M^n$ be a definable set. We define $\dim''(X)$ as follows:

$$\dim''(X) = \max\{i_1 + \cdots + i_n \mid X \text{ contains an } (i_1, \dots, i_n)\text{-cell}\}.$$

The notation $\pi_{i_1,\dots,i_n}$ denotes the projection from M^n to M^d, where $d = i_1 + \cdots + i_n$, forgetting all the j-th coordinates with $i_j = 0$. Then $\pi(X)$ has a nonempty interior. This definition is employed as the definition of sets definable in o-minimal structures in [2].

2.3 Equivalence of definitions

We have recalled five definitions of dimension. As it is announced in Section 1, they all coincide.

Theorem 2.16. *Let $\mathcal{L}$ be a language containing a binary predicate $<$ and T be a definably complete locally o-minimal $\mathcal{L}$-theory. Let $\mathcal{M} = (M, \dots)$ be a model of T. Let A be a subset of M and X be a subset of M^n definable over A. We have*

$$\dim X = \mathrm{D}(X) = \mathrm{rk}_T^{\mathrm{discl}}(X/A) = \dim' X.$$

In addition, if $\mathcal{M}$ is a uniformly locally o-minimal structure of the second kind, we have

$$\dim X = \dim'' X.$$

Proof. The equality $\dim X = \mathrm{D}(X)$ is given in [9, Proposition 4.3]. The equality $\dim X = \mathrm{rk}_T^{\mathrm{discl}}(X/A)$ is given in [9, Theorem 3.5]. The equality $\dim X = \dim' X$ is found in [10, Proposition 2.8(9)]. The equality $\dim' X = \dim'' X$ is found in [4, Theorem 5.4]. □

3 Decomposition into special submanifolds

3.1 Definitions

We first introduce our definition of special manifolds.

Definition 3.1. Consider an expansion of a dense linear order without endpoints $\mathcal{M} = (M, <, \ldots)$. Let $\pi : M^n \to M^d$ be a coordinate projection, where n is a positive integer and d is a non-negative integer with $d \leq n$. We consider that M^0 is a singleton equipped with the trivial topology and the projection $\pi : M^n \to M^0$ is the trivial map when $d = 0$. Let τ be the unique permutation of $\{1, \ldots, n\}$ such that

(a) $\tau(i) < \tau(j)$ for $1 \leq i < j \leq n$ when $\tau(i) > d$ and $\tau(j) > d$.
(b) The composition $\pi \circ \overline{\tau}$ is the projection onto the first d coordinates, where $\overline{\tau} : M^n \to M^n$ is the map defined by $\overline{\tau}(x_1, \ldots, x_n) = (x_{\tau(1)}, \ldots, x_{\tau(n)})$.

Set

$$\mathrm{fib}(X, \pi, x) = \{y \in M^{n-d} \mid (x, y) \in \overline{\tau}^{-1}(X)\}$$

for $x \in \pi(X)$. Note that $\mathrm{fib}(X, \pi, x) = \{y \in M^{n-d} \mid (x, y) \in X\}$ when π is the projection onto the first d coordinates.

When π is the coordinate projection onto the first d coordinate, a definable subset X of M^n is a π-*special submanifold* if, for any $x \in M^d$, there exist an open box U in M^d containing the point x and a family $\{V_y\}_{y \in \mathrm{fib}(X,\pi,x)}$ of mutually disjoint open boxes in M^n indexed by the set $\mathrm{fib}(X, \pi, x)$ such that

(1) $\pi(V_y) = U$ for all $y \in \mathrm{fib}(X, \pi, x)$;
(2) $X \cap \pi^{-1}(U)$ is contained in $\bigcup_{y \in \mathrm{fib}(X,\pi,x)} V_y$, and
(3) $V_y \cap X$ is the graph of a continuous map defined on U for each $y \in \mathrm{fib}(X, \pi, x)$.

We do not require that the union $\bigcup_{y \in \mathrm{fib}(X,\pi,x)} V_y$ is definable.

When π is not the coordinate projection onto the first d coordinate, we say that a definable subset X of M^n is π-*special submanifold* if $\overline{\tau}^{-1}(X)$ is $\pi \circ \overline{\tau}$-special submanifold. We omit the prefix π when it is clear from the context.

Note that a discrete, closed definable subset of M^n is always a π-special submanifold, where $\pi : M^n \to M^0$ is the trivial map.

We next define quasi-special submanifolds.

Definition 3.2. Consider an expansion of a densely linearly order without endpoints $\mathcal{M} = (M, <, \ldots)$. Let $\pi : M^n \to M^d$ be a coordinate projection. A definable subset is a π-*quasi-special submanifold* or simply a *quasi-special submanifold* if, for every point $x \in \pi(X)$, we can take an open box U in M^d containing the point x and a family $\{V_y\}_{y \in \text{fib}(X,\pi,x)}$ of mutually disjoint open boxes in M^n indexed by the set $\text{fib}(X, \pi, x)$ satisfying the conditions (1) and (3) in Definition 3.1.

A quasi-special submanifold is not necessarily a special submanifold.

Example 3.3. Consider the ordered field of reals $(\mathbb{R}, <, 0, 1, +, \cdot)$. The set

$$\{(x, 0) \mid x \in \mathbb{R}\} \cup \{(x, 1/x) \mid x > 0\}$$

is definable and a quasi-special submanifold, but it is not a special submanifold. We can not take an open box U and a family of open boxes $\{V_y\}_{y \in \text{fib}(X,\pi,x)}$ satisfying the condition (2) in Definition 3.1 at $x = 0$.

Miller gave another definition of special submanifolds when the underlying space is the set of reals $\mathbb{R}$.

Definition 3.4 ([15])**.** We consider an expansion of the ordered set of reals $(\mathbb{R}, <)$. Let $\pi : \mathbb{R}^n \to \mathbb{R}^d$ be a coordinate projection. A d-dimensional submanifold X of $\mathbb{R}^n$ (in the usual sense) is π-special if, for each $x \in \pi(X)$, there exists an open box U in $\mathbb{R}^d$ containing the point x such that each connected component C of $X \cap \pi^{-1}(U)$ projects homeomorphically onto U.

Note that there are no connected definable sets other than singletons in some ordered structure whose universe is not $\mathbb{R}$ such as the set of algebraic real numbers $\mathbb{R}_{\text{alg}}$. Definition 3.4 does not make sense in such a structure.

Proposition 3.5. *Consider an expansion of the ordered set of reals* $(\mathbb{R}, <)$. *Let* $\pi : \mathbb{R}^n \to \mathbb{R}^d$ *be a coordinate projection. A definable subset of* $\mathbb{R}^n$ *is a* π*-special submanifold in the sense of Definition 3.4 if it is a* π*-special submanifold in the sense of Definition 3.1. The opposite implication holds true when the structure is locally o-minimal.*

Proof. See [8, Proposition 3.10]. □

Fornasiero gave another definition of special manifolds in [3]. He called them multi-cells in his paper.

Definition 3.6. Let $\mathcal{F} = (F, <, +, 0, \cdot, 1, \ldots)$ be an expansion of an ordered commutative field. Let X be a definable subset of F^n of dimension d and $\pi : F^n \to F^d$ be a coordinate projection. Take the permutation τ of $\{1, \ldots, n\}$ satisfying the conditions (a) and (b) in Definition 3.1. The notation $\overline{\tau}$ denotes the map defined in Definition 3.1.

A point $(a, b) \in M^n$ is (X, π)-*normal* if there exist a definable neighborhood A of a in M^d and a definable neighborhood B of b in M^{n-d} such that either $A \times B$ is disjoint from $\overline{\tau}^{-1}(X)$ or $(A \times B) \cap \overline{\tau}^{-1}(X)$ is the graph of a definable continuous map $f : A \to B$.

We first consider the case in which $\overline{\tau}^{-1}(X) \subseteq F^d \times (0, 1)^{n-d}$. A point $a \in F^d$ is (X, π)-*bad* if it is the projection of a non-(X, π)-normal point; otherwise, the point a is called (X, π)-*good*.

Consider the case in which X does not satisfy the previous condition. Let $\phi : F \to (0, 1)$ be a definable homeomorphism. Consider the map

$$\psi : \mathrm{id}^d \times \phi^{n-d} : F^d \times F^{n-d} \to F^d \times (0, 1)^{n-d}.$$

We say that a is (X, π)-*good* if it is $(\psi(\overline{\tau}^{-1}(X)), \pi \circ \overline{\tau})$-good. We define (X, π)-bad points etc. similarly.

The definable set X is a π-*multi-cell* if every point of $\pi(X)$ is (X, π)-good.

Proposition 3.7. *Let $\mathcal{F} = (F, <, +, 0, \cdot, 1, \ldots)$ be a definably complete locally o-minimal expansion of an ordered field. Let $\pi : F^n \to F^d$ be a coordinate projection. A definable set is a π-special submanifold in the sense of Definition 3.1 if and only if it is a π-multi-cell.*

Proof. See [8, Proposition 3.13]. □

3.2 Decomposition theorem

We recall the definition of decomposition into (quasi-)special submanifolds.

Definition 3.8. Consider an expansion of a densely linearly order without endpoints $\mathcal{M} = (M, <, \ldots)$. Let $\{X_i\}_{i=1}^m$ be a finite family of definable subsets of M^n. A *decomposition of M^n into (quasi-)special submanifolds partitioning* $\{X_i\}_{i=1}^m$ is a finite family of (quasi-)special submanifolds $\{C_i\}_{i=1}^N$ such that

- $\bigcup_{i=1}^N C_i = M^n$,
- $C_i \cap C_j = \emptyset$ when $i \neq j$ and
- either C_i has an empty intersection with X_j or it is contained in X_j

for any $1 \leq i \leq m$ and $1 \leq j \leq N$. A decomposition $\{C_i\}_{i=1}^N$ of M^n into (quasi-)special submanifolds *satisfies the frontier condition* if the closure of any special manifold $\text{cl}(C_i)$ is the union of a subfamily of the decomposition.

We finally give decomposition theorem.

Theorem 3.9. *Consider a definably complete locally o-minimal expansion of an dense linear order without endpoints $\mathcal{M} = (M, <, \ldots)$. Let $\{X_i\}_{i=1}^m$ be a finite family of definable subsets of M^n. There exists a decomposition $\{C_i\}_{i=1}^N$ of M^n into quasi-special submanifolds partitioning $\{X_i\}_{i=1}^m$ and satisfying the frontier condition. Furthermore, the number N of quasi-special submanifolds is not greater than the number uniquely determined only by m and n.*

We can replace the word 'quasi-special' by 'special' when $\mathcal{M}$ is an expansion of an ordered group $\mathcal{M} = (M, <, 0, +, \ldots)$.

Proof. See [10, Proposition 2.11] and [8, Theorem 3.19]. □

When the structure enjoys stronger property called almost o-minimality, a better decomposition theorem than Theorem 3.9 is available, which is called decomposition into multi-cells.*3 We do not treat it in this paper. See [6] for more information.

参考文献

[1] L. van den Dries, *Dimension of definable sets, algebraic boundedness and henselian fields*, Ann. Pure Appl. Logic, **45** (1989), 189-209.

*3 The definition of multi-cells here is not identical with that of Fornasiero's multi-cells.

[2] L. van den Dries, *Tame topology and o-minimal structures*, London Mathematical Society Lecture Note Series, Vol. 248. Cambridge University Press, Cambridge, 1998.

[3] A. Fornasiero, *Locally o-minimal structures and structures with locally o-minimal open core*, Ann. Pure Appl. Logic, **164** (2013), 211-229.

[4] M. Fujita, *Uniformly locally o-minimal structures and locally o-minimal structures admitting local definable cell decomposition*, Ann. Pure Appl. Logic, **171** (2020), 102756.

[5] M. Fujita, *Dimension inequality for a definably complete uniformly locally o-minimal structure of the second kind*, J. Symbolic Logic, **85** (2020), 1654-1663.

[6] M. Fujita, *Almost o-minimal structures and $\mathfrak{X}$-structures*, Ann. Pure Appl. Logic, **173** (2022), 103144.

[7] M. Fujita, *Locally o-minimal structures with tame topological properties*, J. Symbolic Logic, **88** (2023), 219-241.

[8] M. Fujita, *Decomposition into special submanifolds*, Math. Log. Quart., **69** (2023), 104-116.

[9] M. Fujita, *Pregeometry over locally o-minimal structures and dimension*, Math. Log. Quart., **69** (2023), 472-481.

[10] M. Fujita, T. Kawakami and W. Komine, *Tameness of definably complete locally o-minimal structures and definable bounded multiplication*, Math. Log. Quart., **68** (2022), 496-515.

[11] T. Kawakami, K. Takeuchi, H. Tanaka and A. Tsuboi, *Locally o-minimal structures*, J. Math. Soc. Japan, **64** (2012), 783-797.

[12] E. Hrushovski and A. Pillay, *Groups definable in local fields and psudo-finite fields*, Israel J. Math., **85** (1994), 203–262.

[13] L. Mathews, *Cell decomposition and dimension function in first-order topological structures*, Proc. London Math. Soc. (3), **70** (1995), 1–32.

[14] C. Miller, *Expansions of dense linear orders with the intermediate value property*, J. Symbolic Logic, **66** (2001), 1783-1790.

[15] C. Miller, *Tameness in expansions of the real field* in Logic Colloquium '01, eds. M. Baaz, S. -D. Friedman and J. Krajíček (Cambridge University Press, Cambridge, 2005), 281-316.

[16] A. Pillay, *First order topological structures and theories*, J. Symbolic Logic, **52**

(1987), 763–778.

[17] A. Pillay, *On groups and fields definable in o-minimal structures*, J. Pure Appl. Alg., **53** (1988), 239–255.

[18] A. Pillay, *Geometric stability theory*, Oxford logic guides, vol. 32. Clarendon Press, Oxford, 1996.

[19] A. Pillay and C. Steinhorn, *Definable sets in ordered structure I*, Trans. Amer. Math. Soc., **295** (1986), 565-592.

[20] H. Schoutens, O-minimalism, J. Symbolic Logic, **79**, 355-409 (2014).

[21] A. Thamrongthanyalak, *Michael's selection theorem in d-minimal expansions of the real field*, Proc. Amer. Math. Soc., **147**, 1059-1071 (2019).

[22] C. Toffalori and K. Vozoris, *Notes on local o-minimality*, Math. Log. Quart., **55** (2009), 617-632.

[23] K. Tent and M. Ziegler, *A course in model theory*, Lecture notes in logic, vol. 40. Cambridge University Press, Cambridge, 2012.