Uniqueness of prime models in definably complete locally o-minimal theories

小峰 航 (Wataru Komine)
筑波大学 理工情報生命学術院 数理物質科学研究群
(Degree Programs in Pure and Applied Sciences,
Graduate School of Science and Technology,
University of Tsukuba)

Abstract

It is well-known that every o-minima theory has a prime model and it is unique up to isomorphism. We give sufficient conditions for uniqueness of prime models in definably complete locally o-minimal theories.

1 Introduction

We are interested in definably complete locally o-minimal structures and its theories. Pillay and Steinhorn showed the following:

Theorem 1 ([1]). *Every o-minimal thoery has prime model over any sets, that are unique up to isomorphism.*

In this note, we give a condition for uniqueness of prime models in definable complete locally o-minimal theories. This condition is related to form of definable sets.

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2 Preliminaries

First, we define o-minimality, definable completeness and local o-minimality. Let a structure $\mathcal{M} = (M, <, \cdots)$ be expansion of a linear ordered set without endpoints.

Definition 2. $\mathcal{M}$ is o-minimal structure if every definable subset of M is a finite union of points and open intervals. $\mathcal{M}$ is locally o-minimal structure if for any points $a \in M$ and definable subset A of M, there is an open interval I containing a such that $A \cap I$ is a finite union of points and open intervals. $\mathcal{M}$ is definably complete if every definable subset of M has supremum and infimum in $M \cup \{-\infty, +\infty\}$.

Example 3. Dense linear ordered sets without endpoints, real closed fields and $(\mathbb{R}, 0, 1, +, -, \cdot, \exp(-))$ are o-minimal. $(\mathbb{R}, <, 0, +, \sin(-))$ is not o-minimal structure since discrete infinite subset $\mathbb{Z} = \{x \in \mathbb{R} \mid \sin(x\pi) = 0\}$ is definable. But it is definably complete locally o-minimal structure.

In definably complete locally o-minimal structure, an analogy of monotonicity theorem holds as following (it is called strong local monotonicity theorem).

Theorem 4 (strong local monotonicity theorem, [2])**.** *Let I be a A-interval and $f : I \to M$ be a A-definable function. Then there exists a mutually disjoint A-definable partition $I = X_d \cup X_c \cup X_+ \cup X_-$ satisfying the following conditions:*

(1) the A-definable set X_d is discrete and closed;

(2) the A-definable set X_c is open and f is locally constant on X_c;

(3) the A-definable set X_+ is open and f is locally strictly increasing and continuous on X_+;

(4) the A-definable set X_- is open and f is locally strictly decreasing and continuous on X_-.

In this note, we assume a language $\mathcal{L}$ contains $<$ and an interpretation of $<$ is linear ordering without endpoints in $\mathcal{L}$-structures.

Definition 5. $\mathcal{L}$-theory T is o-minimal (resp. definably complete locally o-minimal) if every model of T is o-minimal (resp. definably complete locally o-minimal) structure.

Remark 6. If $\mathcal{M}$ is o-minimal, its complete theory $\mathrm{Th}(\mathcal{M})$ is o-minimal by uniform finiteness theorem. On the other hand, definable completeness and local o-minimality are represented by $\mathcal{L}$-sentences. Therefore, if $\mathcal{M}$ is definably complete locally o-minimal, its complete theoery $\mathrm{Th}(\mathcal{M})$ is definably complete locally o-minimal.

In this note, we assume $\mathcal{L}$-theory T is complete and work in monster model $\mathbb{M}$. On existence and uniqueness of prime models, the following fact is important.

Fact 7 ([1][4])**.** *Let a complete theory T be given.*

(1) Let $A \subset \mathcal{M} \models T$. Any model $\mathcal{M}$ that is constructible over A also is prime over A.

(2) Suppose that for any subset A of a model $\mathcal{M}$ of T and any formula φ having parameters A, there is a complete formula with parameters from A which, relative to $Th(\mathcal{M}, a)_{a\in A}$, implies φ. Then, for any $A \subset \mathcal{M}$, there is a model $\mathcal{M}$ of T that is constructible over A. (it is enough to consider when φ has just one free variable)

(3) Let $A \subset \mathcal{M} \models T$. Then any two models that are constructible over A are isomorphic over A.

3 Main theorem

In this section, let T be a definably complete locally o-minimal theory. By using strong local monotonicity theorem (Theorem 4), We can show the following theorem with a similar argument to o-minimal case. (cf.[1])

Theorem 8. *If $dcl(\emptyset)$ is nonempty, then T has prime model over emptyset and it is unique up to isomorphism. Moreover, T has prime model over nonempty A and it is unique up to isomorphism.*

We showed the following theorem that is related to complete formulas.

Theorem 9. *We assume that T has a complete formula. Then, T has a prime model over $\emptyset$ and it is unique up to isomorphism.*

The following is the main theorem.

Theorem 10. *We assume that there is a prime model $\mathcal{M}$ of T such that it has a discrete definable subset X of M without parameters which isomorphic to $n \times \mathbb{Z}$ as orderd sets for some positive integer n. Then prime models of T are isomorphic.*

sketch of proof. For convenience, we assume $n = 2$.

We take $\mathcal{L}$-formula $\varphi(v)$ which define X. Since X is isomorphic to $\mathbb{Z} \sqcup \mathbb{Z}$ as ordered sets, we get decomposition $X = X_0 \sqcup X_1$ where X_0 corresponds to first part of $\mathbb{Z} \sqcup \mathbb{Z}$ and X_1 corresponds to second part of $\mathbb{Z} \sqcup \mathbb{Z}$.

We take a element $a \in X_0$ from first part. Let $\mathcal{N}$ be a constructible model over a and $X' = \varphi(\mathcal{N})$. Then we get decomposition $X' = X'_0 \sqcup X'_1$ where each X'_i isomorphic to $\mathbb{Z}$. Since $\mathcal{M}$ is prime, we get an elementary embbeding $j : \mathcal{M} \hookrightarrow \mathcal{N}$. So we get decomposition $X' = j(X_0) \sqcup j(X_1)$. Then $j(X_0) = X'_0$ and $j(X_1) = X'_1$. In $\mathcal{N}$, $S^m(a) = j(a)$ for some integer m where S is successor function in X'. Since $j(a)$ is definable over a, $\mathcal{N}$ is prime over $j(a)$. Then $\mathcal{M}$ is prime over a. By uniqueness of prime models over nonemptyset, $\mathcal{M}$ and $\mathcal{N}$ are isomorphic. □

References

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