A REMARK ON QUASI-GEOMETRIC ELIMINATION OF IMAGINARIES

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Abstract. Quasi-geometric elimination of imaginaries is equivalent to geometric elimination of imaginaries.

1. Results, notatnions and the definition of quasi-geometric elimination of imaginaries

In first section we define quasi-geometric elimination of imaginaries. In any theory geometric eliminaries implies quasi-geometric elimination of imaginaries. In the second section we show the converse implication in rosy theories having strict independence relations introduced by H.Adler. In the last section we present latest results without proofs in [Y1]. Let T be a complete L-theory, and $\mathcal{M}$ be a sufficiently saturated model of T. We work in $\mathcal{M}^{\mathrm{eq}} := \{(\bar{a})_E : (\bar{a})_E$ is the E-class of $\bar{a}$, where $\bar{a}$ is a finite tuple of M and $E(x, y)$ is a ϕ-definable equivalence relation with $\mathrm{lh}(\bar{x}) = \mathrm{lh}(\bar{y}) = \mathrm{lh}(\bar{a})\ \}$. $\bar{a}, \bar{b}, \bar{c}$ denote finite tuple of $\mathcal{M}^{\mathrm{eq}}$ and $A, B, C, \cdots$ denote small subsets of $\mathcal{M}^{\mathrm{eq}}$ unless stated otherwise. For any $A, A', B \subset \mathcal{M}^{\mathrm{eq}}$, if there exists $\sigma \in \mathrm{Aut}(\mathcal{M}^{\mathrm{eq}}/B)$ be such that $\sigma(A) = A'$, we write $\mathrm{tp}(A/B) = \mathrm{tp}(A'/B)$ or $A' \models \mathrm{tp}(A/B)$. For $e \in \mathcal{M}^{\mathrm{eq}}$, we write $e \in \mathrm{acl}^{\mathrm{eq}}(A)$ if the orbit of e by automorphisms fixing A pointwise is finite. For finite real tuple $\bar{a} \subset \mathcal{M}$ we we write $\bar{a} \in \mathrm{acl}(A)$ if the orbit of $\bar{a}$ by automorphisms fixing A pointwise is finite. We say that T has Geometric Elimination of Imaginaries over C if for any $e = (\bar{a})_E \in \mathcal{M}^{\mathrm{eq}}$ there exists a finite real tuple $\bar{b} \subset \mathcal{M}$ such that $\mathrm{acl}^{\mathrm{eq}}(eC) = \mathrm{acl}^{\mathrm{eq}}(\bar{b}, C)$. GEI simply means GEI over $\emptyset$.

Definition 1.1. Fix $C \subset \mathcal{M}$. We say that T has *quasi*-Geometric Elimination of Imaginaries over C if we have $\mathrm{acl}^{\mathrm{eq}}(\mathrm{acl}(EC)\cap\mathrm{acl}(FC)) = \mathrm{acl}^{\mathrm{eq}}(EC)\cap\mathrm{acl}^{\mathrm{eq}}(FC)$ for any $E, F \subset \mathcal{M}$. "$\subseteq$" always holds. Quasi-GEI means quasi-GEI over $\emptyset$. Quasi-GEI implies quasi-GEI over any C.

Lemma 1.2. *GEI over C implies quasi-GEI over C. In particular GEI implies quasi-GEI.*

Proof. If $e \in \mathrm{acl}^{\mathrm{eq}}(EC)\cap\mathrm{acl}^{\mathrm{eq}}(FC)$, then by GEI over C, by compactness there exists $\bar{a} \subset \mathcal{M}$ with $\mathrm{acl}^{\mathrm{eq}}(eC) = \mathrm{acl}^{\mathrm{eq}}(\bar{a}, C)$. Then $\bar{a} \subseteq \mathrm{acl}^{\mathrm{eq}}(EC)$ and $\bar{a} \subseteq \mathrm{acl}^{\mathrm{eq}}(FC)$. As $\bar{a}$ is real, we have $\bar{a} \subseteq \mathrm{acl}(EC) \cap \mathrm{acl}(FC)$. So $e \in \mathrm{acl}^{\mathrm{eq}}(\mathrm{acl}(EC) \cap \mathrm{acl}(FC))$. □

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2. Quasi-GEI implies GEI in rosy theories

H.Adler [A] introduces a strict independence relation $* \mathop{\smile\!\!\!\!\!\!|}_{*} *$ for triplet of small subsets of $\mathcal{M}^{\text{eq}}$ satisfying (1)-(9):

(1) invariance: If $A \mathop{\smile\!\!\!\!\!\!|}_{B} C$ and $\text{tp}(ABC) = \text{tp}(A'B'C')$, then $A' \mathop{\smile\!\!\!\!\!\!|}_{B'} C'$.
(2) monotonicity: If $A \mathop{\smile\!\!\!\!\!\!|}_{B} C, A' \subseteq A$ and $C' \subseteq C$, then $A' \mathop{\smile\!\!\!\!\!\!|}_{B} C'$.
(3) right base monotonicity: If $A \mathop{\smile\!\!\!\!\!\!|}_{B} D$ and $B \subseteq C \subseteq D$, then $A \mathop{\smile\!\!\!\!\!\!|}_{C} D$.
(4) left transitivity: If $B \subseteq C \subseteq D, D \mathop{\smile\!\!\!\!\!\!|}_{C} A$ and $C \mathop{\smile\!\!\!\!\!\!|}_{B} A$, then $D \mathop{\smile\!\!\!\!\!\!|}_{B} A$.
(5) left normality: $A \mathop{\smile\!\!\!\!\!\!|}_{B} C$ implies $AB \mathop{\smile\!\!\!\!\!\!|}_{B} C$.
(6) left extension: If $A \mathop{\smile\!\!\!\!\!\!|}_{B} C$ and $C \subseteq D$, then there exists $A' \models \text{tp}(A/BC)$ such that $A' \mathop{\smile\!\!\!\!\!\!|}_{B} D$.
(7) left finite character: If $\bar{a} \mathop{\smile\!\!\!\!\!\!|}_{B} C$ for any finite tuple $\bar{a} \subset A$, then $A \mathop{\smile\!\!\!\!\!\!|}_{B} C$.
(8) local character: For any A there exists a cardinal $\kappa(A)$ such that, for any B, there exists $B_0 \subseteq B$ with $|B_0| < \kappa(A)$ and $A \mathop{\smile\!\!\!\!\!\!|}_{B_0} B$.
(9) anti-reflexivity: $A \mathop{\smile\!\!\!\!\!\!|}_{B} A$ implies $A \subseteq \text{acl}^{\text{eq}}(B)$.

T is said to be *rosy* if there exists a strict independence relation on $\mathcal{M}^{\text{eq}}$. Theorem 1.14 in [A] shows that (1)-(8) imply symmetry: $A \mathop{\smile\!\!\!\!\!\!|}_{B} C \Leftrightarrow C \mathop{\smile\!\!\!\!\!\!|}_{B} A$. From now on we work in a saturated rosy eq-structure $(\mathcal{M}^{\text{eq}}, \mathop{\smile\!\!\!\!\!\!|})$. We use the following fact (Remark 2.1 in [Y]) by symmetry and (1)-(8) except (7). But symmetry needs (7).

Fact 2.1. *For any $A, B, C \subset \mathcal{M}^{\text{eq}}$ we have $A \mathop{\smile\!\!\!\!\!\!|}_{B} C$ iff $\text{acl}^{\text{eq}}(A) \mathop{\smile\!\!\!\!\!\!|}_{\text{acl}^{\text{eq}}(B)} \text{acl}^{\text{eq}}(C)$.*

Lemma 2.2. *(Left extension and anti-reflexivity are needed.) Let $e = (\bar{a})_E$ and take $\bar{b}, \bar{c} \models \text{tp}(\bar{a}/eC)$ such that $\bar{b}, \bar{c}, \bar{a}$ are independent over eC. Then $\bar{a} \mathop{\smile\!\!\!\!\!\!|}_{\text{acl}(\bar{c},C)\cap\text{acl}(\bar{b},C)} \bar{b}, \bar{c}$ implies that $e = (\bar{a})_E$ is interalgebraic with a small real sequence over C.*

Proof. Let $e = (\bar{a})_E$ and take $\bar{b}, \bar{c} \models \text{tp}(\bar{a}/eC)$ such that $\bar{b}, \bar{c}, \bar{a}$ are independent over eC. We have $e = (\bar{b})_E = (\bar{c})_E$. Suppose that $\bar{a} \mathop{\smile\!\!\!\!\!\!|}_{\text{acl}(\bar{c},C)\cap\text{acl}(\bar{b},C)} \bar{b}, \bar{c}$.. Put $X = \text{acl}(\bar{c}, C) \cap \text{acl}(\bar{b}, C) \subset \mathcal{M}$. By $e \mathop{\smile\!\!\!\!\!\!|}_{X} e$, $e \in \text{acl}^{\text{eq}}(X)$ follows by anti-reflexivity. As $\bar{b} \mathop{\smile\!\!\!\!\!\!|}_{eC} \bar{c}$, $X \mathop{\smile\!\!\!\!\!\!|}_{eC} X$ follows, by anti-reflexivity, $\text{acl}^{\text{eq}}(eC) = \text{acl}^{\text{eq}}(XC)$ follows □

Theorem 2.3. *(Left extension and both-sided transitivity are needed.) Quasi-GEI over C implies GEI over C. In particular quasi-GEI implies GEI.*

Proof. Let $e = (\bar{a})_E$. Take $\bar{b}, \bar{c} \models \text{tp}(\bar{a}/Ce)$ such that $\bar{a}, \bar{b}, \bar{c}$ are independent over Ce. As $\bar{b} \mathop{\smile\!\!\!\!\!\!|}_{eC} \bar{c}$ with $eC \subseteq \text{acl}^{\text{eq}}(\text{acl}(\bar{b}, C) \cap \text{acl}(\bar{c}, C)) = \text{acl}^{\text{eq}}(\bar{b}, C) \cap \text{acl}^{\text{eq}}(\bar{c}, C)$ by quasi-GEI over C. By Fact 2.1 we have $\bar{c} \mathop{\smile\!\!\!\!\!\!|}_{\text{acl}(\bar{c},C)\cap\text{acl}(\bar{b},C)} \bar{b}$. We have that $\bar{a} \mathop{\smile\!\!\!\!\!\!|}_{\bar{c},C} \bar{b}, \bar{c}$ and $\bar{a} \mathop{\smile\!\!\!\!\!\!|}_{\bar{b},C} \bar{b}, \bar{c}$. We need to show that $\bar{a} \mathop{\smile\!\!\!\!\!\!|}_{X} \bar{b}, \bar{c}$, where $X = \text{acl}(\bar{c}, C) \cap \text{acl}(\bar{b}, C) (\supseteq C)$. By left transitivity and $\bar{b} \mathop{\smile\!\!\!\!\!\!|}_{X} \bar{c}$, we have $\bar{a}, \bar{c}, C \mathop{\smile\!\!\!\!\!\!|}_{X} \bar{b}$ and $\bar{a}, \bar{b}, C \mathop{\smile\!\!\!\!\!\!|}_{X} \bar{c}$ follow. So we have $\bar{a} \mathop{\smile\!\!\!\!\!\!|}_{X} \bar{b}$ and $\bar{a} \mathop{\smile\!\!\!\!\!\!|}_{X\bar{b}} \bar{c}$. By right transitivity $\bar{a} \mathop{\smile\!\!\!\!\!\!|}_{X} \bar{b}, \bar{c}$ follows. By Lemma 2.2, we see that $\text{acl}^{\text{eq}}(eC) = \text{acl}^{\text{eq}}(XC)$. □

3. Some results in the author's paper working in progress [Y1]

We present results without proofs for the submission [Y1] to the other journal.
● Will Johnson suggests that GEI is equivalent to quasi-GEI in any theory by considering anti-reflexive indepedence relation ($A \mathop{\smile\!\!\!\!\!\!|}^{a}_{B} C \Leftrightarrow_{\text{def}} \text{acl}^{\text{eq}}(AB) \cap \text{acl}^{\text{eq}}(BC) = \text{acl}^{\text{eq}}(C)$) which has the full exisistence condition [A1] ; for any $A, B, C \subset \mathcal{M}^{\text{eq}}$ there exists $A' \models \text{tp}(A/B)$ such that $A' \mathop{\smile\!\!\!\!\!\!|}^{a}_{B} C$. Then we can show that GEI=quasi-GEI

in any theory as in Theorem 2.3.

● Base monotonicity of $\mathop{\smile\hspace{-0.9em}\mid\hspace{0.5em}}^a$ is equivalent to modular law [A1] ; for algebraically closed sets $A, C, D \subset \mathcal{M}^{\mathrm{eq}}$ with $C \subseteq D$ we have $D \cap \mathrm{acl}^{\mathrm{eq}}(AC) = \mathrm{acl}^{\mathrm{eq}}((A \cap D)C) = \mathrm{acl}^{\mathrm{eq}}(D \cap AC)$. If $(\mathcal{M}^{\mathrm{eq}}, \mathop{\smile\hspace{-0.9em}\mid\hspace{0.5em}})$ is one-based and has base monotonicity and anti-reflexivity for $\mathop{\smile\hspace{-0.9em}\mid\hspace{0.5em}}$, modular law holds. Conversely if modular law holds, then $(\mathcal{M}^{\mathrm{eq}}, \mathop{\smile\hspace{-0.9em}\mid\hspace{0.5em}}^a)$ is rosy and one-based with $\mathop{\smile\hspace{-0.9em}\mid\hspace{0.5em}}^a = \mathop{\smile\hspace{-0.9em}\mid\hspace{0.5em}}^{\text{þ}}$, thorn non-forking.

● Will Johnson asks to the author why quasi-GEI is important? The following notion is very related to quasi-GEI. We define modular law *in the real sort* if for algebraically closed sets $A, C, D \subset \mathcal{M}$ with $C \subseteq D$ $D \cap \mathrm{acl}^{\mathrm{eq}}(AC) = \mathrm{acl}^{\mathrm{eq}}((A \cap D)C) = \mathrm{acl}^{\mathrm{eq}}(D \cap AC)$. If $(\mathcal{M}, \mathop{\smile\hspace{-0.9em}\mid\hspace{0.5em}})$ is modular and has base monotonicity and anti-reflexivity for $\mathop{\smile\hspace{-0.9em}\mid\hspace{0.5em}}$, then modular law in the real sort and GEI hold. Conversely if modular law in the real sort holds, then $(\mathcal{M}, \mathop{\smile\hspace{-0.9em}\mid\hspace{0.5em}}^a)$ is rosy and modular with GEI. In particular modular law in the real sort is equivalent to modular law with GEI. It is a similar result that one-basedness is equivalent to modularity with GEI.

● We say that $(\mathcal{M}^{\mathrm{eq}}, \mathop{\smile\hspace{-0.9em}\mid\hspace{0.5em}})$ has the real intersection property if $\bar{a} \mathop{\smile\hspace{-0.9em}\mid\hspace{0.5em}}_A B$ and $\bar{a} \mathop{\smile\hspace{-0.9em}\mid\hspace{0.5em}}_B A$ imply $\bar{a} \mathop{\smile\hspace{-0.9em}\mid\hspace{0.5em}}_{A \cap B} AB$ for any $\bar{a}, A = \mathrm{acl}(A), B = \mathrm{acl}(B) \subset \mathcal{M}$. Quasi-GEI implies the real intersection property for $\mathop{\smile\hspace{-0.9em}\mid\hspace{0.5em}}^a$. If modular law holds, then GEI holds iff the real intersection property for $\mathop{\smile\hspace{-0.9em}\mid\hspace{0.5em}}^a$ holds. Quasi-GEI does not hold for the theory T of infinitely many equivalence E-classes with each E-class is also infinite. Suppose that $E(a, b)$ and $a \neq b$. Then we have $\mathrm{acl}(a) = a, \mathrm{acl}(b) = b$. Then $e = (a)_E \in \mathrm{acl}^{\mathrm{eq}}(a) \cap \mathrm{acl}^{\mathrm{eq}}(b)$ and $e \notin \mathrm{acl}^{\mathrm{eq}}(\mathrm{acl}(a) \cap \mathrm{acl}(b)) = \mathrm{acl}^{\mathrm{eq}}(\emptyset)$. T does not have quasi-GEI. As T is stable and one-based, so modular law does not imply quasi-GEI. This is an example which has modular law without modular law in the real sort.

● We say that quasi-modular law holds if $\mathrm{acl}^{\mathrm{eq}}(\mathrm{acl}(D) \cap \mathrm{acl}(AC)) \subseteq \mathrm{acl}^{\mathrm{eq}}(\mathrm{acl}^{\mathrm{eq}}(D) \cap (\mathrm{acl}^{\mathrm{eq}}(A)\mathrm{acl}^{\mathrm{eq}}(C)))$ for any $A, C, D \subset \mathcal{M}$ with $C \subseteq D$. Then modular law implies quasi-modular law. Quasi-modular law with GEI implies modular law in the real sort.

Question 3.1.
(1) *Find non one-based* $(\mathcal{M}^{\mathrm{eq}}, \mathop{\smile\hspace{-0.9em}\mid\hspace{0.5em}})$ *with modular law and* $\mathop{\smile\hspace{-0.9em}\mid\hspace{0.5em}} \neq \mathop{\smile\hspace{-0.9em}\mid\hspace{0.5em}}^a$.
(2) *Find one-based* $(\mathcal{M}^{\mathrm{eq}}, \mathop{\smile\hspace{-0.9em}\mid\hspace{0.5em}})$ *without modular law.* $\mathop{\smile\hspace{-0.9em}\mid\hspace{0.5em}}$ *will not have base monotonicity and anti-reflexivity, so* $(\mathcal{M}^{\mathrm{eq}}, \mathop{\smile\hspace{-0.9em}\mid\hspace{0.5em}})$ *is not rosy.*
(3) *Is there any connection between quasi-modular law and CM-triviality?*

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