A WALK ON TYPE SPACE

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Abstract. We introduce a family of random walks on the type space associated to a first-order structure. We study some basic properties connected to stationary distributions as well as provide a model theoretic perspective on the simple symmetric random walk on the integers.

Methods from topological dynamics have been fundamental in shaping model theory and in particular, the model theory of groups [6, 10, 11]. Over the past couple of years, ideas from harmonic analysis and convolution dynamics have played an interesting role in this development [3, 4, 5]. From one perspective, some of these recent results can be reinterpreted as statements concerning a definable variant of convolution random walks on definable groups. Hence, we jump to two conclusions:

(1) There is a natural progression to generalize this study and explore other random processes on first-order structures, especially in relation to localized notions from neostability, e.g., generically stable types and fim measures.
(2) Ideas from the theory of random walks have the potential to positively impact model theory akin to the influence that topological dynamics has already had on the field.

This note is primarily focused on a simple implementation of random walks on type spaces. We remark that the more interesting variants, the *generic variants* of the random walks as well as connections to generic stability will be studied in upcoming research.

1. Preliminaries

Our notation is relatively standard. Throughout this note, $\mathcal{L}$ is a first-order language and M is an $\mathcal{L}$-structure. We let $\mathcal{L}_x(M)$ be the collection of $\mathcal{L}$-formulas with parameters from M and free variable(s) x. We will also work modulo logical equivalence, i.e., we identify two formulas in $\mathcal{L}_x(M)$ if and only if they define the same definable subset of M^x. We always identify definable sets with the formulas which define them.

Definition 1.1. In this note, a Keisler measure is a finitely additive probability measure on $\mathcal{L}_x(M)$. We use $\mathfrak{M}_x(M)$ to denote the collection of Keisler measures. We also recall that there is a one-to-one correspondence between Keisler measures and regular Borel probability measures on the corresponding type space, $S_x(M)$. In particular, if μ is a Keisler measure, then there exists a unique, regular, countably additive Borel probability measure $\tilde{\mu}$ on $S_x(M)$ such that for every $\varphi(x) \in \mathcal{L}_x(M)$,

$$\mu(\varphi(x)) = \tilde{\mu}([\varphi(x)]),$$

where $[\varphi(x)] = \{p \in S_x(M) : \varphi(x) \in p\}$. We regularly identify Keisler measures with their corresponding regular Borel probability measures without comment. We also remark that $\mathfrak{M}_x(M)$ is a compact Hausdorff space under the induced topology from $[0,1]^{\mathcal{L}_x(M)}$.

Definition 1.2. A Markov kernel is a Borel function from $\mathcal{P} : S_x(M) \to \mathfrak{M}_x(M)$.

A *random walk on type space* is given by two pieces of data: (1) An initial distribution, i.e., a fixed Keisler measure μ in $\mathfrak{M}_x(M)$, and (2) a Markov kernel $\mathcal{P} : S_x(M) \to \mathfrak{M}_x(M)$. One should think of the Markov kernel as a rule which does the following: *If I find myself standing on a type p, then the probability that my next step satisfies the formula $\theta(x)$ is precisely $\mathcal{P}(p)(\theta(x))$.* We now recall how to apply a Markov kernel to an arbitrary Keisler measure.

Proposition 1.3. *Fix a measure μ in $\mathfrak{M}_x(M)$ and a Markov kernel $\mathcal{P} : S_x(M) \to \mathfrak{M}_x(M)$. Then the Keisler measure $\mathcal{P}(\mu)$ given by*

$$\mathcal{P}(\mu)(\theta(x)) := \int_{q \in S_x(M)} \mathcal{P}_q(\theta(x)) d\mu(q),$$

is well-defined, where $\mathcal{P}_q(\theta(x)) = \mathcal{P}(q)(\theta(x))$.

Proof. Note that $q \to \mathcal{P}_q(\theta(x))$ is the composition of a Borel function and a continuous function, i.e., $q \to \mathcal{P}(q)$ and $\lambda \to \lambda(\theta(x))$. Hence for each formula the map is Borel and the integral is well-defined. It is straightfoward to check that $\mathcal{P}(\mu)$ is a Keisler measure via linearity of integration. □

Definition 1.4 (Law)**.** Formally speaking, if we are given a Markov kernel $\mathcal{P} : S_x(M) \to \mathfrak{M}_x(M)$ and an initial distribution μ, we can construct *the law associated to the Markov kernel and initial distribution*. This is the measure which *corresponds* to the random walk generated by these two pieces of data. While we are not focused on this object in this note, we describe its construction for the reader: The *law* is the measure $\mathbb{P}_\mu$ on $\prod_{i=0}^{\omega} S_x(M)$ where $\mathbb{P}_\mu := \prod_{i=0}^{\omega} \mu_i$ where $\mu_0 = \mu$ and $\mu_{i+1} = \mathcal{P}(\mu_i)$. The probability that an event occurs is precisely its $\mathbb{P}_\mu$-measure.

2. $\mathcal{L}$-chains and uniform $\mathcal{L}$-chains

In this section, we introduce a natural family of Markov chains on first-order structures and describe how to lift them to Markov kernels on the associated type space. There are two different kinds of Markov kernels which we will discuss: $\mathcal{L}$-chains and uniform $\mathcal{L}$-chains. $\mathcal{L}$-chains are Markov kernels which are identified by some simple abstract property while uniform $\mathcal{L}$-chains are explicitly encoded in the language and lifted from the structure to the type space.

2.1. Basic defintions.

Definition 2.1. Let M be an $\mathcal{L}$-structure. Then a Markov kernel $\mathcal{P} : S_x(M) \to \mathfrak{M}_x(M)$ is called an *$\mathcal{L}$-chain on $S_x(M)$* if

(1) For each $a \in M^x$, $\mathcal{P}(\text{tp}(a/M)) \in \text{conv}(M)$ where
$$\text{conv}(M^x) = \left\{ \sum_{i=1}^{n} r_i \delta_{\text{tp}(a_i/M)} : n \in \mathbb{N}_{\geq 1}, r_i \in [0,1], \sum_{i=1}^{n} r_i = 1, a_i \in M^x \right\}.$$
(2) $\mathcal{P}$ is continuous.

When M is unambiguous, we refer to $\mathcal{P}$ simply as an $\mathcal{L}$-chain.

Definition 2.2. Let M be an $\mathcal{L}$-structure. Then a Markov kernel $\mathcal{P} : S_x(M) \to \mathfrak{M}_x(M)$ is called a *uniform $\mathcal{L}$-chain on $S_x(M)$* if there exists a single formula $\varphi(x, y_1, ..., y_n) \in \mathcal{L}_{x,\bar{y}}(M)$ where $|x| = |y_i|$ for each $i \leq n$ with the following properties:

(1) For each $b \in M^x$, there exists a unique $\bar{a} \in M^y$ (up to permutation of indices) such that $M \models \varphi(b, \bar{a})$. In other words, if $M \models \varphi(b, \bar{a})$ and $M \models \varphi(b, \bar{c})$, then $(a_{\sigma(1)}, ..., a_{\sigma(n)}) = (c_1, ..., c_n)$ for some σ in $\text{Sym}(n)$.
(2) For each $q \in S_x(M)$, if $M \prec M'$, $b \in M'$ and $b \models q$, then
$$\mathcal{P}(q) = \text{Av}(a_1, ..., a_n)|_M.$$
where $M' \models \varphi(b, a_1, ..., a_n)$.

Moreover, we say that the formula $\varphi(x, \bar{y})$ witnesses the uniformity of $\mathcal{P}$ and will sometimes write $\mathcal{P}$ as $\mathcal{P}^\varphi$. When M is unambiguous, we refer to $\mathcal{P}$ simply as a uniform $\mathcal{L}$-chain.

Let us first establish that uniform $\mathcal{L}$-chains are indeed $\mathcal{L}$-chains.

Proposition 2.3. *Let $\mathcal{P}$ be a uniform $\mathcal{L}$-chain on $S_x(M)$. Then $\mathcal{P}$ is an $\mathcal{L}$-chain on $S_x(M)$.*

Proof. It suffices to show that the map $\mathcal{P} : S_x(M) \to \mathfrak{M}_x(M)$ is continuous. Let $\varphi(x, y_1, ..., y_n)$ be a formula witnessing the uniformity of $\mathcal{P}$. Let $(p_i)_{i \in I}$ be a net of types in $S_x(M)$ such that $\lim_{i \in I} p_i = p$. It suffices to show that for any $\theta(x) \in \mathcal{L}_x(M)$, $\lim_{i \in I} \mathcal{P}_{p_i}(\theta(x)) = \mathcal{P}_p(\theta(x))$. Let $M \prec M'$, $b \models p$, and $M' \models \varphi(b, a_1, ..., a_n)$. Then,
$$\mathcal{P}_p(\theta(x)) = \text{Av}(a_1, ..., a_n)(\theta(x)) = \frac{|\{i \leq n : M' \models \theta(a_i)\}|}{n} = k/n,$$
for some $k \leq n$. Now consider the formula,
$$\psi_\theta(x) := \forall y_1 ... y_n \left(\varphi(x, y_1, ..., y_n) \to \left(\bigvee_{\substack{A \subseteq n \\ |A| = k}} \bigwedge_{i \in A} \theta(y_i) \wedge \bigwedge_{i \notin A} \neg\theta(y_i) \right) \right).$$

Notice that $\psi_\theta(x) \in p$. Since $[\psi_\theta(x)]$ is a open set, there exists some p_j such that for all $l \geq j$, $p_l \in [\psi_\theta(x)]$. Now notice that if $l \geq j$, we have that $\mathcal{P}_{p_l}(\theta(x)) = k/n$ and so we conclude that $\lim_{i \in I} \mathcal{P}(p_i) = \mathcal{P}(p)$. □

The following is the quintessential example of how one usually comes across $\mathcal{L}$-chains *in the wild*. Given a formula $\varphi(x, y)$ which looks like an edge relation on a graph of bounded degree, one can construct a corresponding uniform $\mathcal{L}$-chain.

Example 2.4. Let M be an $\mathcal{L}$-structure and $\varphi(x, y)$ be an $\mathcal{L}_{xy}(M)$ formula such that $|x| = |y|$ and for every $b \in M^x$, there exists some natural number d such that $0 < |\varphi(b, M^y)| \leq d$. Then consider the Markov kernel given by

$$\mathcal{P}(p) = \frac{1}{|\varphi(b, M')|} \sum_{a \in \varphi(b, M')} \delta_{\text{tp}(a/M)},$$

where $M \prec M'$, $b \in M'$ and $b \models p$. By a straightforward coding argument using $d!$ many variables, one see that $\mathcal{P}$ is a uniform $\mathcal{L}$-chain.

The following is a non-example.

Example 2.5. Consider the structure $M = (\mathbb{N}; =)$ and the Markov kernel $\mathcal{P}$ given by $\text{tp}(n/M) \to \delta_{\text{tp}(n+1/M)}$, $p \to \delta_{\text{tp}(0/M)}$ where p is the unique non-realized type. Then $\mathcal{P}$ is not an $\mathcal{L}$-chain.

We now define two basic kinds of uniform $\mathcal{L}$-chains. The definition of a mutually algebraic formula originates from [8].

Definition 2.6. Suppose that $\mathcal{P}$ is a uniform $\mathcal{L}$-chain on $S_x(M)$ witnessed by $\varphi(x, \bar{y})$.

(1) We say that $\mathcal{P}$ is *mutually algebraic* if $\varphi(x, \bar{y})$ is mutually algebraic, i.e., there exists some natural number d such that for any $i \leq n$ and any $a_i \in M^{y_i}$, we have that

$$|\varphi(M^x, M^{y_1}, ..., a_i, ..., M^{y_n})| \leq d,$$

as well as for every $b \in M^x$, $|\varphi(b, M^{y_1}, ..., M^{y_n})| \leq d$. However, we remark that this second condition is already satisfied by the definition of a uniform $\mathcal{L}$-chain.

(2) We say that $\mathcal{P}$ is *strongly injective* if $\models \varphi(b, a_1, ..., a_n) \wedge \varphi(e, c_1, ..., c_n)$ and $\{a_1, ..., a_n\} \cap \{c_1, ..., c_n\} \neq \emptyset$, then $b = e$.

Proposition 2.7. *Let $\mathcal{P}$ be a uniform $\mathcal{L}$-chain on $S_x(M)$.*

(1) If f is an injective definable function from M^x to itself, then $\mathcal{P}^{f(x)=y}$ is strongly injective.

(2) If $\mathcal{P}$ is strongly injective, then $\mathcal{P}$ is mutually algebaic.

Proof. We prove the statements.

(1) Clear.

(2) Suppose that the uniformity of $\mathcal{P}$ is witnessed by $\varphi(x, y_1, ..., y_n)$ and $M \models \varphi(b, a_1, ..., a_n)$. Then for any $i \leq n$, notice

$$\begin{aligned} |\{\varphi(M^x, M^{y_1}, ..., a_i, ..., M^{y_n})\}| &= |\{\varphi(b, M^{y_1}, ..., a_i, ..., M^{y_n}\}| \\ &\leq |\text{Sym}(n-1)|. \end{aligned}$$

If no such b exists for a particular a_i, then the size of the solution set is 0 and hence also bounded. □

2.2. **Stationary distributions.** We recall the definition of stationary distributions and unique ergodicity in this context and prove a few general properties. In general, the definition of *uniquely ergodic* slightly varies from text to text, and so our definition is consistent with the one given in [1]. Stationary distributions are fixed points of Markov kernels and provide fundamental information about the associated random walk.

Definition 2.8. Suppose that $\mathcal{P} : S_x(M) \to \mathfrak{M}_x(M)$ is a Markov kernel. Then a stationary distribution for $\mathcal{P}$ is a Keisler measure μ in $\mathfrak{M}_x(M)$ such that $\mathcal{P}(\mu) = \mu$. Moreover, we say that $\mathcal{P}$ is *uniquely ergodic* if it admits a unique stationary distribution.

Remark 2.9. It is a direct consequence of Markov-Kakutani fixed point theorem that any $\mathcal{L}$-chain admits a stationary distribution. In particular, the map $\mathcal{P} : \mathfrak{M}_x(M) \to \mathfrak{M}_x(M)$ is continuous and affine and thus admits a fixed point.

Furthermore, if $\mathcal{P}$ is a uniform $\mathcal{L}$-chain witnessed by a formula $\varphi(x,\bar{y})$ which is mutually algebraic, then $\mathcal{P}$ admits a *non-realized* stationary distribution, i.e., $\mathcal{P}(\mu)=\mu$ and $\text{supp}(\mu)\cap\{\text{tp}(a/M): a\in M\}=\emptyset$. This again follows directly from the Markov-Kakutani fixed point theorem, but now applied to $\mathcal{P}|_X : X\to X$ where $X=\mathfrak{M}_x(M)\backslash\bigcup_{a\in M}\{\mu:\mu(x=a)>0\}$.

Lemma 2.10. *Suppose that $\mathcal{P}$ is a uniform $\mathcal{L}$-chain on $S_x(M)$ and ν is $\mathcal{P}$-stationary. Suppose that uniformity is witnessed by $\varphi(x,\bar{y})$. For any definable set $D(x)$, we let*

$$D_\varphi(x) := \exists w\exists y_1,...,y_n\left(D(w)\wedge\varphi(w,y_1,...,y_n)\wedge\bigvee_{i=1}^{n} x=y_i\right).$$

Then,

1. *For any $D(x)$, $\nu(D(x))\le\nu(D_\varphi(x))$.*
2. *If $\mathcal{P}$ is strongly injective, then for any $D(x)$, $\nu(D(x))=\nu(D_\varphi(x))$.*

Proof. We first prove claim (1). Notice

$$\begin{aligned}\nu(D_\varphi(x))=\mathcal{P}(\nu)(D_\varphi(x)) &= \int_{S_x(M)}\mathcal{P}_p(D_\varphi(x))d\nu\\ &= \int_{[D(x)]}\mathcal{P}_p(D_\varphi(x))d\nu+\int_{[\neg D(x)]}\mathcal{P}_p(D_\varphi(x))d\nu\\ &\overset{(*)}{=} \int_{[D(x)]}\mathbf{1}d\nu+\int_{[\neg D(x)]}\mathcal{P}_p(D_\varphi(x))d\nu\\ &= \nu(D(x))+\int_{[\neg D(x)]}\mathcal{P}_p(D_\varphi(x))d\nu\ge\nu(D(x)).\end{aligned}$$

We justify equation $(*)$. Suppose that $p\in[D(x)]$. Then if $b\models p$, and $M\prec M'\models\varphi(b,a_1,...,a_n)$, then $M'\models D_\varphi(a_i)$. Therefore, for any $p\in[D(x)]$,

$$\mathcal{P}_p(D_\varphi(x))=\text{Av}(a_1,...,a_n)(D_\varphi(x))=1.$$

To prove claim (2), notice that if $\mathcal{P}$ is strongly injective, then for every $p\in[\neg D(x)]$, $\mathcal{P}_p(D_\varphi(x))=0$. Hence $\ge$ is replaced with $=$ in the final term. □

The next proposition shows that if $\mathcal{P}$ is an uniquely ergodic uniform $\mathcal{L}$-chain, and μ is the unique stationary distribution, then the sets of μ-positive measure are in some sense generic, i.e., a variant of finitely many translations cover the space.

Proposition 2.11. *Suppose that $\mathcal{P}$ is an uniquely ergodic uniform $\mathcal{L}$-chain, witnessed by the formula $\varphi(x,\bar{y})$ where $\bar{y}=(y_1,...,y_n)$ and the stationary distribution μ. Then for any definable set $D(x)$, we define $D^{-m}(x)$ where*

$$D^{-1}(x):=\exists\bar{y}(\varphi(x,\bar{y})\wedge\bigvee_{i=1}^{n}D(y_i))\ \text{and}\ D^{-m-1}(x)=(D^{-m})^{-1}(x).$$

We let $D^0(x)=D(x)$. If $\mu(D(x))>0$ then there exists some natural number d such that $\bigcup_{k=0}^{d}[D^{-k}(x)]=S_x(M)$.

Proof. If $\bigcup_{k=0}^{\infty}[D^{-k}(x)]=S_x(M)$, then the statement holds from compactness. If not, then $K:=\bigcap_{k=0}^{\infty}[\neg D^{-k}(x)]$ is a non-empty closed subset of $S_x(M)$. If $p\in K$, then $\text{supp}(\mathcal{P}_p)\subseteq K$ and so $\mathcal{P}|_K : K\to K$. Indeed, suppose there exists $q\in\text{supp}(\mathcal{P}_p)\cap K^c$. Then $D^{-k}(x)\in q$ for some $k\ge 0$. Notice that

$$M\models\forall x\forall\bar{y}([\varphi(x,\bar{y})\wedge\bigvee_{i=1}^{n}D^{-k}(y_i)]\to D^{-k-1}(x)).$$

Thus, $D^{-k-1}(x)\in p$ – a contradiction. Therefore $\mathcal{P}|_K$ induces a continuous affine map from $\mathcal{M}(K)\to\mathcal{M}(K)$. By Markov-Kakutani, we have a stationary distribution ν which concentrates on K. Notice that $K\cap[D(x)]=\emptyset$ and so $\nu\neq\mu$ – this contradiction the assumption that $\mathcal{P}$ is uniquely ergodic. □

Finally, we make a comment about the totally transcendental setting. For simplicity, we restrict to a countable language. We use $RM(-)$ to denote the Morley Rank.

Fact 2.12. *Suppose that T is totally transcendental, $M \prec M' \models T$. Let $a, b \in M'$, and suppose that $b \in \text{acl}(Ma)$. Then the Morley rank of $\text{tp}(b/M)$ is less than or equal to the Morley rank of $\text{acl}(a/M)$.*

Proposition 2.13. *Assume that T it totally transcendental with rank α and $M \prec M' \models T$. Suppose that $\mathcal{P}$ is a mutually algebraic uniform $\mathcal{L}$-chain witnessed by $\varphi(x, \bar{y})$. Then there exists a stationary distribution ν such that for every $q \in \text{supp}(\nu)$, $RM(q) = \alpha$.*

Proof. We first show that if $RM(p) = \alpha$, then $\mathcal{P}(p)$ concentrates on types of maximal rank. Suppose that $RM(\theta(x)) < \alpha$ and let $b \models p$ and $M' \models \varphi(b, a_1, ..., a_n)$. Then

$$\mathcal{P}(p)(\theta(x)) = \text{Av}(a_1, ..., a_n)(\theta(x)) = \sum_{i=1}^{n} \delta_{\text{tp}(a_i/M)}(\theta(x)) = 0.$$

Indeed, notice that for each $i \leq n$, we have that $b \in \text{acl}(Ma_i)$ since $\varphi(x, \bar{y})$ is mutually algebraic, i.e., consider $\exists \bar{y}(\varphi(x, y_1, ..., a_i, ..., y_n))$. By Fact 2.12, $RM(\text{tp}(a_i/M)) \geq RM(\text{tp}(b/M)) = \alpha$ and so $\theta(x) \notin \text{tp}(a_i/M)$ which implies that the sum is 0. Now, the collection of types of maximal rank, say gen, is a compact subset (even finite) of $S_x(M)$. We can consider the Markov kernel $\mathcal{Q} : \text{gen} \to \text{gen}$ via $\mathcal{Q}(q) = \mathcal{P}|_{\text{gen}}(q)$. The corresponding extension of $\mathcal{Q}$ to $\mathcal{M}(\text{gen}) \to \mathcal{M}(\text{gen})$ is continuous and affine and so by the Markov-Kakutani fixed point theorem, it admits a stationary distribution, say ν. The measure ν is also a stationary for $\mathcal{P}$ since

$$\begin{aligned}\mathcal{P}(\nu)(\theta(x)) &= \int_{q \in S_x(M)} \mathcal{P}(q)(\theta(x)) d\nu(q) = \int_{q \in \text{supp}(\nu)} \mathcal{P}(p)(\theta(x)) d\nu(q) \\ &= \int_{q \in \text{supp}(\nu)} \mathcal{Q}(q)(\theta(x)) d\nu(q) = \mathcal{Q}(\nu)(\theta(x)) = \nu(\theta(x)).\end{aligned}$$ □

3. Simple symmetric random walk on the integers

In this section, we consider the simple symmetric random walk on the integers. This random walk can be encoded many different ways in the model theoretic context and may have different properties depending on the encoding. We provide two examples where the encoding is uniquely ergodic (one which is interesting, one which is not) and an example where the encoding is not uniquely ergodic. In this section, the $\mathcal{L}$-chains are described using the trick from Example 2.4. We begin with a *trivial* encoding.

Lemma 3.1. *Suppose that M is strongly minimal and $\mathcal{P}$ is an $\mathcal{L}$-chain on $S_x(M)$ such that the discrete random walk $(M, \mathcal{P}|_M)$ is irreducible but not positive recurrent. Then the Markov kernel $\mathcal{P}$ admits a unique stationary distribution, namely the Dirac measure concentrating on the unique non-realized type.*

Proof. Since the walk is not positive recurrent, there does not exist a stationary distribution concentrating on realized points (See e.g. [9, Theorem 6.4.3]). By Proposition 2.13 (or Remark 2.9), there exists a stationary distribution concentrating on $\{p\}$ where p is the unique non-realized type. Hence δ_p is a stationary distribution. Finally, suppose that $\mu = r\mu_1 + s\mu_2$ where $r + s = 1$, $r, s \in (0, 1)$, $\mu_0 = \delta_p$ and μ_1 concentrates on M. Then

$$\mathcal{P}(r\mu_0 + s\mu_1) = r\mu_0 + s\mathcal{P}(\mu_1) \Longrightarrow \mu_1 = \mathcal{P}(\mu_1).$$

But since $(M, P|_M)$ is not positive recurrent, μ_1 cannot be stationary, and hence μ_1 does not exist. □

Proposition 3.2. *Consider the structure $M = (\mathbb{Z}; S)$ and the formula $\varphi(x, y) := (x = S(y) \vee S(y) = x)$. Then the associated uniform $\mathcal{L}$-chain on $S_x(M)$ is uniquely ergodic.*

Proof. The walk $(M, \mathcal{P}|_M)$ is the standard simple random walk on the integers. It is irreducible, recurrent (due to Póya in 1921), but not positive recurrent. Since the structure is also strongly minimal, Lemma 3.1 applies. □

We now consider a more interesting encoding which takes the group structure of $\mathbb{Z}$ into account. Again, our Markov kernel is uniquely ergodic, but now the unique stationary distribution is non-trivial. The following elementary fact about harmonic functions on $\mathbb{Z}$ is used. We remark that this is an instance of classical non-trivial theorems concerning random walks on the integers/abelian groups [2, 7].

Fact 3.3. *Consider the measure* $\lambda = \frac{1}{2}\delta_1 + \frac{1}{2}\delta_{-1}$ *on* $\mathbb{Z}$. *There are no non-constant bounded* λ*-harmonic functions. In other words, there are no non-constant bounded functions* $f : \mathbb{Z} \to \mathbb{R}$ *such that for any* $k \in \mathbb{Z}$,

$$f(k) = \int_{t \in \mathbb{Z}} f(k+t) d\lambda.$$

Notice that term on the right hand side nicely reduces to $\frac{1}{2}f(k-1) + \frac{1}{2}f(k+1)$ *because of the definition of* λ.

Proposition 3.4. *Consider the structure* $M = (\mathbb{Z}; +, 0, 1)$ *and the formula* $\varphi(x, y) := (x = y + 1 \vee y = x + 1)$. *Then the associated uniform* $\mathcal{L}$*-chain* $\mathcal{P}$ *on* $S_x(M)$ *is uniquely ergodic.*

Proof. We recall that every definable subset of $\mathbb{Z}$ is eventually periodic, i.e., if $A \subseteq \mathbb{Z}$ is definable, then A is a disjoint union of cosets of subgroups of $\mathbb{Z}$, plus or minus finitely many points. We let the formula,

$$D_n(x + k) := \exists y(\underbrace{y + ... + y}_{n-times} = x - k).$$

We claim that there exists a unique measure $\mu \in \mathfrak{M}_x(M)$ such that for every $n \geq 2$ and $k \geq 0$,

$$\mu(D_n(x + k)) = \frac{1}{n}.$$

Notice that the condition above forces the measure of any finite subset of $\mathbb{Z}$ to be 0. We first prove that μ is $\mathcal{P}$-stationary. By above, it suffices to show that $(\mathcal{P}\mu)(D_n(x + k)) = \mu(D_n(x + k))$ for all $n \geq 2$, $k \geq 0$. Notice

$$\begin{aligned}(\mathcal{P}\mu)(D_n(x + k)) &= \int_{p \in S_x(M)} \mathcal{P}_p(D_n(x + k)) d\mu \\ &= \int_{p \in S_x(M)} \frac{1}{2}\chi_{D_n(x+k-1)} + \frac{1}{2}\chi_{D_n(x+k+1)} d\mu \\ &= \frac{1}{2}\mu(D_n(x + k - 1)) + \frac{1}{2}\mu(D_n(x + k + 1)) \\ &= \frac{1}{n}.\end{aligned}$$

By uniqueness, $\mathcal{P}(\mu) = \mu$. We now show that μ is the unique stationary measure. Suppose that ν is another. Since $\nu \neq \mu$, there exists some $n_* \geq 2$ and $k_* \geq 0$ such that $\nu(D_{n_*}(x + k_*)) \neq 1/n_*$. We claim that the function $f_\nu : \mathbb{Z} \to \mathbb{R}$ via $f_\nu(k) = \nu(D_{n_*}(x + k))$ must be non-constant bounded λ-harmonic function where $\lambda = \frac{1}{2}\delta_1 + \frac{1}{2}\delta_{-1}$. This contradict Fact 3.3. ☐

Finally, we observe that if we add all subsets of $\mathbb{Z}$, the resulting Markov kernel is no longer uniquely ergodic. There are many distinct stationary distributions.

Proposition 3.5. *Consider the language* $\mathcal{L} = \{+, 0, 1, \{A(x)\}_{A \in \mathcal{P}(\mathbb{Z})}\}$ *and let* M *be the natural interpretation of the symbols on* $\mathbb{Z}$. *Consider the sequence of measures* ν_n *where*

$$\nu_n(A(x)) = \frac{|\{0 \leq j \leq n : \models (A(j))\}|}{n}.$$

Given an ultrafilter D *on* ω, *we let* $\nu_D = \lim_D \nu_n$. *We claim that* ν_D *is* $\mathcal{P}$*-stationary where* $\mathcal{P} = \mathcal{P}^\varphi$ *and* $\varphi(x, y) := (x = y + 1 \vee y = x + 1)$. *Using this construction, and choosing ultrafilters appropriately, one can derive that* $\mathcal{P}$ *is not uniquely ergodic.*

Proof. Notice

$$\begin{aligned}\mathcal{P}(\nu_D)(A(x)) = \int_{q \in S_x(M)} \mathcal{P}(p)(A(x)) d\nu_D &= \frac{1}{2}\nu_D(A(x + 1)) + \frac{1}{2}\nu_D(A(x - 1)) \\ &= \frac{1}{2}\nu_D(A(x)) + \frac{1}{2}\nu_D(A(x)) = \nu_D(A(x)).\end{aligned}$$

☐

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