Mean Stability and Bifurcation in Random Dynamical Systems of Polynomial Automorphisms on $\mathbb{C}^2$

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Abstract

We consider random dynamical systems of polynomial automorphisms (complex generalized Hénon maps and their conjugate maps) of $\mathbb{C}^2$. We show that a generic random dynamical system of polynomial automorphisms has "mean stablity" on $\mathbb{C}^2$. Further, we show that if a system has mean stability, then (1) for each $z \in \mathbb{C}^2$ and for almost every sequence $\gamma = (\gamma_n)_{n=1}^{\infty}$ of maps, the maximal Lyapunov exponent of γ at z is negative, (2) there are only finitely many minimal sets of the system, (3) each minimal set is attracting, (4) for each $z \in \mathbb{C}^2$ and for almost every sequence γ of maps, the orbit $\{\gamma_n \cdots \gamma_1(z)\}_{n=1}^{\infty}$ tends to one of the minimal sets of the system. Note that none of (1)–(4) can hold for any deterministic iteration dynamical system of a single complex generalized Hénon map. To show the density of mean stable systems, we consider the bifurcation and stability of families of random dynamical systems of polynomial automorphisms of $\mathbb{C}^2$. We observe many new phenomena in random dynamical systems of polynomial automorphisms of $\mathbb{C}^2$ and observe the mechanisms. We provide new strategies and methods to study higher-dimensional random holomorphic dynamical systems.

The results in this presentation are included in [S24].

Motivation.

- Nature has a lot of random (noise) terms. Thus it is natural and important to consider random dynamical systems.

- Holomorphic dynamical systems have been deeply investigated. The study of them helps us to investigate real dynamical systems.

- Combining the above two ideas, we consider random holomorphic dynamical systems.

- We want to find new phenomena (so called randomness-induced phenomena) in random dynamical systems which cannot hold in deterministic iteration dynamical systems of single maps.

- Other motivations: Random Newton's method (in which we can find roots of polynomials more easily than the deterministic methods, see S., 2021 ([S21], Comm. Math. Phys.)). The action of holomorphic automorphisms on complex manifolds. The action of mapping class groups of the Riemann surfaces on the character varieties, etc.

Definition 1.

(1) Let $\mathbb{C}^2$ be the 2-dimensional complex Euclidean space. Let $f : \mathbb{C}^2 \to \mathbb{C}^2$ be a polynomial map, i.e., if we write $f(x, y) = (g(x, y), h(x, y))$, then $g(x, y)$ and $h(x, y)$ are polynomials of (x, y). We say that f is a polynomial automorphism on $\mathbb{C}^2$ if f is a holomorphic automorphism on $\mathbb{C}^2$. Let $\mathrm{PA}(\mathbb{C}^2)$ be the space of all polynomial automorphisms on $\mathbb{C}^2$.

Remark: if $f \in \mathrm{PA}(\mathbb{C}^2)$ then $f^{-1} \in \mathrm{PA}(\mathbb{C}^2)$.

Remark: [MNTT00]. If $f \in \mathrm{PA}(\mathbb{C}^2)$ then f is conjugate by an element $g \in \mathrm{PA}(\mathbb{C}^2)$ to one of the following maps:

(a) an affine map $(x, y) \mapsto (ax + by + c, a'x + b'y + c'), ab' - a'b \neq 0$.

(b) an elementary map $(x, y) \mapsto (ax + b, sy + p(x)), as \neq 0$, where $p(x)$ is a polynomial of x.

(c) a finite composition of some generalized Hénon maps

$$(x, y) \mapsto (y, p(y) - \delta x), \delta \neq 0,$$

where $p(y)$ is a polynomial of y with $\deg(p) \geq 2$.

(2) Let X^+ be the space of all maps $f : \mathbb{C}^2 \to \mathbb{C}^2$ of the form

$$f(x, y) = (y + \alpha, p(y) - \delta x)$$

where $\alpha \in \mathbb{C}, \delta \in \mathbb{C} \setminus \{0\}$, and $p(y)$ is a polynomial of y with $\deg(p) \geq 2$. Note that $X^+ \subset \mathrm{PA}(\mathbb{C}^2)$. We endow X^+ with the topology such that a sequence $\{f_j(x, y) = (y + \alpha_j, p_j(y) - \delta_j x)\}_{j=1}^{\infty}$ in X^+ converges to an element $f(x, y) = (y + \alpha, p(y) - \delta x)$ in X^+ if and only if

(i) $\alpha_j \to \alpha$ $(j \to \infty)$,

(ii) $\delta_j \to \delta$ $(j \to \infty)$,

(iii) $\deg(p_j) = \deg(p)$ for each large number j and

(iv) the coefficients of p_j converge to the coefficients of p appropriately as $j \to \infty$.

Also, we set $X^- := \{f^{-1} \in \mathrm{PA}(\mathbb{C}^2) \mid f \in X^+\}$ endowed with the topology similar to that of X^+. Note that $X^- \cong X^+$ via $f^{-1} \leftrightarrow f$.

Remark. (i) If $f \in X^{\pm}$ then f is conjugate to a generalized Hénon map by an element $g \in \mathrm{PA}(\mathbb{C}^2)$. (ii) If $f \in X^+$ (resp. X^-) then f can be extended to a holomorphic self-map on $\mathbb{P}^2 \setminus \{[1:0:0]\}$ (resp. $\mathbb{P}^2 \setminus \{[0:1:0]\}$). (iii) For each $f \in X^+$ (resp. X^-), the point $[0:1:0]$ (resp. $[1:0:0]$) is an attracting fixed point of f.

(3) Let $\mathfrak{M}_1(X^{\pm})$ be the space of all Borel probabiliy measures on $X^{\pm}$. Also, we set

$$\mathfrak{M}_{1,c}(X^{\pm}) := \{\tau \in \mathfrak{M}_1(X^{\pm}) \mid \ \mathrm{supp}\, \tau \text{ is a compact subset of } X^{\pm}\}.$$

We endow $\mathfrak{M}_{1,c}(X^{\pm})$ with a topology $\mathcal{O}$ which satisfies that

$\tau_n \to \tau$ as $n \to \infty$ if and only if

(a) for each bounded continuous function $\varphi : X^{\pm} \to \mathbb{C}$, we have $\int \varphi\, d\tau_n \to \int \varphi\, d\tau$ as $n \to \infty$, and

(b) $\mathrm{supp}\, \tau_n \to \mathrm{supp}\, \tau$ as $n \to \infty$ with respect to the Hausdorff metric in the space of all non-empty compact subsets of $X^{\pm}$.

For each $\tau \in \mathfrak{M}_{1,c}(X^+)$, we consider i.i.d. random dynamical system on $\mathbb{P}^2 \setminus \{[1:0:0]\}$ such that at every step we choose a map $f \in X^+$ according to τ. This defines a Markov process whose state space is $\mathbb{P}^2 \setminus \{[1:0:0]\}$ and whose transition probability $p(z, A)$ from a point $z \in \mathbb{P}^2 \setminus \{[1:0:0]\}$ to a Borel subset A of $\mathbb{P}^2 \setminus \{[1:0:0]\}$ satisfies

$$p(z, A) = \tau(\{f \in X^+ \mid f(z) \in A\}).$$

(4) For each $\tau \in \mathfrak{M}_{1,c}(X^{\pm})$, let

$$G_\tau := \{\gamma_n \circ \cdots \circ \gamma_1 \mid n \in \mathbb{N}, \gamma_j \in \mathrm{supp}\, \tau (\forall j)\}.$$

This is a semigroup with the semigroup operation being the functional composition. (It is important to study the dynamics of G_τ.)

(5) Let Λ be an open subset of $\mathbb{P}^2$. We say that an element $\tau \in \mathfrak{M}_{1,c}(X^{\pm})$ is mean stable on Λ if each $f \in \mathrm{supp}\, \tau$ is defined on Λ and $f(\Lambda) \subset \Lambda$ and there exist an $n \in \mathbb{N}$, an $m \in \mathbb{N}$, non-empty open subsets $U_1, \ldots, U_m$ of Λ, a non-empty compact subset K of $\cup_{j=1}^m U_j$, and a constant c with $0 < c < 1$ such that the following (a) and (b) hold.

(a) For each $(\gamma_1, \ldots, \gamma_n) \in (\mathrm{supp}\, \tau)^n$, we have

$$\gamma_n \circ \cdots \circ \gamma_1(\cup_{j=1}^m U_j) \subset K.$$

Moreover, for each $j = 1, \ldots, m$, for all $x, y \in U_j$ and for each $(\gamma_1, \ldots, \gamma_n) \in (\mathrm{supp}\, \tau)^n$, we have

$$d(\gamma_n \circ \cdots \circ \gamma_1(x), \gamma_n \circ \cdots \circ \gamma_1(y)) \leq cd(x, y),$$

where d denotes the distance induced by the Fubini-Study metric on $\mathbb{P}^2$.

(b) For each $z \in \Lambda$, there exists an element $f_z \in G_\tau$ such that $f_z(z) \in \cup_{j=1}^m U_j$.

(6) Let $\mathcal{MS}$ be the set of all $\tau \in \mathfrak{M}_{1,c}(X^+)$ satisfying that

(i) τ is mean stable on $\mathbb{P}^2 \setminus \{[1:0:0]\}$ and

(ii) τ^{-1} is mean stable on $\mathbb{P}^2 \setminus \{[0:1:0]\}$, where τ^{-1} is the element of $\mathfrak{M}_{1,c}(X^-)$ such that $\tau^{-1}(A) = \tau(\{f \in X^+ \mid f^{-1} \in A\})$ for each Borel subset A of X^-.

Remark 2. $\mathcal{MS}$ is open in $(\mathfrak{M}_{1,c}(X^+), \mathcal{O})$.

Theorem 3 ([S24])**.** *$\mathcal{MS}$ is open and dense in $\mathfrak{M}_{1,c}(X^+)$.*
Moreover, for each $\tau \in \mathcal{MS}$, we have all of the following (1)–(5).

(1) *There exists a constant c_τ with $c_\tau < 0$ such that the following holds.*

– *For each $z \in \mathbb{P}^2 \setminus \{[1:0:0]\}$, there exists a Borel subset $B^+_{\tau,z}$ of $(X^+)^{\mathbb{Z}}$ with $(\otimes_{n=-\infty}^{\infty} \tau)(B^+_{\tau,z}) = 1$ such that for each $\gamma = (\gamma_j)_{j \in \mathbb{Z}} \in B^+_{\tau,z}$, we have*

$$\limsup_{n \to \infty} \frac{1}{n} \log \|D(\gamma_{n-1} \circ \cdots \circ \gamma_0)_z\| \leq c_\tau < 0.$$

Also, for each $z \in \mathbb{P}^2 \setminus \{[0:1:0]\}$, there exists a Borel subset $B^-_{\tau,z}$ of $(X^+)^{\mathbb{Z}}$ with $(\otimes_{n=-\infty}^{\infty} \tau)(B^-_{\tau,z}) = 1$ such that for each $\gamma = (\gamma_j)_{j \in \mathbb{Z}} \in B^-_{\tau,z}$, we have

$$\limsup_{n \to \infty} \frac{1}{n} \log \|D(\gamma_{-n}^{-1} \circ \cdots \circ \gamma_{-1}^{-1})_z\| \leq c_\tau < 0.$$

Here, for each rational map f on $\mathbb{P}^2$ and for each $z \in \mathbb{P}^2$ where f is defined, we denote by $\|Df_z\|$ the norm of the differential of f at z w.r.t. the Fubiny-Study metric in $\mathbb{P}^2$.

(2) *For each $z \in \mathbb{P}^2 \setminus \{[1:0:0]\}$, there exists a Borel subset $C^+_{\tau,z}$ of $(X^+)^{\mathbb{Z}}$ with $(\otimes_{n=-\infty}^{\infty} \tau)(C^+_{\tau,z}) = 1$ such that for each $\gamma = (\gamma_j)_{j \in \mathbb{Z}} \in C^+_{\tau,z}$, there exists a number $r^+ = r^+(\tau, z, \gamma) > 0$ satifying that*

$$diam(\gamma_{n-1} \circ \cdots \circ \gamma_0(B(z, r^+))) \to 0 \text{ as } n \to \infty$$

exponentially fast, and for each $z \in \mathbb{P}^2 \setminus \{[0:1:0]\}$, there exists a Borel subset $C^-_{\tau,z}$ of $(X^+)^{\mathbb{Z}}$ with $(\otimes_{n=-\infty}^{\infty} \tau)(C^-_{\tau,z}) = 1$ such that for each $\gamma = (\gamma_j)_{j \in \mathbb{Z}} \in C^-_{\tau,z}$, there exists a number $r^- = r^-(\tau, z, \gamma) > 0$ satifying that

$$diam(\gamma_{-n}^{-1} \circ \cdots \circ \gamma_{-1}^{-1}(B(z, r^-))) \to 0 \text{ as } n \to \infty$$

exponentially fast, where $B(z, r)$ denotes the ball with center z and radius r with respect to the distance d induced by the Fubini-Study metric on $\mathbb{P}^2$, and for each subset A of $\mathbb{P}^2$, we set $diam A := \sup_{x,y \in A} d(x, y)$.

(3)

(a) *Let* $\mathrm{Min}(\tau)$ *be the set of all minimal sets of* τ *in* $\mathbb{P}^2 \setminus \{[1:0:0]\}$. *Then*

$$1 \leq \sharp \mathrm{Min}(\tau) < \infty.$$

Here, a non-empty compact subset L *of* $\mathbb{P}^2 \setminus \{[1:0:0]\}$ *is said to be a minimal set of* τ *if* $L = \overline{\cup_{h \in G_\tau} \{h(z)\}}$ *for each* $z \in L$.

(b) *For* $\forall z \in \mathbb{P}^2 \setminus \{[1:0:0]\}$, *for* $(\otimes_{n=-\infty}^{\infty} \tau)$*-a.e.* $(\gamma_j)_{j \in \mathbb{Z}} \in (X^+)^{\mathbb{Z}}$, *we have* $d(\gamma_{n-1} \circ \cdots \circ \gamma_0(z), \cup_{L \in \mathrm{Min}(\tau)} L) \to 0$ *as* $n \to \infty$.

(4) *For each* $L \in \mathrm{Min}(\tau)$, *let* $T_{L,\tau} : \mathbb{P}^2 \setminus \{[1:0:0]\} \to [0,1]$ *be the function of probability of tending to* L, *i.e., for each* $z \in \mathbb{P}^2 \setminus \{[1:0:0]\}$, *we set*

$$T_{L,\tau}(z) := (\otimes_{n=1}^{\infty} \tau)(\{(\gamma_j)_{j \in \mathbb{N}} \in (X^+)^{\mathbb{N}} \mid d(\gamma_n \cdots \gamma_1(z), L) \to 0 \ (n \to \infty)\}).$$

Then, $T_{L,\tau}$ *is locally Hölder continuous on* $\mathbb{P}^2 \setminus \{[1:0:0]\}$. *Also,* $T_{L,\tau}$ *is constant on each connected component of* $F(G_\tau)$. *Here,* $F(G_\tau)$ *denotes the set of points* $z \in \mathbb{P}^2 \setminus \{[1:0:0]\}$ *for which there exists a neighborhood* U *of* z *such that* $\{h : U \to \mathbb{P}^2 \setminus \{[1:0:0]\}\}_{h \in G_\tau}$ *is equicontinuous on* U. *(We call* $F(G_\tau)$ *the Fatou set of semigroup* G_τ*.)*

(5) *For each* $\gamma = (\gamma_j)_{j \in \mathbb{Z}} \in (X^+)^{\mathbb{Z}}$, *let* F_γ^+ *be the set of elements*

$$z \in \mathbb{P}^2 \setminus \{[1:0:0]\}$$

for which there exists a neighborhood U *of* z *in* $\mathbb{P}^2 \setminus \{[1:0:0]\}$ *such that* $\{\gamma_n \circ \cdots \circ \gamma_0 : U \to \mathbb{P}^2\}_{n=0}^{\infty}$ *is equicontinuous on* U. *Similarly, let* F_γ^- *be the set of elements*

$$z \in \mathbb{P}^2 \setminus \{[0:1:0]\}$$

for which there exists a neighborhood U *of* z *in* $\mathbb{P}^2 \setminus \{[0:1:0]\}$ *such that* $\{\gamma_{-n}^{-1} \circ \cdots \circ \gamma_{-1}^{-1} : U \to \mathbb{P}^2\}_{n=1}^{\infty}$ *is equicontinuous on* U. *Also, let*

$$J_\gamma^{\pm} := \mathbb{P}^2 \setminus F_\gamma^{\pm}.$$

Then there exists a Borel subset D_τ *of* $(X^+)^{\mathbb{Z}}$ *with* $(\otimes_{n=-\infty}^{\infty} \tau)(D_\tau) = 1$ *such that for each* $\gamma \in D_\tau$, *we have*

$$\mathrm{Leb}_4(J_\gamma^{\pm}) = 0,$$

where Leb_4 *denotes the* 4*-dimensional Lebesgue measure on* $\mathbb{P}^2$.

Remark 4. None of statements (1)(2)(3)(4) in Theorem 3 can hold for deterministic dynamics of a single $f \in X^{\pm}$.

To prove the density of $\mathcal{MS}$ in $\mathfrak{M}_{1,c}(X^+)$ in Theorem 3, we need the following.

Theorem 5 ([S24]). *Let* $\{\tau_t\}_{t\in[0,1]}$ *be a family of elements of* $\mathfrak{M}_{1,c}(X^+)$ *such that all of the following* (1)(2)(3) *hold.*

(1) $t \in [0,1] \mapsto \tau_t \in \mathfrak{M}_{1,c}(X^+)$ *is continuous w.r.t the topology* $\mathcal{O}$.

(2) *If* $t_1, t_2 \in [0,1]$ *and* $t_1 < t_2$, *then* $\mathrm{supp}\,\tau_{t_1} \subset \mathrm{int}(\mathrm{supp}\,\tau_{t_2})$. *Here, int denotes the set of interior points with respect to the topology in* X^+.

(3) $\mathrm{int}(\mathrm{supp}\,\tau_0) \neq \emptyset$.

Let $B := \{t \in [0,1] \mid \tau_t$ *is not mean stable on* $\mathbb{P}^2 \setminus \{[1:0:0]\}\}$. *Then*

$$\sharp B \leq \sharp \mathrm{Min}(\tau_0) - 1 < \infty$$

and $C := \{t \in [0,1] \mid s \mapsto \sharp\mathrm{Min}(\tau_s)$ *is constant in a neighborhood of* $t\}$ *satisfies*

$$C = [0,1] \setminus B.$$

We give the rough ideas of the proof of Theorem 5 as follows.

(i) Under the assumptions of Theorem 5, by Zorn's lemma, we can show that if $t_1, t_2 \in [0,1], t_1 < t_2$ then $1 \leq \sharp\mathrm{Min}(\tau_{t_2}) \leq \sharp\mathrm{Min}(\tau_{t_1})$.

(ii) Moreover, we can easily show that $\sharp\mathrm{Min}(\tau_0) < \infty$.

(iii) It follows that $\sharp([0,1] \setminus C) \leq \sharp\mathrm{Min}(\tau_0) - 1 < \infty$.

(iv) (Key) We can show that if $t_0 \in C$, then any $L \in \mathrm{Min}(\tau_{t_0})$ is "attracting" for τ_{t_0}, by using the "Carathéodory distance".
It follows that if $t_0 \in C$ then τ_{t_0} is mean stable on $\mathbb{P}^2 \setminus \{[0:1:0]\}$.

(v) By (iv), $C \subset [0,1] \setminus B$. Also it is easy to see $[0,1] \setminus B \subset C$.
Thus $C = [0,1] \setminus B$.

(vi) Combining (iii) and (v), we obtain $\sharp B \leq \sharp\mathrm{Min}(\tau_0) - 1 < \infty$.

We give the rough ideas of the proof of the density of $\mathcal{MS}$ in $\mathfrak{M}_{1,c}(X^+)$ in Theorem 3 as follows.

(i) Let $\zeta \in \mathfrak{M}_{1,c}(X^+)$ and let U be an open neighborhood of ζ in $\mathfrak{M}_{1,c}(X^+)$.

(ii) Then, there exists an element $\zeta_0 \in U$ with $\sharp\mathrm{supp}\,\zeta_0 < \infty$.

(iii) By enlarging the support of ζ_0, we can construct a family $\{\tau_t\}_{t\in[0,1]}$ of elements in U such that $\{\tau_t\}_{t\in[0,1]}$ satisfies the conditions (1)(2)(3) in Theorem 5.

(iv) Then, by Theorem 5, there exists a $t > 0$ such that $\tau_t \in U \cap \mathcal{MS}$. This argument shows the density of $\mathcal{MS}$ in $\mathfrak{M}_{1,c}(X^+)$.

Summary

(1) We introduce the notion of mean stability in i.i.d. random (holomorphic) 2-dimensional dynamical systems.

(2) We can see that a generic random dynamical system of polynomial automorphisms on $\mathbb{C}^2$ having some conditions, is mean stable.

(3) If a random holomorphic dynamical system on $\mathbb{P}^2 \setminus \{[1:0:0]\}$ is mean stable then for each $z \in \mathbb{P}^2 \setminus \{[1:0:0]\}$, for a.e. orbit starting with z, the maximal Lyapunov exponent is negative.

(4) Note that the statement of (3) cannot hold for deterministic dynamics of a single polynomial automorphism f on $\mathbb{C}^2$ which is conjugate to a generalized Hénon map by a polynomial automorphism.

(5) In the proof of the density of elements τ in $\mathfrak{M}_{1,c}(X^+)$ which are mean stable on $\mathbb{P}^2 \setminus \{[1:0:0]\}$, we consider a family $\{\tau_t\}$ in $\mathfrak{M}_{1,c}(X^+)$ and we analyse the bifurcation.

We see many randomness-induced phenomena (phenomena in random dynamical systems which cannot hold for iteration dynamics of single maps). In this talk, we have seen randomness-induced order.

Many kinds of maps in one random dynamical sytstem automatically cooperate together to make the chaoticity weaker. We call such phenomena

Cooperation Principle.

Even if a random dynamical system has a randomness-induced order, the system still can have some complexity. We have to investigate the

gradation between chaos and order

in random dynamical systems.

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