Cardinal invariants associated with the combinatorics of the uniformity number of the ideal of meager-additive sets

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Abstract

In [CMRM24], it was proved that it is relatively consistent that *bounding number* $\mathfrak{b}$ is smaller than the uniformity of $\mathcal{MA}$, where $\mathcal{MA}$ denotes ideal of the meager-additive sets of 2^ω. To prove this, it was introduced certain cardinal invariant, which we call $\mathfrak{b}_b^{\mathsf{eq}}$ regarding closely to Bartoszyński's and Judah's characterization of uniformity of $\mathcal{MA}$. In this survey, we will study this cardinal invariant and its dual (we call $\mathfrak{d}_b^{\mathsf{eq}}$). In particular, we show its relation with the cardinals in Cichoń's diagram. Additionally, we present a number of open problems regarding these cardinals.

1 Introduction and preliminaries

We first review our terminology. Let $\mathcal{I}$ be an ideal of subsets of X such that $\{x\} \in \mathcal{I}$ for all $x \in X$. Throughout this paper, we demand that all ideals satisfy this latter requirement. We introduce the following four *cardinal invariants associated with* $\mathcal{I}$:

$$\begin{aligned}
\mathrm{add}(\mathcal{I}) &= \min\left\{|\mathcal{J}| :\ \mathcal{J} \subseteq \mathcal{I}, \bigcup \mathcal{J} \notin \mathcal{I}\right\},\\
\mathrm{cov}(\mathcal{I}) &= \min\left\{|\mathcal{J}| :\ \mathcal{J} \subseteq \mathcal{I}, \bigcup \mathcal{J} = X\right\},\\
\mathrm{non}(\mathcal{I}) &= \min\{|A| : A \subseteq X,\ A \notin \mathcal{I}\}, \text{ and}\\
\mathrm{cof}(\mathcal{I}) &= \min\{|\mathcal{J}| : \mathcal{J} \subseteq \mathcal{I},\ \forall A \in \mathcal{I}\ \exists B \in \mathcal{J} \colon A \subseteq B\}.
\end{aligned}$$

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These cardinals are referred to as the *additivity, covering, uniformity* and *cofinality of* $\mathcal{I}$, respectively. The relationship between the cardinals defined above is illustrated in Figure 1.

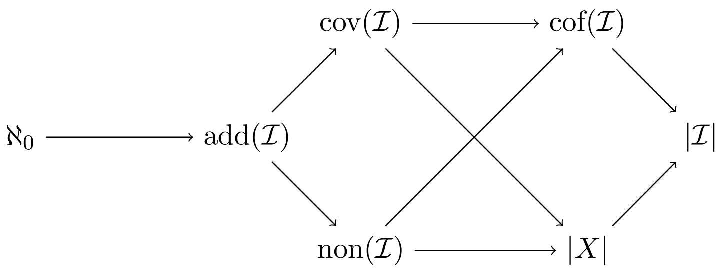

Figure 1: Diagram of the cardinal invariants associated with $\mathcal{I}$. An arrow $\mathfrak{x} \to \mathfrak{y}$ means that (provably in ZFC) $\mathfrak{x} \leq \mathfrak{y}$.

For $f, g \in \omega^\omega$ define

$$f \leq^* g \text{ iff } \exists m < \omega \, \forall n \geq m \colon f(n) \leq g(n).$$

Let

$$\mathfrak{b} := \min\{|F| : F \subseteq \omega^\omega \text{ and } \neg\exists y \in \omega^\omega \, \forall x \in F \colon x \leq^* y\}$$

the *bounding number*, and let

$$\mathfrak{d} := \min\{|D| : D \subseteq \omega^\omega \text{ and } \forall x \in \omega^\omega \, \exists y \in D \colon x \leq^* y\}$$

the *dominating number*. As usual, $\mathfrak{c} := 2^\omega$ denotes the *size of the continuum*.

Definition 1.1. Let $\mathcal{I} \subseteq \mathcal{P}(2^\omega)$ be an ideal.

(1) We say that $\mathcal{I}$ is *translation invariant*[1] if $A + x \in \mathcal{I}$ for each $A \in \mathcal{I}$ and $x \in 2^\omega$.

(2) A set $X \subseteq 2^\omega$ is termed $\mathcal{I}$-*additive* if, for every $A \in \mathcal{I}$, $A + X \in \mathcal{I}$. Denote by $\mathcal{IA}$ the collection of the $\mathcal{I}$-additive subsets of 2^ω. Notice that $\mathcal{IA}$ is a (σ-)ideal and $\mathcal{IA} \subseteq \mathcal{I}$ when $\mathcal{I}$ is a translation invariant (σ-)ideal.

When $\mathcal{I}$ is either $\mathcal{M}$ or $\mathcal{N}$, the ideal $\mathcal{IA}$ has attracted a lot of attention. Bartoszyński and Judah [BJ94], Pawlikowski [Paw85], Shelah [She95], Zindulka [Zin19] and the author, Mejía, and Rivera-Madrid [CMRM24], for example, were among the many who looked into them.

Denote by $\mathbb{I}$ the set of partitions of ω into finite non-empty intervals.

Theorem 1.2 ([She95, Thm. 13])**.** *Let $X \subseteq 2^\omega$. Then $X \in \mathcal{NA}$ iff for all $I = \langle I_n : n \in \omega \rangle \in \mathbb{I}$ there is some $\varphi \in \prod_{n\in\omega} \mathcal{P}(2^{I_n})$ such that $\forall n \in \omega \colon |\varphi(n)| \leq n$ and $X \subseteq H_\varphi$, where*

$$H_\varphi := \{x \in 2^\omega : \forall^\infty n \in \omega \colon x{\restriction}I_n \in \varphi(n)\}.$$

[1]This paper considers the Cantor space 2^ω as a topological group with the standard modulo 2 coordinatewise addition.

The following lemma is an immediate consequence of Definition 1.1.

Lemma 1.3 ([CMRM24, Lem. 1.3]). *For any translation invariant ideal $\mathcal{I}$ on 2^ω, we have:*

1. $\mathrm{add}(\mathcal{I}) \leq \mathrm{add}(\mathcal{IA})$.
2. $\mathrm{non}(\mathcal{IA}) \leq \mathrm{non}(\mathcal{I})$.

The cardinal $\mathrm{non}(\mathcal{IA})$ has been studied in [Paw85, Kra02] under the different name *transitive additivity of* $\mathcal{I}$:[2]

$$\mathrm{add}^*_t(\mathcal{I}) = \min\{|X| : X \subseteq 2^\omega \text{ and } \exists A \in \mathcal{I} \colon A + X \notin \mathcal{I}\}.$$

It is clear from the definition that $\mathrm{non}(\mathcal{IA}) = \mathrm{add}^*_t(\mathcal{I})$.

Pawlikowski [Paw85] characterized $\mathrm{add}^*_t(\mathcal{N})$ (i.e. $\mathrm{non}(\mathcal{NA})$) employing slaloms.

Definition 1.4. Given a sequence of non-empty sets $b = \langle b(n) : n \in \omega \rangle$ and $h \colon \omega \to \omega$, define

$$\prod b := \prod_{n \in \omega} b(n), \text{ and}$$
$$\mathcal{S}(b,h) := \prod_{n\in\omega} [b(n)]^{\leq h(n)}.$$

For two functions $x \in \prod b$ and $\varphi \in \mathcal{S}(b,h)$ write

$$x \in^* \varphi \text{ iff } \forall^\infty n \in \omega : x(n) \in \varphi(n).$$

Theorem 1.5 ([Paw85, Lem. 2.2], see also [CM23, Thm. 8.3]). $\mathrm{non}(\mathcal{NA})$ *is the size of the minimal bounded family $F \subseteq \omega^\omega$ such that*

$$\forall \varphi \in \mathcal{S}(b, \mathrm{id}_\omega)\, \exists x \in F \colon x \notin^* \varphi.$$

Stated differently, the uniformity of $\mathcal{NA}$ can be described using localization cardinals as follows.

For b and h as in Definition 1.4, define

$$\mathfrak{b}^{\mathsf{Lc}}_{b,h} := \min\left\{|F| : F \subseteq \prod b \text{ and } \neg\exists \varphi \in \mathcal{S}(b,h)\, \forall x \in F \colon x \in^* \varphi\right\},$$

and set $\mathrm{minLc} := \min\{\mathfrak{b}^{\mathsf{Lc}}_{b,\mathrm{id}_\omega} : b \in \omega^\omega\}$. Here, id_ω denotes the identity function on ω.

Hence, we obtain that $\mathrm{non}(\mathcal{NA}) = \mathrm{minLc}$. Another characterization of minLc is the following.

Lemma 1.6 ([CM19, Lemma 3.8]). $\mathrm{minLc} = \min\{\mathfrak{b}^{\mathsf{Lc}}_{b,h} : b \in \omega^\omega\}$ *when h goes to infinity.*

Hence, we can infer:

[2] In [BJ95] is denoted by $\mathrm{add}^\star(\mathcal{I})$.

Corollary 1.7. $\mathrm{non}(\mathcal{NA}) = \min\{\mathfrak{b}^{\mathsf{Lc}}_{b,h} : b \in \omega^\omega\}$ *when h goes to infinity.*

Moreover, it recently was proved that

Lemma 1.8 ([CMRM24, Thm. A])**.** $\mathrm{non}(\mathcal{NA}) = \mathrm{add}(\mathcal{NA})$.

The characterization of $\mathrm{add}(\mathcal{N})$ by Pawlikowski can be expressed as follows as a direct result of the previous result:

Theorem 1.9 ([Paw85, Lem. 2.3])**.** $\mathrm{add}(\mathcal{N}) = \min\{\mathfrak{b}, \mathrm{add}(\mathcal{NA})\}$.

We below focus on the σ-ideal of meager-additive sets and its uniformity. Just as in Theorem 1.2, we have one characterization for $\mathcal{MA}$ due to Bartoszyński and Judah

Theorem 1.10 ([BJ94, Thm. 2.2])**.** *Let $X \subseteq 2^\omega$. Then $X \in \mathcal{MA}$ iff for all $I \in \mathbb{I}$ there are $J \in \mathbb{I}$ and $y \in 2^\omega$ such that*

$$\forall x \in X\, \forall^\infty n < \omega\, \exists k < \omega \colon I_k \subseteq J_n \text{ and } x{\restriction}I_k = y{\restriction}I_k.$$

Furthermore, Shelah [She95, Thm. 18] proved that J can be found coarser than I, i.e, all members of J are the union of members of I

They also established a characterization of the uniformity of the meager-additive ideal:

Theorem 1.11 ([BJ94, Thm. 2.2], see also [BJ95, Thm. 2.7.14])**.**
The cardinal $\mathrm{non}(\mathcal{MA})$ is the largest cardinal κ such that, for every bounded family $F \subseteq \omega^\omega$ of size $<\kappa$,

(✿) $$\exists r, h \in \omega^\omega\, \forall f \in F\, \exists n \in \omega\, \forall m \geq n\, \exists k \in [r(m), r(m+1)) \colon f(k) = h(k).$$

We below introduce two cardinal invariants motivated by (✿), which were introduced by by the author along with Mejía and Rivera-Madrid in [CMRM24].

Definition 1.12. Let $b \in \omega^\omega$. For $I \in \mathbb{I}$, and for $f, h \in \prod b$, define

$$f \sqsubset^\bullet (I, h) \text{ iff } \forall^\infty n \in \omega\, \exists k \in I_n \colon f(k) = h(k).$$

We define the following cardinal invariants associated with $\sqsubset^\bullet$.

$$\mathfrak{b}^{\mathsf{eq}}_b := \min\{|F| : F \subseteq \prod b \text{ and } \neg\exists I \in \mathbb{I}\, \exists h \in \prod b\, \forall f \in F \colon f \sqsubset^\bullet (I, h)\}$$

and

$$\mathfrak{d}^{\mathsf{eq}}_b := \min\{|D| : D \subseteq \mathbb{I} \times \prod b \text{ and } \forall f \in \prod b\, \exists (I, h) \in D \colon f \sqsubset^\bullet (I, h)\}.$$

The study of uniformity of $\mathcal{MA}$ was better understood due to these cardinals, which for instance, were utilized by the author along with Mejía and Rivera-Madrid [CMRM24] to prove the consistency of $\mathrm{non}(\mathcal{MA}) > \mathfrak{b}$ and $\mathrm{cov}(\mathcal{MA}) < \mathrm{non}(\mathcal{N})$.

It also turns out that Theorem 1.11 can be reformulated as

(🍃) $$\mathrm{non}(\mathcal{MA}) = \min\{\mathfrak{b}^{\mathsf{eq}}_b : b \in \omega^\omega\}.$$

To be thorough, we provide a proof of (🍃) (see Lemma 2.8 and Lemma 2.9).

This survey aims to study the cardinals invariants $\mathfrak{b}^{\mathsf{eq}}_b$ and $\mathfrak{d}^{\mathsf{eq}}_b$, so one of the goal of this article is to establish:

Theorem A. *The following relations in Figure 2 hold, where $\mathfrak{x} \to \mathfrak{y}$ means $\mathfrak{x} \leq \mathfrak{y}$.*

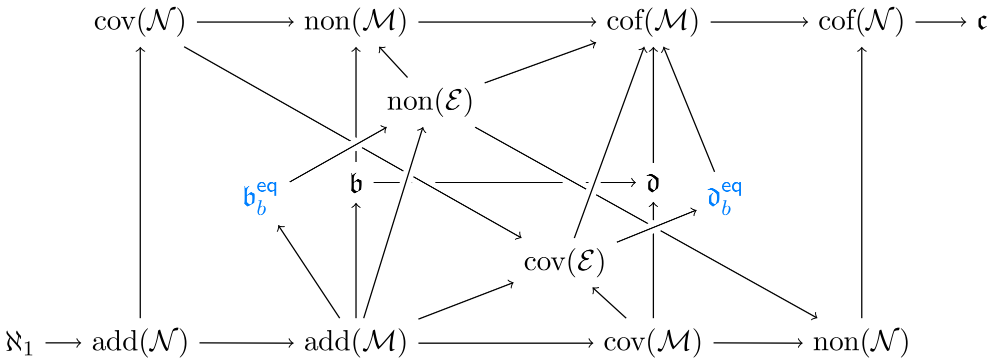

Figure 2: Including $\mathfrak{b}^{\mathsf{eq}}_b$ and $\mathfrak{d}^{\mathsf{eq}}_b$ to Cichoń's diagram.

Theorem B. *The following relations in Figure 3 hold, where $\mathfrak{x} \to \mathfrak{y}$ means $\mathfrak{x} \leq \mathfrak{y}$.*

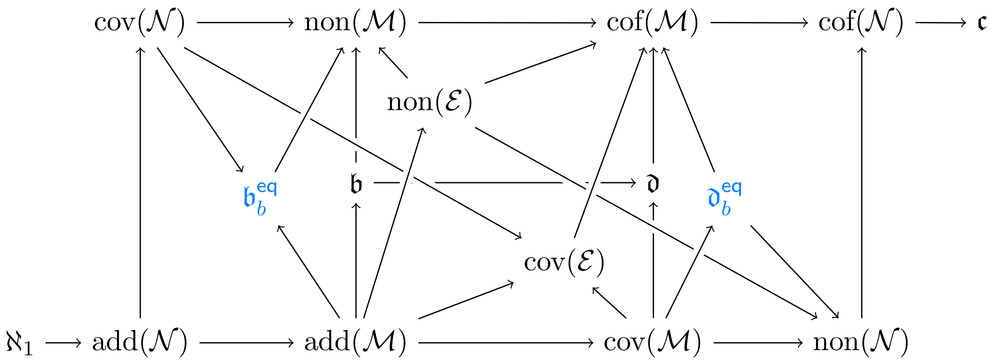

Figure 3: Including $\mathfrak{b}^{\mathsf{eq}}_b$ and $\mathfrak{d}^{\mathsf{eq}}_b$ to Cichoń's diagram. Additionally, if $\sum_{k<\omega} \frac{1}{b(k)} = \infty$ then $\mathrm{cov}(\mathcal{N}) \leq \mathfrak{b}^{\mathsf{eq}}_b$ and $\mathfrak{d}^{\mathsf{eq}}_b \leq \mathrm{non}(\mathcal{N})$.

Theorem A-B will be proved in Section 2. In Section 3, we present one forcing notion closely related to the $\mathfrak{b}^{\mathsf{eq}}_b$, which we call $\mathbb{P}_b$ and illustrate the effect of iterating $\mathbb{P}_b$ on Cichoń's diagram. In addition, we prove the following:

Theorem C (Theorem 3.23)**.** *Let $\aleph_1 \leq \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \lambda_4$ be regular cardinals, and assume λ_5 is a cardinal such that $\lambda_5 \geq \lambda_4$ $\lambda_5 = \lambda_5^{\aleph_0}$ and $\mathrm{cf}([\lambda_5]^{<\lambda_i}) = \lambda_5$ for $i = 1, \ldots, 3$. Then, we can construct a FS iteration of length λ_5 of ccc posets forcing Figure 4.*

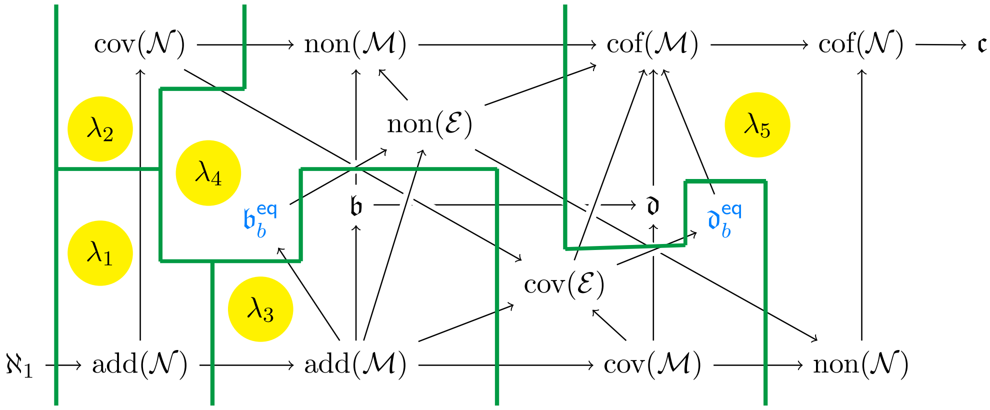

Figure 4: The constellation of Cichoń's diagram forced in Theorem C.

We close this section by reviewing everything needed to develop our targets.

Definition 1.13. We say that $\mathsf{R} = \langle X, Y, \sqsubset\rangle$ is a *relational system* if it consists of two non-empty sets X and Y and a relation $\sqsubset$.

(1) A set $F \subseteq X$ is R-*bounded* if $\exists y \in Y\ \forall x \in F\colon x \sqsubset y$.

(2) A set $E \subseteq Y$ is R-*dominating* if $\forall x \in X\ \exists y \in E\colon x \sqsubset y$.

We associate two cardinal invariants with this relational system R:

$\mathfrak{b}(\mathsf{R}) := \min\{|F|\colon F \subseteq X \text{ is } \mathsf{R}\text{-unbounded}\}$ the *unbounding number of* R, and

$\mathfrak{d}(\mathsf{R}) := \min\{|D|\colon D \subseteq Y \text{ is } \mathsf{R}\text{-dominating}\}$ the *dominating number of* R.

Note that $\mathfrak{d}(\mathsf{R}) = 1$ iff $\mathfrak{b}(\mathsf{R})$ is undefined (i.e. there are no R-unbounded sets, which is the same as saying that X is R-bounded). Dually, $\mathfrak{b}(\mathsf{R}) = 1$ iff $\mathfrak{d}(\mathsf{R})$ is undefined (i.e. there are no R-dominating families).

Directed preorders provide a very representative broad example of relational systems.

Definition 1.14. We say that $\langle S, \leq_S\rangle$ is a *directed preorder* if it is a preorder (i.e. $\leq_S$ is a reflexive and transitive relation on S) such that

$$\forall x, y \in S\ \exists z \in S\colon x \leq_S z \text{ and } y \leq_S z.$$

A directed preorder $\langle S, \leq_S\rangle$ is seen as the relational system $S = \langle S, S, \leq_S\rangle$, and its associated cardinal invariants are denoted by $\mathfrak{b}(S)$ and $\mathfrak{d}(S)$. The cardinal $\mathfrak{d}(S)$ is actually the *cofinality of* S, typically denoted by $\mathrm{cof}(S)$ or $\mathrm{cf}(S)$.

Example 1.15. Define the following relation on $\mathbb{I}$:

$$I \sqsubseteq J \text{ iff } \forall^\infty n < \omega\ \exists m < \omega\colon I_m \subseteq J_n.$$

Note that $\sqsubseteq$ is a directed preorder on $\mathbb{I}$, so we think of $\mathbb{I}$ as the relational system with the relation $\sqsubseteq$. In Blass [Bla10], it is proved that $\mathbb{I} \cong_{\mathrm{T}} \omega^\omega$. Hence, $\mathfrak{b} = \mathfrak{b}(\mathbb{I})$ and $\mathfrak{d} = \mathfrak{d}(\mathbb{I})$.

Example 1.16. We consider the following relational systems for any ideal $\mathcal{I}$ on X.

(1) $\mathcal{I} := \langle \mathcal{I}, \subseteq \rangle$ is a directed partial order. Note that

$$\mathfrak{b}(\mathcal{I}) = \mathrm{add}(\mathcal{I})$$
$$\mathfrak{d}(\mathcal{I}) = \mathrm{cof}(\mathcal{I})$$

(2) $\mathbf{C}_{\mathcal{I}} := \langle X, \mathcal{I}, \in \rangle$. When $\bigcup \mathcal{I} = X$,

$$\mathfrak{b}(\mathbf{C}_{\mathcal{I}}) = \mathrm{non}(\mathcal{I})$$
$$\mathfrak{d}(\mathbf{C}_{\mathcal{I}}) = \mathrm{cov}(\mathcal{I})$$

Example 1.17. For $b \in \omega^\omega$ define the relational system $\mathsf{R}_b := \langle \prod b, \mathbb{I} \times \prod b, \sqsubset^\bullet \rangle$. Notice that

(1) $\mathsf{R}_b \cong_{\mathrm{T}} \langle \prod b, \mathbb{I} \times \omega^\omega, \sqsubset^\bullet \rangle$. Then $\mathfrak{b}(\mathsf{R}_b) = \mathfrak{b}_b^{\mathsf{eq}}$ and $\mathfrak{d}(\mathsf{R}_b) = \mathfrak{d}_b^{\mathsf{eq}}$.

(2) If $b' \in \omega^\omega$ and $b \leq^* b'$, then $\mathsf{R}_b \preceq_{\mathrm{T}} \mathsf{R}_{b'}$. In particular, $\mathfrak{b}_{b'}^{\mathsf{eq}} \leq \mathfrak{b}_b^{\mathsf{eq}}$ and $\mathfrak{d}_b^{\mathsf{eq}} \leq \mathfrak{d}_{b'}^{\mathsf{eq}}$.

Remark 1.18. If $b \not\geq^* 2$ then we can find some $(I, h) \in \mathbb{I} \times \prod b$ such that $f \sqsubset^\bullet (I, h)$ for all $f \in \prod b$, so $\mathfrak{d}(\mathsf{R}_b) = 1$ and $\mathfrak{b}(\mathsf{R}_b)$ is undefined.

We now review the products of relational systems.

Definition 1.19. Let $\bar{\mathsf{R}} = \langle \mathsf{R}_i : i \in K \rangle$ be a sequence of relational systems $\mathsf{R}_i = \langle X_i, Y_i, \sqsubset_i \rangle$. Define $\prod \bar{\mathsf{R}} = \prod_{i \in K} \mathsf{R}_i := \left\langle \prod_{i \in K} X_i, \prod_{i \in K} Y_i, \sqsubset^\times \right\rangle$ where $x \sqsubset^\times y$ iff $x_i \sqsubset_i y_i$ for all $i \in K$.

For two relational systems R and R', write $\mathsf{R} \times \mathsf{R}'$ to denote their product, and when $\mathsf{R}_i = \mathsf{R}$ for all $i \in K$, we write $\mathsf{R}^K := \prod \bar{\mathsf{R}}$.

Fact 1.20 ([CM25])**.** *Let $\bar{\mathsf{R}}$ be as in Definition 1.19. Then* $\sup_{i \in K} \mathfrak{d}(\mathsf{R}_i) \leq \mathfrak{d}(\prod \bar{\mathsf{R}}) \leq \prod_{i \in K} \mathfrak{d}(\mathsf{R}_i)$ *and* $\mathfrak{b}(\prod \bar{\mathsf{R}}) = \min_{i \in K} \mathfrak{b}(\mathsf{R}_i)$.

We use the composition of relational systems to prove Lemma 2.5.

Definition 1.21 ([Bla10, Sec. 4])**.** Let $\mathsf{R}_e = \langle X_e, Y_e, \sqsubset_e \rangle$ be a relational system for $e \in \{0, 1\}$. The *composition of* R_0 *with* R_1 is defined by $(\mathsf{R}_0; \mathsf{R}_1) := \langle X_0 \times X_1^{Y_0}, Y_0 \times Y_1, \sqsubset_* \rangle$ where

$$(x, f) \sqsubset_* (y, b) \text{ iff } x \sqsubset_0 y \text{ and } f(y) \sqsubset_1 b.$$

Fact 1.22. *Let R_i be a relational system for $i < 3$. If $\mathsf{R}_0 \preceq_{\mathrm{T}} \mathsf{R}_1$, then $\mathsf{R}_0 \preceq_{\mathrm{T}} \mathsf{R}_1 \times \mathsf{R}_2 \preceq_{\mathrm{T}} (\mathsf{R}_1, \mathsf{R}_2)$ and $\mathsf{R}_1 \times \mathsf{R}_2 \cong_{\mathrm{T}} \mathsf{R}_2 \times \mathsf{R}_1$.*

The following theorem describes the effect of the composition on cardinal invariants.

Theorem 1.23 ([Bla10, Thm. 4.10])**.** *Let R_e be a relational system for $e \in \{0, 1\}$. Then* $\mathfrak{b}(\mathsf{R}_0; \mathsf{R}_1) = \min\{\mathfrak{b}(\mathsf{R}_0), \mathfrak{b}(\mathsf{R}_1)\}$ *and* $\mathfrak{d}(\mathsf{R}_0; \mathsf{R}_1) = \mathfrak{d}(\mathsf{R}_0) \cdot \mathfrak{d}(\mathsf{R}_1)$.

Instead of dealing with all meager sets, we will consider a suitably chosen cofinal family below.

Definition 1.24. Let $I \in \mathbb{I}$ and let $x \in 2^\omega$. Define

$$B_{x,I} := \{y \in 2^\omega : \forall^\infty n \in \omega \colon y{\restriction}I_n \neq x{\restriction}I_n\}.$$

For $n \in \omega$, define

$$B^n_{x,I} := \{y \in 2^\omega : \forall m \geq n \colon x{\restriction}I_m \neq y{\restriction}I_m\}.$$

Then $B^m_{x,I} \subseteq B^n_{x,I}$ whenever $m < n < \omega$. Thus, $B_{x,I} = \bigcup_{n\in\omega} B^n_{x,I}$.
Denote by B_I the set $B_{0,I} = \{y \in 2^\omega : \forall^\infty n \in \omega \colon y{\restriction}I_n \neq 0)\}$.

A pair $(x, I) \in 2^\omega \times \mathbb{I}$ is known as a *chopped real*, and these are used to produce a cofinal family of meager sets. It is clear that $B_{x,I}$ is a meager subset of 2^ω (see, e.g. [Bla10]).

Theorem 1.25 (Talagrand [Tal80], see also e.g. [BJS93, Prop. 13])**.** *For each meager set $F \subseteq 2^\omega$ and $I \in \mathbb{I}$ there are $x \in 2^\omega$ and $I' \in \mathbb{I}$ such that $F \subseteq B_{I',x}$ and each I'_n is the union of finitely many I_k's.*

Lemma 1.26 ([BJS93, Prop 9])**.** *For $x, y \in 2^\omega$ and for $I, J \in \mathbb{I}$, the following statements are equivalent:*

(1) $B_{I,x} \subseteq B_{J,y}$.

(2) $\forall^\infty n < \omega\, \exists k < \omega \colon I_k \subseteq J_n$ *and* $x{\restriction}I_k = y{\restriction}I_k$.

Definition 1.27. Given a sequence $b = \langle b(n) : n \in \omega\rangle$ of non-empty sets, denote

$$\mathrm{seq}_{<\omega} b := \bigcup_{n<\omega} \prod_{i<n} b(i).$$

For each $\sigma \in \mathrm{seq}_{<\omega}(b)$ define

$$[s] := [s]_b := \{x \in \prod b \colon s \subseteq x\}.$$

As a topological space, $\prod b$ has the product topology with each $b(n)$ endowed with the discrete topology. Note that $\{[s]_b : s \in \mathrm{seq}_{<\omega} b\}$ forms a base of clopen sets for this topology. When each $b(n)$ is countable we have that $\prod b$ is a Polish space and, in addition, if $|b(n)| \geq 2$ for infinitely many n, then $\prod b$ is a perfect Polish space. The most relevant instances are:

- The Cantor space 2^ω, when $b(n) = 2$ for all n, and
- The Baire space ω^ω, when $b(n) = \omega$ for all n.

We now review the Lebesgue measure on $\prod b$ when each $b(n) \leq \omega$ is an ordinal. For any ordinal $0 < \eta \leq \omega$, the probability measure μ_η on the power set of η is defined by:

- when $\eta = n < \omega$, μ_n is the measure such that, for all $k < n$, $\mu_n(\{k\}) = \frac{1}{n}$, and
- when $\eta = \omega$, μ_ω is the measure such that, for $k < \omega$, $\mu_\omega(\{k\}) = \frac{1}{2^{k+1}}$.

Denote by Lb_b the product measure of $\langle \mu_{b(n)} : n < \omega \rangle$, which we call *the Lebesgue Measure on* $\prod b$, so Lb_b is a probability measure on the Borel σ-algebra of $\prod b$. More concretely, Lb_b is the unique measure on the Borel σ-algebra such that, for any $s \in \mathrm{seq}_{<\omega} b$, $\mathsf{Lb}_b([s]) = \prod_{i<|s|} \mu_{b(i)}(\{s(i)\})$. In particular, denote by Lb, Lb_2 and Lb_ω the Lebesgue measure on $\mathbb{R}$, on 2^ω, and on ω^ω, respectively.

Let X be a topological space. Denote by $\mathcal{M}(X)$ the collection of all meager subsets of X, and let $\mathcal{M} := \mathcal{M}(\mathbb{R})$. If X is a perfect Polish space, then $\mathcal{M}(X) \cong_{\mathrm{T}} \mathcal{M}(\mathbb{R})$ and $\mathsf{Cv}_{\mathcal{M}(X)} \cong_{\mathrm{T}} \mathsf{Cv}_{\mathcal{M}(\mathbb{R})}$ (see [Kec95, Ex. 8.32 & Thm. 15.10]). Therefore, the cardinal invariants associated with the meager ideal are independent of the perfect Polish space used to calculate it. When the space is clear from the context, we write $\mathcal{M}$ for the meager ideal.

On the other hand, denote by $\mathcal{B}(X)$ the σ-algebra of Borel subsets of X, and assume that $\mu\colon \mathcal{B}(X) \to [0,\infty]$ is a σ-finite measure such that $\mu(X) > 0$ and every singleton has measure zero. Denote by $\mathcal{N}(\mu)$ the ideal generated by the μ-measure zero sets, which is also denoted by $\mathcal{N}(X)$ when the measure on X is clear. Then $\mathcal{N}(\mu) \cong_{\mathrm{T}} \mathcal{N}(\mathsf{Lb})$ and $\mathsf{Cv}_{\mathcal{N}(\mu)} \cong_{\mathrm{T}} \mathsf{Cv}_{\mathcal{N}(\mathsf{Lb})}$ where Lb is the Lebesgue measure on $\mathbb{R}$ (see [Kec95, Thm. 17.41]). Therefore, the four cardinal invariants associated with both measure zero ideals are the same. When $b = \langle b(n) : n < \omega \rangle$, each $b(n) \leq \omega$ is a non-zero ordinal, and $\prod b$ is perfect, we have that Lb_b satisfies the properties of μ above. When the measure space is understood, we just write $\mathcal{N}$ for the null ideal.

Definition 1.28. For b as above, denote by $\mathcal{E}(\prod b)$ the ideal generated by the F_σ measure zero subsets of $\prod b$. Likewise, define $\mathcal{E}(\mathbb{R})$ and $\mathcal{E}([0,1])$. When $\prod b$ is perfect, the map $F_b\colon \prod b \to [0,1]$ defined by

$$F_b(x) := \sum_{n<\omega} \frac{x(n)}{\prod_{i\leq n} b(i)}$$

is a continuous onto function, and it preserves measure. Hence, this map preserves sets between $\mathcal{E}(\prod b)$ and $\mathcal{E}([0,1])$ via images and pre-images. Therefore, $\mathcal{E}(\prod b) \cong_{\mathrm{T}} \mathcal{E}([0,1])$ and $\mathsf{Cv}_{\mathcal{E}(\prod b)} \cong_{\mathrm{T}} \mathsf{Cv}_{\mathcal{E}([0,1])}$. We also have $\mathcal{E}(\mathbb{R}) \cong_{\mathrm{T}} \mathcal{E}([0,1])$ and $\mathsf{Cv}_{\mathbb{R}} \cong_{\mathrm{T}} \mathsf{Cv}_{\mathcal{E}([0,1])}$, as well as $\mathcal{E}(\omega^\omega) \cong_{\mathrm{T}} \mathcal{E}(2^\omega)$ and $\mathsf{Cv}_{\mathcal{E}(\omega^\omega)} \cong_{\mathrm{T}} \mathsf{Cv}_{\mathcal{E}(2^\omega)}$.

When the space is clear, we write $\mathcal{E}$. Therefore, the cardinal invariants of $\mathcal{E}$ do not depend on the previous spaces.

2 ZFC results

This section aims to display the new arrows that appear in Cichon's diagram. All of the contents in this section are taken almost verbatim from [CMRM24, Sec. 2].

Lemma 2.1. $\mathsf{Cv}_{\mathcal{M}} \preceq_{\mathrm{T}} \mathsf{R}_b$ *whenever* $b \geq^* 2$*. In particular,* $\mathfrak{b}^{\mathsf{eq}}_b \leq \mathrm{non}(\mathcal{M})$ *and* $\mathrm{cov}(\mathcal{M}) \leq \mathfrak{d}^{\mathsf{eq}}_b$*.*

Proof. We work with $\mathcal{M}(\prod b)$ instead of $\mathcal{M}$ (see Section 1). Let $F\colon \prod b \to \prod b$ be the identity function and define $G\colon \mathbb{I}\times\prod b\to\mathcal{M}(\prod b)$ as follows.

$$\begin{aligned} G\colon \mathbb{I}\times\prod b &\to \mathcal{M}\big(\prod b\big)\\ (J,h) &\mapsto \{x\in\prod b:\ x\sqsubset^\bullet (J,h)\}\end{aligned}$$

Observe that $\{x\in\prod b:\ x\sqsubset^\bullet (J,h)\}\in\mathcal{M}(\prod b)$, since

$$\{x\in\prod b:\ x\sqsubset^\bullet (J,h)\}=\bigcup_{m<\omega}\bigcap_{n\geq m}\bigcup_{k\in J_n}A_k^{h(k)},$$

where $A_k^\ell:=\{x\in\prod b:\ x(k)=\ell\}$ for $\ell<b(k)$, and each A_k^ℓ is clopen. In fact, it is F_σ-set. It is clear that if $x\sqsubset^\bullet (J,h)$, then $x\in\{x\in\prod b:\ x\sqsubset^\bullet (J,h)\}$. □

We below present connections between R_b and measure zero.

Lemma 2.2. *Let $b\in\omega^\omega$.*

(1) *If $\sum_{k<\omega}\frac{1}{b(k)}<\infty$ then $\mathsf{Cv}_\mathcal{E}\preceq_{\mathrm{T}}\mathsf{R}_b$. In particular, $\mathfrak{b}_b^{\mathsf{eq}}\leq\mathrm{non}(\mathcal{E})$ and $\mathrm{cov}(\mathcal{E})\leq\mathfrak{d}_b^{\mathsf{eq}}$.*

(2) *If $\sum_{k<\omega}\frac{1}{b(k)}=\infty$. Then $\mathsf{Cv}_\mathcal{N}\preceq_{\mathrm{T}}\mathsf{R}_b^\perp$. As a consequence, $\mathrm{cov}(\mathcal{N})\leq\mathfrak{b}_b^{\mathsf{eq}}$ and $\mathfrak{d}_b^{\mathsf{eq}}\leq\mathrm{non}(\mathcal{N})$.*

Proof. To prove (1)–(2), we work with $\mathcal{N}(\prod b)$ instead of $\mathcal{N}$ (see Section 1).

(1): Observe that

$$\mathsf{Lb}_b([s]_b)=\mathsf{Lb}\Big(\{x\in\prod b:\ s\subseteq x\}\Big)=\prod_{i<|s|}\frac{1}{|b(i)|}$$

for any $s\in\mathrm{seq}_{<\omega}b$. Let F and G be as in Lemma 2.1. To complete the proof it suffices to prove

$$\mathsf{Lb}_b(\{x\in\prod b:\ x\sqsubset^\bullet (J,h)\})=0.$$

Recall

$$\{x\in\prod b:\ x\sqsubset^\bullet (J,h)\}=\bigcup_{m<\omega}\bigcap_{n\geq m}\bigcup_{k\in J_n}A_k^{h(k)}.$$

Notice that $\mathsf{Lb}_b(A_k^\ell)=\frac{1}{b(k)}$, so we obtain

$$\mathsf{Lb}_b\Big(\{x\in\prod b:\ x\sqsubset^\bullet (J,h)\}\Big)\leq\lim_{m\to\infty}\prod_{n\geq m}\sum_{k\in J_n}\frac{1}{b(k)}.$$

This limit above is 0 because $\sum_{k<\omega}\frac{1}{b(k)}<\infty$.

(2): Since $\sum_{k<\omega}\frac{1}{b(k)}=\infty$, find $J\in\mathbb{I}$ such that $\sum_{k\in J_n}\frac{1}{b(k)}\geq n$ for all $n<\omega$. Observe that

$$\prod b\smallsetminus\{x\in\prod b:\ x\sqsubset^\bullet (J,h)\}=\bigcap_{m<\omega}\bigcup_{n\geq m}\bigcap_{k\in J_n}\Big(\prod b\smallsetminus A_k^{h(k)}\Big),$$

so

$$\mathsf{Lb}_b\left(\prod b \smallsetminus \{x \in \prod b : x \sqsubset^\bullet (J,h)\}\right) \leq \lim_{m\to\infty} \sum_{n\geq m} \prod_{k\in J_n} \left(1 - \frac{1}{b(k)}\right)$$
$$\leq \lim_{m\to\infty} \sum_{n\geq m} e^{-\sum_{k\in J_n} \frac{1}{b(k)}}.$$

Since $\sum_{k\in J_n} \frac{1}{b(k)} \geq n$, $\mathsf{Lb}_b\left(\prod b \smallsetminus \{x \in \prod b : x \sqsubset^\bullet (J,h)\}\right) = 0$.

Now we define the functions Ψ_- and Ψ_+ as follows.

$$\Psi_-\colon \prod b \longrightarrow \mathbb{I} \times \prod b \text{ and}$$
$$\Psi_+\colon \prod b \longmapsto \mathcal{N}(\prod b)$$

for each $h \in \prod b$ and $x \in \prod b$ by the assignments

$$\Psi_-\colon x \longmapsto (J,x)$$
$$\Psi_+\colon h \longmapsto \prod b \smallsetminus \{x \in \prod b : x \sqsubset^\bullet (J,h)\}$$

It is not hard to see that for any $x \in \prod b$ and for any $h \in \prod b$, if $h \not\sqsubset^\bullet (J,x)$ then $x \in \prod b \smallsetminus \{x \in \prod b : x \sqsubset^\bullet (J,h)\}$. □

We introduce the following relational system for combinatorial purposes.

Definition 2.3. Let $b := \langle b(n) : n < \omega\rangle$ be a sequence of non-empty sets. Define the relational system $\mathsf{Ed}_b := \langle \prod b, \prod b, \neq^\infty\rangle$ where $x =^\infty y$ means $x(n) = y(n)$ for infinitely many n. The relation $x \neq^\infty y$ means that *x and y are eventually different.* Denote $\mathfrak{b}^{\mathsf{aLc}}_{b,1} := \mathfrak{b}(\mathsf{Ed}_b)$ and $\mathfrak{d}^{\mathsf{aLc}}_{b,1} := \mathfrak{d}(\mathsf{Ed}_b)$.

Recall the following characterization of the cardinal invariants associated with $\mathcal{M}$. The one for $\mathrm{add}(\mathcal{M})$ is due to Miller [Mil81].

Theorem 2.4 ([CM19, Sec. 3.3])**.**

$$\mathrm{add}(\mathcal{M}) = \min(\{\mathfrak{b}\} \cup \{\mathfrak{d}^{\mathsf{aLc}}_{b,1} : b \in \omega^\omega\}) \textit{ and } \mathrm{cof}(\mathcal{M}) = \sup(\{\mathfrak{d}\} \cup \{\mathfrak{b}^{\mathsf{aLc}}_{b,1} : b \in \omega^\omega\})$$

Following, we are establishing a connection between R_b and $(\mathsf{Ed}_b^\perp, \mathbb{I})$.

Lemma 2.5. *For $b \in \omega^\omega$, $\mathsf{R}_b \preceq_{\mathrm{T}} (\mathsf{Ed}_b^\perp, \mathbb{I})$. As a consequence, $\mathfrak{d}^{\mathsf{eq}}_b \leq \max\{\mathfrak{b}^{\mathsf{aLc}}_{b,1}, \mathfrak{d}\}$ and $\min\{\mathfrak{d}^{\mathsf{aLc}}_{b,1}, \mathfrak{b}\} \leq \mathfrak{b}^{\mathsf{eq}}_b$.*

Proof. Define $\Psi_-\colon \prod b \to \prod b \times \mathbb{I}^{\prod b}$ by $\Psi_-(x) := (x, F_x)$ where, for $y \in \mathbb{I}$, if $y =^\infty x$ then $F_x(y) := I^y_x \in \mathbb{I}$ is chosen such that $\forall k < \omega\, \exists i \in I^y_{x,k}\colon y(i) = x(i)$; otherwise, $F_x(y)$ can be anything (in $\mathbb{I}$).

Define $\Psi_+\colon \prod b \times \mathbb{I} \to \mathbb{I} \times \prod b$ by $\Psi_+(y,J) = (J,y)$. We check that (Ψ_-, Ψ_+) is a Tukey connection. Assume that $x, y \in \prod b$, $J \in \mathbb{I}$ and that $\Psi_-(x) \sqsubset_* (y,J)$, i.e. $x =^\infty y$ and $I^y_x \sqsubseteq J$. Since each $I^y_{x,k}$ contains a point where x and y coincide, $I^y_x \sqsubseteq J$ implies that, for all but finitely many $n < \omega$, J_n contains a point where x and y coincide, which means that $x \sqsubset^\bullet (J,y) = \Psi_+(y,J)$. □

Theorem 2.4 and Lemma 2.5 together yield:

Corollary 2.6. *For all* $b \in \omega^\omega$, $\mathfrak{d}^{\mathsf{eq}}_b \leq \operatorname{cof}(\mathcal{M})$.

Note that $\operatorname{add}(\mathcal{M}) \leq \min\{\mathfrak{b}^{\mathsf{eq}}_b : b \in \omega^\omega\}$ already follows from Lemma 1.3 and (🍃).

We close this section with the proof of (🍃). Before, a natural question regarding (🍃) that arises is

Question 2.7. *Does* $\operatorname{cov}(\mathcal{MA}) = \sup\{\mathfrak{d}^{\mathsf{eq}}_b : b \in \omega^\omega\}$ *hold?*

One negative answer to the prior question was given by the author along with Mejía and Rivera-Madrid [CMRM24]. Concretely, they proved the consistency of

$$\sup\{\mathfrak{d}^{\mathsf{eq}}_b : b \in \omega^\omega\} < \operatorname{cov}(\mathcal{MA}).$$

We prove (🍃) by using the subsequent two lemmas.

Lemma 2.8. *Let* $b \in \omega^\omega$. *Then* $\mathsf{R}_b \preceq_{\mathrm{T}} \mathsf{Cv}_{\mathcal{MA}}$. *As a consequence,*

$$\operatorname{non}(\mathcal{MA}) \leq \min\{\mathfrak{b}^{\mathsf{eq}}_b : b \in \omega^\omega\} \text{ and } \sup\{\mathfrak{d}^{\mathsf{eq}}_b : b \in \omega^\omega\} \leq \operatorname{cov}(\mathcal{MA}).$$

Proof. Given $b \in \omega^\omega$, thanks to Example 1.17 we may assume that there is some $I^b \in \mathbb{I}$ such that $b(n) = 2^{|I^b_n|}$. Then, we can identify numbers $<b(n)$ with 0-1 sequences of length $|I^b_n|$. We will find maps $\Psi_-\colon \prod b \to 2^\omega$ and $\Psi_+\colon \mathcal{MA} \to \mathbb{I} \times \prod b$ such that, for any $f \in \prod b$ and for any $X \in \mathcal{MA}$, $\Psi_-(f) \in X$ implies $f \sqsubset^\bullet \Psi_+(X)$.

Define $\Psi_-\colon \prod b \to 2^\omega$ as follows.

$$\begin{aligned} \Psi_-\colon \prod b &\to 2^\omega \\ x &\mapsto x^{I^b}_f = \underbrace{f(0)}_{\text{length } |I^b_0|} {}^\frown \cdots\cdots {}^\frown \underbrace{f(n)}_{\text{length } |I^b_n|} {}^\frown \cdots \end{aligned}$$

For $X \in \mathcal{MA}$, $X + B_{I^b} \in \mathcal{M}$. Note that

$$X + B_{I^b} = \bigcup_{x \in X} B_{x, I^b}.$$

Then, by Theorem 1.25, there are $y \in 2^\omega$ and $J \in \mathbb{I}$ such that

$$\bigcup_{x \in X} B_{x, I^b} \subseteq B_{y,J}.$$

Let $h \in \prod b$ such that $y = x^{I^b}_h$ (recall that $b(n) = 2^{|I^b_n|}$), so put $\Psi_+(X) := (J', h)$ where

$$k \in J'_n \text{ iff } \min J_n < \max I^b_k \leq \max J_n.$$

It remains to prove that, for any $f \in \prod b$ and for any $X \in \mathcal{MA}$, $\Psi_-(f) \in X$ implies $f \sqsubset^\bullet \Psi_+(X)$. Suppose that $x^{I_b}_f \in X$ and $\Psi_+(X) = (J', h)$. Then $B_{x^{I_b}_f, I^b} \subseteq B_{x^{I^b}_h, J}$. Hence, by using Lemma 1.26,

$$\forall^\infty n\, \exists k\colon I^b_k \subseteq J_n \text{ and } x^{I^b}_f \restriction I^b_k = x^{I^b}_h \restriction I^b_k.$$

Since $I^b_k \subseteq J_n$ implies $k \in J'_n$, the equation above implies that $f \sqsubset^\bullet (J', h)$. □

Lemma 2.9. *For any dominating family $D \subseteq \omega^\omega$, $\mathbf{C}_{\mathcal{MA}} \preceq_{\mathrm{T}} \prod_{b\in D} \mathsf{R}_b$. In particular, $\min_{b\in D} \mathfrak{b}^{\mathsf{eq}}_b \leq \mathrm{non}(\mathcal{MA})$ and $\mathrm{cov}(\mathcal{MA}) \leq \prod_{b\in D} \mathfrak{d}^{\mathsf{eq}}_b$.*

Proof. Without loss of generality, we may assume that there is some $\mathbb{I}$-dominating family D_0, i.e. $\forall I \in \mathbb{I}\, \exists J \in D_0 \colon I \sqsubseteq J$, such that for each $b \in D$ there is some $I \in D_0$ such that $b = 2^I$, i.e. $b(n) = 2^{I_n}$ for all $n < \omega$.

Define $\Psi_-\colon 2^\omega \to \prod_{I\in D_0} 2^I$ by $\Psi_-(x)(I) := \langle x{\restriction}I_n : n < \omega\rangle$; and define $\Psi_+\colon \prod_{I\in D_0} \mathbb{I} \times 2^I \to \mathcal{MA}$ such that, for $z = \langle (J^I, z^I) : I \in \mathbb{I}\rangle$,

$$\Psi_+(z) := \{x \in 2^\omega : \forall I \in D_0\, \forall^\infty n < \omega\, \exists k \in J^I_k \colon x{\restriction}I_k = z^I(k)\}.$$

For each $I \in D_0$ let $I'_n := \bigcup_{k\in J^I_n} I_k$ and $y^I \in 2^\omega$ the concatenation of all the $z^I(k) \in 2^{I_k}$ for $k < \omega$, i.e., $y^I{\restriction}I_k = z^I(k)$. Then $I' := \langle I'_n : n < \omega\rangle \in \mathbb{I}$, $I \sqsubseteq I'$ and

$$\forall^\infty n < \omega\, \exists k < \omega \colon I_k \subseteq I'_n \text{ and } x{\restriction}I_k = y{\restriction}I_k.$$

Therefore, by Theorem 1.10, $\Psi_+(z) \in \mathcal{MA}$. (Ψ_-, Ψ_+) is clearly the required Tukey connection. $\square$

3 Consistent results

The main goal of this section is to establish Theorem C, which is based on [CMRM24, Sec. 3].

We now present one forcing notion closely related to $\mathfrak{b}^{\mathsf{eq}}_b$, that is to say, that increases $\mathfrak{b}^{\mathsf{eq}}_b$.

Definition 3.1 ([CMRM24, Def. 3.20])**.** Given $b \in \omega^\omega$, the poset $\mathbb{P}_b$ is defined as follows: A condition $p = (s, t, F) \in \mathbb{P}_b$ if it fulfills the following:

- $s \in \omega^{<\omega}$ is increasing with $s(0) > 0$ (when $|s| > 0$),
- $t \in \mathrm{seq}_{<\omega}(b) := \bigcup_{n<\omega} \prod_{i<n} b(i)$, and
- $F \in [\prod b]^{<\aleph_0}$.

We order $\mathbb{P}_b$ by setting $(s', t', F') \leq (s, t, F)$ iff $s \subseteq s'$, $t \subseteq t'$, $F \subseteq F'$ and,

$$\forall f \in F\, \forall n \in |s'| \smallsetminus |s|\, \exists k \in [s'(n-1), s'(n)) \colon f(k) = t'(k). \text{ (Here } s'(-1) := 0.)$$

Fact 3.2. *Let $b \in \omega^\omega$. Then $\mathbb{P}_b$ is σ-centered.*

Proof. For $s \in \omega^{<\omega}$ increasing, and for $t \in \mathrm{seq}_{<\omega}(b)$, set

$$P_{s,t} := \{(s', t', F) \in \mathbb{P}_b : s' = s \text{ and } t' = t\}$$

It is not hard to verify that $P_{s,t}$ is centered and $\bigcup_{s\in\omega^{<\omega},\, t\in\mathrm{seq}_{<\omega}(b)} P_{s,t} = \mathbb{P}_b$. $\square$

Let G be a $\mathbb{P}_b$-generic filter over V. In $V[G]$, define

$$r_{\mathrm{gn}} := \bigcup\{s : \exists t, F\colon (s,t,F) \in G\} \text{ and } h_{\mathrm{gn}} := \bigcup\{t : \exists s, F\colon (s,t,F) \in G\}.$$

Then $(r_{\mathrm{gn}}, h_{\mathrm{gn}}) \in \omega^\omega \times \prod b$ and, for every $f \in \prod b \cap V$, and for all but finitely many $n \in \omega$ there is some $k \in [r_{\mathrm{gn}}(n), r_{\mathrm{gn}}(n+1)]$ such that $f(k) = h_{\mathrm{gn}}(k)$. We can identify the generic real with $(J_{\mathrm{gn}}, h_{\mathrm{gn}}) \in \mathbb{I} \times \prod b$ where $J_{\mathrm{gn},n} := [r_{\mathrm{gn}}(n-1), r_{\mathrm{gn}}(n))$, which satisfies that, for every $f \in \prod b \cap V$, $f \sqsubset^\bullet (J_{\mathrm{gn}}, h_{\mathrm{gn}})$.

Definition 3.3 ([Mej19, BCM21]). Let $F \subseteq \mathcal{P}(\omega)$ be a filter. We assume that all filters are *free*, i.e. they contain the *Frechet filter* $\mathsf{Fr} := \{\omega \smallsetminus a : a \in [\omega]^{<\aleph_0}\}$. A set $a \subseteq \omega$ is *F-positive* if it intersects every member of F. Denote by F^+ the collection of F-positive sets.

Given a poset $\mathbb{P}$ and $Q \subseteq \mathbb{P}$, Q is F-linked if, for any $\langle p_n \colon n < \omega\rangle \in Q^\omega$, there is some $q \in \mathbb{P}$ such that

$$q \Vdash \{n < \omega \colon p_n \in \dot{G}\} \in F^+, \text{ i.e. it intersects every member of } F.$$

Note that, in the case $F = \mathsf{Fr}$, the previous equation is "$q \Vdash \{n < \omega \colon p_n \in \dot{G}\}$ is infinite".

We say that Q is *uf-linked (ultrafilter-linked)* if it is F-linked for any filter F on ω containing the *Frechet filter* Fr.

For an infinite cardinal μ, $\mathbb{P}$ is *μ-F-linked* if $\mathbb{P} = \bigcup_{\alpha<\mu} Q_\alpha$ for some F-linked Q_α $(\alpha < \mu)$. When these Q_α are uf-linked, we say that $\mathbb{P}$ is *μ-uf-linked.*

Note that if $F \subseteq F'$ are filters on ω, then σ-uf-linked $\Rightarrow$ σ-F'-linked $\Rightarrow$ σ-F-linked $\Rightarrow$ σ-Fr-linked. For ccc posets, however, we have:

Lemma 3.4 ([Mej19, Lem 5.5]). *If $\mathbb{P}$ is ccc then any subset of $\mathbb{P}$ is uf-linked iff it is Fr-linked.*

Below are presented a few well-known and basic instances of σ-uf-linked posets.

Example 3.5.

(1) Let $\mathbb{P}$ be a poset and $Q \subseteq \mathbb{P}$. Note that a sequence $\langle p_n : n < \omega\rangle$ in Q witnesses that Q is <u>not</u> Fr-linked iff the set $\{q \in \mathbb{P} : \forall^\infty n < \omega \colon q \perp p_n\}$ is dense.

(2) Any singleton is uf-linked. Hence, any poset $\mathbb{P}$ is $|\mathbb{P}|$-uf-linked. In particular, Cohen forcing is σ-uf-linked.

(3) Random forcing $\mathbb{B}$ is σ-uf-linked [Mej19].

(4) The forcing eventually different real forcing $\mathbb{E}$ (see [Mej19]) is σ-uf-linked. This poset satisfies a stronger property see Example 3.9 (2).

The upcoming lemma indicates that σ-Fr-linked poset does not add dominating reals.

Lemma 3.6 ([Mej19]). *Any μ-Fr-linked poset is μ^+-ω^ω-good.*

We now focus on reviewing one linkedness property stronger than ultrafilter linkedness.

Definition 3.7 ([GMS16, BCM21, CMRM24])**.** Given a (non-principal) ultrafilter D on ω and $Q \subseteq \mathbb{P}$, say that Q is *D-lim-linked* if there are a $\mathbb{P}$-name $\dot{D}'$ of an ultrafilter on ω extending D and a map $\lim^D : Q^\omega \to \mathbb{P}$ such that, whenever $\bar{p} = \langle p_n : n < \omega \rangle \in Q^\omega$,

$$\lim^D \bar{p} \Vdash \{n < \omega : p_n \in \dot{G}\} \in \dot{D}'.$$

A set $Q \subseteq \mathbb{P}$ has *uf*-lim-linked if it is D-lim-linked for any ultrafilter D.

In addition, for an infinite cardinal θ, the poset $\mathbb{P}$ is *uniformly μ-D-lim-linked* if $\mathbb{P} = \bigcup_{\alpha<\theta} Q_\alpha$ where each Q_α is D-lim-linked and the $\mathbb{P}$-name $\dot{D}'$ above mentioned only depends on D (and not on Q_α, although we have different limits for each Q_α. When these Q_α are *uf*-lim-linked, we say that $\mathbb{P}$ is *uniformly μ-uf-lim-linked*

Remark 3.8. Any uf-lim-linked set $Q \subseteq \mathbb{P}$ is clearly uf-linked, which implies that it is Fr-linked.

Example 3.9.

1. Any singleton is uf-lim-linked. As a consequence, any poset $\mathbb{P}$ is uniformly $|\mathbb{P}|$-uf-lim-linked, witnessed by its singletons.
2. $\mathbb{E}$ is uniformly σ-uf-lim-linked (see [GMS16], see also [Mil81]).
3. $\mathbb{B}$ is not σ-uf-lim-linked (see [BCM21, Rem. 3.10]).

Next, we show another example of uniformly σ-uf-lim-linked.

Lemma 3.10 ([CMRM24, Thm. 3.21])**.** *Let $b \in \omega^\omega$. Then $\mathbb{P}_b$ is uniformly σ-uf-lim-linked.*

Proof. For $s \in \omega^{<\omega}$, $t \in \mathrm{seq}_{<\omega}(b)$ and $m < \omega$

$$P_{s,t,m} := P_b(s,t,m) = \{(s',t',F) \in \mathbb{P}_b : s' = s,\ t' = t \text{ and } |F| \leq m\}.$$

For an ultrafilter D on ω, and $\bar{p} = \langle p_n : n \in \omega \rangle \in P_{s,t,m}$, we show how to define $\lim^D \bar{p}$. Let $p_n = (s,t,F_n) \in P_{s,t,m}$. Considering the lexicographic order $\lhd$ of $\prod b$, and let $\{x_{n,k} : k < m_n\}$ be a $\lhd$-increasing enumeration of F_n where $m_n \leq m$. Next find an unique $m_* \leq m$ such that $A := \{n \in \omega : m_n = m_*\} \in D$. For each $k < m_*$, define $x_k := \lim^D_n x_{n,k}$ in $\prod b$ where $x_k(i)$ is the unique member of $b(i)$ such that $\{n \in A : x_{n,k}(i) = x_k(i)\} \in D$ (this coincides with the topological D-limit). Therefore, we can think of $F := \{x_k : k < m_*\}$ as the D-limit of $\langle F_n : n < \omega \rangle$, so we define $\lim^D \bar{p} := (s,t,F)$. Note that $\lim^D \bar{p} \in P_{s,t,m}$.

The sequence $\langle P_{s,t,m} : s \in \omega^{<\omega}, t \in \mathrm{seq}_{<\omega}(b), m < \omega \rangle$ witnesses that $\mathbb{P}_b$ is uniformly σ-D-lim-linked for any ultrafilter D on ω. This is a consequence of the following claim:

Claim 3.11 ([CMRM24, Claim. 3.22])**.** *The set*

$$D \cup \bigcup_{s,m} \{\{n < \omega : p_n \in G\} : \bar{p} \in P^\omega_{s,t,m} \cap V,\ \lim^D \bar{p} \in G\}$$

has the finite intersection property whenever G is $\mathbb{P}$-generic over V. ☐

We below review briefly the preservation theory of unbounded families presented in [CM19, Sect. 4]. This a generalization of Judah and Shelah's [JS90] and Brendle's [Bre91] preservation theory.

Definition 3.12. Let $\mathsf{R} = \langle X, Y, \sqsubset\rangle$ be a relational system and let θ be a cardinal.

(1) For a set M,

  (i) An object $y \in Y$ is R-*dominating over* M if $x \sqsubset y$ for all $x \in X \cap M$.

  (ii) An object $x \in X$ is R-*unbounded over* M if it $\mathsf{R}^\perp$-dominating over M, that is, $x \not\sqsubset y$ for all $y \in Y \cap M$.

(2) A family $\{x_i : i \in I\} \subseteq X$ is *strongly* θ-R-*unbounded* if $|I| \geq \theta$ and, for any $y \in Y$, $|\{i \in I : x_i \sqsubset y\}| < \theta$.

We look at the following type of well-defined relational systems.

Definition 3.13. Say that $\mathsf{R} = \langle X, Y, \sqsubset\rangle$ is a *Polish relational system (Prs)* if

(1) X is a Perfect Polish space,

(2) Y is a non-empty analytic subspace of some Polish Z, and

(3) $\sqsubset = \bigcup_{n<\omega} \sqsubset_n$ where $\langle \sqsubset_n : n \in \omega\rangle$ is some increasing sequence of closed subsets of $X \times Z$ such that, for any $n < \omega$ and for any $y \in Y$, $(\sqsubset_n)^y = \{x \in X : x \sqsubset_n y\}$ is closed nowhere dense.

Remark 3.14. By Definition 3.13 (3), $\langle X, \mathcal{M}(X), \in\rangle$ is Tukey below R where $\mathcal{M}(X)$ denotes the σ-ideal of meager subsets of X. Therefore, $\mathfrak{b}(\mathsf{R}) \leq \mathrm{non}(\mathcal{M})$ and $\mathrm{cov}(\mathcal{M}) \leq \mathfrak{d}(\mathsf{R})$.

For the rest of this section, fix a Prs $\mathsf{R} = \langle X, Y, \sqsubset\rangle$ and an infinite cardinal θ.

Definition 3.15 (Judah and Shelah [JS90], Brendle [Bre91])**.** A poset $\mathbb{P}$ is θ-R-*good* if, for any $\mathbb{P}$-name $\dot{h}$ for a member of Y, there is a non-empty set $H \subseteq Y$ (in the ground model) of size $<\theta$ such that, for any $x \in X$, if x is R-unbounded over H then $\Vdash x \not\sqsubset \dot{h}$.

We say that $\mathbb{P}$ is R-*good* if it is $\aleph_1$-R-good.

The previous is a standard property associated with preserving $\mathfrak{b}(\mathsf{R})$ small and $\mathfrak{d}(\mathsf{R})$ large after forcing extensions.

Remark 3.16. Notice that $\theta < \theta_0$ implies that any θ-R-good poset is θ_0-R-good. Also, if $\mathbb{P} \lessdot \mathbb{Q}$ and $\mathbb{Q}$ is θ-R-good, then $\mathbb{P}$ is θ-R-good.

Lemma 3.17 ([CM19, Lemma 2.7])**.** *Assume that θ is a regular cardinal. Then any poset of size $<\theta$ is θ-R-good. In particular, Cohen forcing $\mathbb{C}$ is R-good.*

We now present the instances of Prs and the corresponding good posets that we use in our applications.

Example 3.18.

(1) Define $\Omega_n := \{a \in [2^{<\omega}]^{<\aleph_0} : \mathsf{Lb}(\bigcup_{s\in a}[s]) \leq 2^{-n}\}$ (endowed with the discrete topology) and put $\Omega := \prod_{n<\omega}\Omega_n$ with the product topology, which is a perfect Polish space. For every $x \in \Omega$ denote

$$N_x := \bigcap_{n<\omega}\bigcup_{m\geq n}\bigcup_{s\in x(m)}[s],$$

which is clearly a Borel null set in 2^ω.

Define the Prs $\mathsf{Cn} := \langle \Omega, 2^\omega, \sqsubset^{\mathrm{n}}\rangle$ where $x \sqsubset^{\mathrm{n}} z$ iff $z \notin N_x$. Recall that any null set in 2^ω is a subset of N_x for some $x \in \Omega$, so Cn and $\mathbf{C}_{\mathcal{N}}^{\perp}$ are Tukey-Galois equivalent. Hence, $\mathfrak{b}(\mathsf{Cn}) = \mathrm{cov}(\mathcal{N})$ and $\mathfrak{d}(\mathsf{Cn}) = \mathrm{non}(\mathcal{N})$.

Any μ-centered poset is μ^+-Cn-good ([Bre91]). In particular, σ-centered posets are Cn-good.

(2) The relational system Ed_b is Polish when $b = \langle b(n) : n < \omega\rangle$ is a sequence of non-empty countable sets such that $|b(n)| \geq 2$ for infinitely many n. Consider $\mathsf{Ed} := \langle \omega^\omega, \omega^\omega, \neq^\infty\rangle$. By [BJ95, Thm. 2.4.1 & Thm. 2.4.7] (see also [CM23, Thm. 5.3]), $\mathfrak{b}(\mathsf{Ed}) = \mathrm{non}(\mathcal{M})$ and $\mathfrak{d}(\mathsf{Ed}) = \mathrm{cov}(\mathcal{M})$.

(3) The relational system $\omega^\omega = \langle \omega^\omega, \omega^\omega, \leq^*\rangle$ is Polish. Any μ-Fr-linked poset (see Definition 3.3) is μ^+-ω^ω-good (see Lemma 3.6).

(4) For each $k < \omega$, let $\mathrm{id}^k : \omega \to \omega$ such that $\mathrm{id}^k(i) = i^k$ for all $i < \omega$ and $\mathcal{H} := \{\mathrm{id}^{k+1} : k < \omega\}$. Let $\mathsf{Lc}^* := \langle \omega^\omega, \mathcal{S}(\omega,\mathcal{H}), \in^*\rangle$ be the Polish relational system where

$$\mathcal{S}(\omega,\mathcal{H}) := \{\varphi\colon \omega \to [\omega]^{<\aleph_0} : \exists h \in \mathcal{H}\,\forall i < \omega\colon |\varphi(i)| \leq h(i)\},$$

and recall that $x \in^* \varphi$ iff $\forall^\infty n\colon x(n) \in \varphi(n)$. As a consequence of [BJ95, Thm. 2.3.9] (see also [CM23, Thm. 4.2]), $\mathfrak{b}(\mathsf{Lc}^*) = \mathrm{add}(\mathcal{N})$ and $\mathfrak{d}(\mathsf{Lc}^*) = \mathrm{cof}(\mathcal{N})$.

Any μ-centered poset is μ^+-Lc^*-good (see [Bre91, JS90]) so, in particular, σ-centered posets are Lc^*-good. Besides, Kamburelis [Kam89] showed that any Boolean algebra with a strictly positive finitely additive measure is Lc^*-good (in particular, any subalgebra of random forcing).

(5) For $b \in \omega^\omega$, R_b is a Polish relational system when $b \geq^* 2$ (see Example 1.17).

(6) Let $\mathsf{M} := \langle 2^\omega, \mathbb{I} \times 2^\omega, \sqsubset^{\mathrm{m}}\rangle$ where

$$x \sqsubset^{\mathrm{m}} (I, y) \text{ iff } \forall^\infty n\colon x{\restriction}I_n \neq y{\restriction}I_n.$$

This is a Polish relational system and $\mathsf{M} \cong_{\mathrm{T}} \mathbf{C}_{\mathcal{M}}$ (by Theorem 1.25).

Note that, whenever M is a transitive model of ZFC, $c \in 2^\omega$ is a Cohen real over M iff c is M-unbounded over M.

(7) Define the relational system $\mathsf{Ce} = \langle 2^\omega, \mathsf{NE}, \sqsubset^\star\rangle$ where NE is the collection of sequences $\bar{T} = \langle T_n\colon n < \omega\rangle$ such that each T_n is a subtree of ${}^{<\omega}2$ (not necessarily well-pruned), $T_n \subseteq T_{n+1}$ and $\mathsf{Lb}([T_n]) = 0$, i.e. $\lim_{n\to\infty}\dfrac{|T \cap {}^n2|}{2^n} = 0$, and $x \sqsubset^\star \bar{T}$ iff $x \in [T_n]$ for some $n < \omega$.

Good posets are preserved along FS iterations as follows.

Lemma 3.19 ([BCM23, Sec. 4]). *Let* $\langle \mathbb{P}_\xi, \dot{\mathbb{Q}}_\xi : \xi < \pi \rangle$ *be a FS iteration such that, for* $\xi < \pi$, $\mathbb{P}_\xi$ *forces that* $\dot{\mathbb{Q}}_\xi$ *is a non-trivial* θ*-cc* θ*-*R*-good poset. Let* $\{\gamma_\alpha : \alpha < \delta\}$ *be an increasing enumeration of* 0 *and all limit ordinals smaller than* π *(note that* $\gamma_\alpha = \omega\alpha$*), and for* $\alpha < \delta$ *let* $\dot{c}_\alpha$ *be a* $\mathbb{P}_{\gamma_{\alpha+1}}$*-name of a Cohen real in* X *over* V_{γ_α}*.*

Then $\mathbb{P}_\pi$ *is* θ*-*R*-good. Moreover, if* $\pi \geq \theta$ *then* $\mathbf{C}_{[\pi]^{<\theta}} \preceq_{\mathrm{T}} \mathsf{R}$*,* $\mathfrak{b}(\mathsf{R}) \leq \theta$ *and* $|\pi| \leq \mathfrak{d}(\mathsf{R})$*.*

To force a lower bound of $\mathfrak{b}(\mathsf{R})$, we use:

Lemma 3.20 ([CM22, Thm. 2.12]). *Let* $\mathsf{R} = \langle X, Y, \sqsubset \rangle$ *be a Polish relational system,* θ *an uncountable regular cardinal, and let* $\mathbb{P}_\pi = \langle \mathbb{P}_\xi, \dot{\mathbb{Q}}_\xi : \xi < \pi \rangle$ *be a FS iteration of* θ*-cc posets with* $\mathrm{cf}(\pi) \geq \theta$*. Assume that, for all* $\xi < \pi$ *and any* $A \in [X]^{<\theta} \cap V_\xi$*, there is some* $\eta \geq \xi$ *such that* $\dot{\mathbb{Q}}_\eta$ *adds an* R*-dominating real over* A*. Then* $\mathbb{P}_\pi$ *forces* $\theta \leq \mathfrak{b}(\mathsf{R})$*, i.e.* $\mathsf{R} \preceq_{\mathrm{T}} \mathbf{C}_{[X]^{<\theta}}$*.*

The following results illustrates the effect of adding cofinally many R-dominating reals along a FS iteration.

Lemma 3.21 ([CM22, Lem. 2.9]). *Let* R *be a definable relational system of the reals, and let* λ *be a limit ordinal of uncountable cofinality. If* $\mathbb{P}_\lambda = \langle \mathbb{P}_\xi, \dot{\mathbb{Q}}_\xi : \xi < \lambda \rangle$ *is a FS iteration of* $\mathrm{cf}(\lambda)$*-cc posets that adds* R*-dominating reals cofinally often, then* $\mathbb{P}_\lambda$ *forces* $\mathsf{R} \preceq_{\mathrm{T}} \lambda$*.*

In addition, if R *is a Prs and all iterands are non-trivial, then* $\mathbb{P}_\lambda$ *forces* $\mathsf{R} \cong_{\mathrm{T}} \mathsf{M} \cong_{\mathrm{T}} \lambda$*. In particular,* $\mathbb{P}_\lambda$ *forces* $\mathfrak{b}(\mathsf{R}) = \mathfrak{d}(\mathsf{R}) = \mathrm{non}(\mathcal{M}) = \mathrm{cov}(\mathcal{M}) = \mathrm{cf}(\lambda)$*.*

Next, we illustrate the effect of iterating $\mathbb{P}_b$ on Cichoń's diagram.

Theorem 3.22. *Let* π *be an ordinal of uncountable cofinality such that* $|\pi|^{\aleph_0} = |\pi|$*. The FS iteration of* $\mathbb{P}$ *of length* π *(i.e. the FS iteration* $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \pi \rangle$ *where each* $\dot{\mathbb{Q}}_\alpha$ *is a* $\mathbb{P}_\alpha$*-name of* $\mathbb{P}_b$*) forces* $\mathfrak{c} = |\pi|$*,* $\mathsf{Cv}_{\mathcal{M}} \cong_{\mathrm{T}} \pi$ *and* $\mathsf{Cv}_{\mathcal{N}}^{\perp} \cong_{\mathrm{T}} \omega^\omega \cong_{\mathrm{T}} \mathbf{C}_{[\mathbb{R}]^{<\aleph_1}}$*. In particular, it forces* $\mathrm{cov}(\mathcal{N}) = \mathfrak{b} = \aleph_1$*,* $\mathfrak{b}_b^{\mathsf{eq}} = \mathrm{non}(\mathcal{M}) = \mathrm{cov}(\mathcal{M}) = \mathfrak{d}_b^{\mathsf{eq}} = \mathrm{cf}(\pi)$ *and* $\mathrm{non}(\mathcal{N}) = \mathfrak{d} = \mathfrak{c} = |\pi|$ *(see Figure 5).*

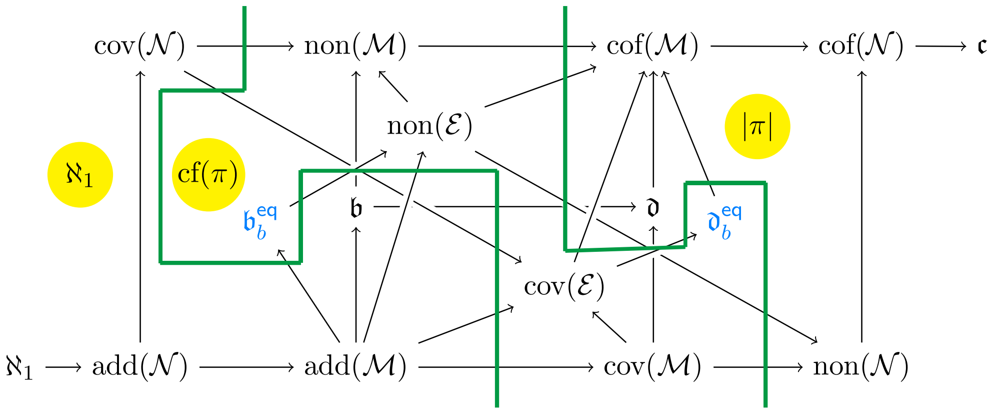

Figure 5: Cichoń's diagram after adding π-many generic reals with $\mathbb{P}_b$, where π has uncountable cofinality and $|\pi|^{\aleph_0} = |\pi|$.

The proof of the above theorem is a consequence of Theorem C, so we proceed to prove Theorem C.

Theorem 3.23. *Let $\aleph_1 \leq \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \lambda_4$ be regular cardinals, and assume λ_5 is a cardinal such that $\lambda_5 > \lambda_4$ and $\lambda_5 = \lambda_5^{\aleph_0}$ and $\mathrm{cf}([\lambda_5]^{<\lambda_i}) = \lambda_5$ for $i = 1, \dots, 4$. Then, we can construct a FS iteration of length λ of ccc posets forcing $\mathfrak{c} = \lambda_5$, $\mathsf{Lc}^* \cong_{\mathrm{T}} \mathbf{C}_{[\lambda_5]^{<\lambda_1}}$, $\mathsf{Cv}_{\mathcal{N}}^{\perp} \cong_{\mathrm{T}} \mathbf{C}_{[\lambda_5]^{<\lambda_3}}$, $\omega^\omega \cong_{\mathrm{T}} \mathbf{C}_{[\lambda_5]^{<\lambda_4}}$, and $\mathsf{R}_b \cong_{\mathrm{T}} \mathsf{M} \cong_{\mathrm{T}} \lambda_4$. In particular, it forced Figure 4.*

Proof. Make a FS iteration $\mathbb{P} = \langle \mathbb{P}_\xi, \dot{\mathbb{Q}}_\xi : \xi < \lambda \rangle$ of length $\lambda := \lambda_5 \lambda_4$ (ordinal product) as follows. Fix a partition $\langle C_i : 1 \leq i \leq 3 \rangle$ of $\lambda_5 \smallsetminus \{0\}$ where each set has size λ_5. For each $\rho < \lambda_4$ denote $\eta_\rho := \lambda_5 \rho$. We define the iteration at each $\xi = \eta_\rho + \varepsilon$ for $\rho < \lambda_4$ and $\varepsilon < \lambda_5$ as follows:

$$\dot{\mathbb{Q}}_\xi := \begin{cases} \mathbb{P}_b & \text{if } \varepsilon = 0, \\ \mathbb{LOC}^{\dot{N}_\xi} & \text{if } \varepsilon \in C_1, \\ \mathbb{B}^{\dot{N}_\xi} & \text{if } \varepsilon \in C_2, \\ \mathbb{D}^{\dot{N}_\xi} & \text{if } \varepsilon \in C_3, \end{cases}$$

where $\dot{N}_\xi$ is a $\mathbb{P}_\xi$-name of a transitive model of ZFC of size $<\lambda_i$ when $\varepsilon \in C_i$.

Additionally, by a book-keeping argument, we make sure that all such models N_ξ are constructed such that, for any $\rho < \lambda_4$:

($\bullet_1$) if $A \in V_{\eta_\rho}$ is a subset of ω^ω of size $<\lambda_1$, then there is some $\varepsilon \in C_1$ such that $A \subseteq N_{\eta_\rho + \varepsilon}$;

($\bullet_2$) if $A \in V_{\eta_\rho}$ is a subset of Ω of size $<\lambda_2$, then there is some $\varepsilon \in C_2$ such that $A \subseteq N_{\eta_\rho + \varepsilon}$; and

($\bullet_3$) if $A \in V_{\eta_\rho}$ is a subset of ω^ω of size $<\lambda_3$, then there is some $\varepsilon \in C_3$ such that $A \subseteq N_{\eta_\rho + \varepsilon}$.

We prove that $\mathbb{P}$ is as required. Clearly, $\mathbb{P}$ forces $\mathfrak{c} = \lambda_5$.

On the one side, notice that all iterands are λ_1-Lc^*-good (see Lemma 3.17 and Example 3.18 (4)), λ_2-$\mathsf{Cv}_{\mathcal{N}}^{\perp}$-good (see Lemma 3.17 and Example 3.18 (1)) and λ_3-ω^ω-good (see Lemma 3.17, Lemma 3.10, and Lemma 3.6), so by Lemma 3.19 we obtain $\mathbb{P}$ forces $\mathbf{C}_{[\lambda_5]^{<\lambda_1}} \preceq_{\mathrm{T}} \mathsf{Lc}^*$, $\mathbf{C}_{[\lambda_5]^{<\lambda_1}} \preceq_{\mathrm{T}} \mathsf{Cv}_{\mathcal{N}}^{\perp}$ and $\mathbf{C}_{[\lambda_5]^{<\lambda_1}} \preceq_{\mathrm{T}} \omega^\omega$. On the other hand, by using ($\bullet_1$) and Lemma 3.20, $\mathbb{P}$ forces $\mathsf{Lc}^* \preceq_{\mathrm{T}} \mathbf{C}_{[\lambda_5]^{<\lambda_1}}$.

In a similar way to the previous argument, $\mathbb{P}$ forces $\mathsf{Cv}_{\mathcal{N}}^{\perp} \preceq_{\mathrm{T}} \mathbf{C}_{[\lambda_5]^{<\lambda_3}}$, $\omega^\omega \preceq_{\mathrm{T}} \mathbf{C}_{[\lambda_5]^{<\lambda_4}}$.

Finally, since $\mathrm{cf}(\lambda) = \lambda_4$, by Lemma 3.21 $\mathbb{P}$ forces $\mathsf{R}_b \cong_{\mathrm{T}} \mathsf{M} \cong_{\mathrm{T}} \lambda_4$ because $\mathbb{P}_b$ adds R_b-dominating reals. □

According to the preceding theorem, $\mathfrak{b}_b^{\mathsf{eq}} > \mathrm{cov}(\mathcal{N})$ is consistent. However, what about $\mathrm{cov}(\mathcal{N}) > \mathfrak{b}_b^{\mathsf{eq}}$? We then give a positive answer to this question.

A notion proceeding ultrafilter-limits, which is more powerful, is finitely additive measures (fams)-limits introduced implicitly in the proof of the consistency of $\mathrm{cf}(\mathrm{cov}(\mathcal{N})) = \omega$ by Shelah [She00] and was formalized in [KST19]. Recently, the author refined this in general setting along with Mejía, and Uribe-Zapata [CMUZ24].

Definition 3.24 ([CMUZ24])**.** Let $\mathbb{P}$ be a poset and let $\Xi\colon \mathcal{P}(\omega) \to [0,1]$ be a fam (with $\Xi(\omega) = 1$ and $\Xi(\{n\}) = 0$ for all $n < \omega$), $I = \langle I_n\colon n < \omega\rangle$ be a partition of ω into finite sets, and $\varepsilon > 0$.

(1) A set $Q \subseteq \mathbb{P}$ is (Ξ, I, ε)-*linked* if there is a function $\lim\colon Q^\omega \to \mathbb{P}$ and a $\mathbb{P}$-name $\dot{\Xi}'$ of a fam on $\mathcal{P}(\omega)$ extending Ξ such that, for any $\bar{p} = \langle p_\ell : \ell < \omega\rangle \in Q^\omega$,

$$\lim \bar{p} \Vdash \int_\omega \frac{|\{\ell \in I_k : p_\ell \in \dot{G}\}|}{|I_k|} d\dot{\Xi}' \geq 1 - \varepsilon.$$

(2) The poset $\mathbb{P}$ is μ-FAM-*linked*, witnessed by $\langle Q_{\alpha,\varepsilon}\colon \alpha < \mu,\ \varepsilon \in (0,1) \cap \mathbb{Q}\rangle$, if:

   (i) Each $Q_{\alpha,\varepsilon}$ is (Ξ, I, ε)-linked for any Ξ and I.

   (ii) For $\varepsilon \in (0,1) \cap \mathbb{Q}$, $\bigcup_{\alpha<\omega} Q_{\alpha,\varepsilon}$ is dense in $\mathbb{P}$.

(3) The poset $\mathbb{P}$ is *uniformly* μ-FAM-*linked* if there is some $\langle Q_{\alpha,\varepsilon}\colon \alpha < \mu,\ \varepsilon \in (0,1)\cap\mathbb{Q}\rangle$ as above, such that in (1) the name $\dot{\Xi}'$ only depends on Ξ (and not on any $Q_{\alpha,\varepsilon}$).

Example 3.25.

(1) Any singleton is (Ξ, I, ε)-linked. Hence, any poset $\mathbb{P}$ is uniformly $|\mathbb{P}|$-FAM-linked. In particular, Cohen forcing is uniformly σ-FAM-linked.

(2) Shelah [She00] proved implicitly that random forcing is uniformly σ-FAM-linked. More generally, any measure algebra of Maharam type μ is uniformly μ-FAM-linked [MUZ24].

(3) The creature ccc forcing from [HS16] adding eventually different reals is (uniformly) σ-FAM-linked. This is proved in [KST19], witmore general setting in [Mej24].

The author with Mejía proved that fam-limits below help to control $\mathrm{non}(\mathcal{E})$. Concretely, they proved:

Lemma 3.26 ([CMUZ24])**.** *σ-FAM-linked posets are Ce-good.*

The following results answered our question.

Lemma 3.27 ([BS92], see also [Car23])**.** *Assume $\aleph_1 \leq \kappa \leq \lambda = \lambda^{\aleph_0}$ with κ regular and assume that $b \in \omega^\omega$ satisfies $\sum_{k<\omega} \frac{1}{b(k)} < \infty$. Let $\mathbb{B}_\pi$ be a FS iteration of random forcing of length $\pi = \lambda\kappa$. Then, in $V^{\mathbb{B}_\pi}$, $\mathsf{Lc}^* \cong_{\mathrm{T}} \omega^\omega \cong_{\mathrm{T}} \mathsf{Cv}_{\mathcal{E}} \cong_{\mathrm{T}} \mathbf{C}_{[\lambda]^{<\aleph_1}}$ and $\mathsf{Cv}_{\mathcal{N}}^\perp \cong_{\mathrm{T}} \mathsf{M} \cong_{\mathrm{T}} \kappa$.*

Proof. Since $\mathbb{B}$ adds random reals, these are $\mathsf{Cv}_{\mathcal{N}}$-unbounded reals, which are precisely the $\mathsf{Cn}^\perp$-dominating reals. So by Lemma 3.21 $\mathbb{B}_\pi$ forces $\mathsf{Cv}_{\mathcal{N}}^\perp \cong_{\mathrm{T}} \mathsf{M} \cong_{\mathrm{T}} \lambda_4$ because $\mathrm{cf}(\lambda) = \lambda_4$.

Notice that $\mathbb{B}$ is σ-uf-linked and σ-FAM-linked (see Example 3.5 (3) and Example 3.25 (2), respectively), so by Lemma 3.6 and Lemma 3.26 $\mathbb{B}$ is ω^ω-good and Ce-good, respectively. Thus, $\mathbb{B}$ is $\mathsf{Lc}*$-good by Example 3.18 (4). Hence, by Lemma 3.19, $\mathbb{B}_\pi$ forces $\mathbf{C}_{[\lambda_5]^{<\aleph_1}} \preceq_{\mathrm{T}} \mathsf{Lc}^*$, $\mathbf{C}_{[\lambda_5]^{<\aleph_1}} \preceq_{\mathrm{T}} \omega^\omega$, $\mathbf{C}_{[\lambda_5]^{<\aleph_1}} \preceq_{\mathrm{T}} \mathsf{Cv}_{\mathcal{E}}$.

On the other hand, clearly $\mathsf{Lc}^* \preceq_{\mathrm{T}} \mathbf{C}_{[\lambda_5]^{<\aleph_1}}$, $\omega^\omega \preceq_{\mathrm{T}} \mathbf{C}_{[\lambda_5]^{<\aleph_1}}$, $\mathsf{Cv}_{\mathcal{E}} \preceq_{\mathrm{T}} \mathbf{C}_{[\lambda_5]^{<\aleph_1}}$ are forced. Consequently, $\mathbb{B}_\pi$ forces $\mathsf{Lc}^* \cong_{\mathrm{T}} \omega^\omega \cong_{\mathrm{T}} \mathsf{Cv}_{\mathcal{E}} \cong_{\mathrm{T}} \mathbf{C}_{[\lambda]^{<\aleph_1}}$. ☐

4 Open problems

We know the consistency $\mathfrak{b}^{\mathsf{eq}}_b > \mathfrak{b}$, but the following is not known:

Problem 4.1. *Is* $\mathfrak{b}^{\mathsf{eq}}_b < \mathfrak{b}$ *consistently. Dually, Is* $\mathfrak{d} < \mathfrak{d}^{\mathsf{eq}}_b$ *consistent?*

Notice that for $b \in \omega^\omega$, $\mathfrak{b}^{\mathsf{aLc}}_{b,1} \leq \mathrm{non}(\mathcal{M})$ and $\mathrm{cov}(\mathcal{M}) \leq \mathfrak{d}^{\mathsf{aLc}}_{b,1}$. On the other hand, after a FS (finite support) iteration of uncountable cofinality lentgh of ccc non-trivial posets, $\mathrm{non}(\mathcal{M}) \leq \mathrm{cov}(\mathcal{M})$, which implies by Lemma 2.5 that $\mathfrak{b} \leq \mathfrak{b}^{\mathsf{eq}}_b$ and $\mathfrak{d}^{\mathsf{eq}}_b \leq \mathfrak{d}$. Hence, FS iterations cannot solve Problem 4.1.

Despite the fact that $\mathfrak{b}^{\mathsf{eq}}_b \leq \mathrm{non}(\mathcal{E})$ (Lemma 2.2 (1)), we do not know the following:

Problem 4.2. *Is* $\mathfrak{b}^{\mathsf{eq}}_b < \mathrm{non}(\mathcal{E})$ *consistent for any (some) b?*

Brendle [Bre99] (see also [Car24, Lem. 2.6]) proved the consistency of $\mathrm{non}(\mathcal{E}) > \mathfrak{d}$, so we ask:

Problem 4.3. *Is* $\mathfrak{b}^{\mathsf{eq}}_b > \mathfrak{d}$ *consistent for any (some) b?*

In relation to $\mathfrak{b}^{\mathsf{eq}}_b$ and $\mathrm{non}(\mathcal{E})$ when $\sum_{k<\omega} \frac{1}{b(k)} = \infty$, we do not know the following:

Problem 4.4. *Are each of the following statements consistent with* ZFC*?*

1. (1) $\mathrm{non}(\mathcal{E}) < \mathfrak{b}^{\mathsf{eq}}_b$ *for any (some) b. Dually,* $\mathfrak{d}^{\mathsf{eq}}_b < \mathrm{cov}(\mathcal{E})$ *for any (some) b.*
2. (2) $\mathfrak{b}^{\mathsf{eq}}_b < \mathrm{non}(\mathcal{E})$ *for any (some) b. Dually,* $\mathrm{cov}(\mathcal{E}) < \mathfrak{d}^{\mathsf{eq}}_b$ *for any (some) b.*

Recently, Yamazoe used uf-limits on intervals (introduced by Mejía [Mej24]) along FS iterations to construct a poset to force

$$\aleph_1 < \mathrm{add}(\mathcal{N}) < \mathrm{cov}(\mathcal{N}) < \mathfrak{b} < \mathrm{non}(\mathcal{E}) < \mathrm{non}(\mathcal{M}) < \\ < \mathrm{cov}(\mathcal{M}) < \mathfrak{d} < \mathrm{non}(\mathcal{N}) < \mathrm{cof}(\mathcal{N}).$$

The above model can be modified to get the following:

$$\aleph_1 < \mathrm{add}(\mathcal{N}) < \mathrm{cov}(\mathcal{N}) < \mathfrak{b} < \mathfrak{b}^{\mathsf{eq}}_b = \mathrm{non}(\mathcal{E}) < \mathrm{non}(\mathcal{M}) < \\ < \mathrm{cov}(\mathcal{M}) < \mathfrak{d} < \mathrm{non}(\mathcal{N}) < \mathrm{cof}(\mathcal{N}).$$

The point is that $\mathfrak{b}^{\mathsf{eq}}_b \leq \mathrm{non}(\mathcal{E})$ when $\sum_{k<\omega} \frac{1}{b(k)} < \infty$ and the forcing that increases $\mathfrak{b}^{\mathsf{eq}}_b$ has uniformly σ-uf-lim-linked (Lemma 3.10). So we ask:

Problem 4.5. (1) *Is it consistent* ZFC *with*

$$\aleph_1 < \mathrm{add}(\mathcal{N}) < \mathrm{cov}(\mathcal{N}) < \mathfrak{b} < \mathfrak{b}^{\mathsf{eq}}_b < \mathrm{non}(\mathcal{E}) < \mathrm{non}(\mathcal{M}) < \\ < \mathrm{cov}(\mathcal{M}) < \mathfrak{d} < \mathrm{non}(\mathcal{N}) < \mathrm{cof}(\mathcal{N}).$$

(2) *Is it consistent* ZFC *with*

$$\aleph_1 < \mathrm{add}(\mathcal{N}) < \mathrm{cov}(\mathcal{N}) < \mathfrak{b}^{\mathsf{eq}}_b < \mathfrak{b} < \mathrm{non}(\mathcal{E}) < \mathrm{non}(\mathcal{M}) < \\ < \mathrm{cov}(\mathcal{M}) < \mathfrak{d} < \mathrm{non}(\mathcal{N}) < \mathrm{cof}(\mathcal{N}).$$

Notice that FS iterations cannot solve (2) of Problem 4.5 (see discussion after Problem 4.1). A positive answer of Problem 4.4 could help solve the following:

Problem 4.6. (1) *Is it consistent* ZFC *with*

$$\aleph_1 < \mathrm{add}(\mathcal{N}) < \mathrm{cov}(\mathcal{N}) < \mathfrak{b} < \mathrm{non}(\mathcal{E}) < \mathfrak{b}_b^{\mathsf{eq}} < \mathrm{non}(\mathcal{M}) < \\ < \mathrm{cov}(\mathcal{M}) < \mathfrak{d} < \mathrm{non}(\mathcal{N}) < \mathrm{cof}(\mathcal{N}).$$

(2) *Is it consistent* ZFC *with*

$$\aleph_1 < \mathrm{add}(\mathcal{N}) < \mathrm{cov}(\mathcal{N}) < \mathfrak{b}_b^{\mathsf{eq}} < \mathfrak{b} < \mathrm{non}(\mathcal{E}) < \mathrm{non}(\mathcal{M}) < \\ < \mathrm{cov}(\mathcal{M}) < \mathfrak{d} < \mathrm{non}(\mathcal{N}) < \mathrm{cof}(\mathcal{N}).$$

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