COMPARING FORCING APPROACHES TO DENSE IDEALS

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ABSTRACT. We analyze some posets involved in forcing constructions for dense ideals, showing that the Anonymous Collapse and the Dual Shioya Collapse are equivalent for collapsing a large cardinal to ω_2. We also give a somewhat simplified construction of a normal ideal I on ω_2 such that $\mathcal{P}(\omega_2)/I \sim \mathrm{Col}(\omega_1, \omega_2)$.

1. INTRODUCTION

For a regular cardinal κ, a κ-complete ideal I on κ is said to be *saturated* if $\mathcal{P}(\kappa)/I$ has the κ^+-c.c., which is the smallest possible chain condition when κ is a successor cardinal. The study of saturated ideals on successor cardinals was initiated by Kunen [14], who proved that it is consistent relative to a huge cardinal that there is a saturated ideal on ω_1. If I is a saturated ideal on a successor cardinal κ, and $G \subseteq \mathcal{P}(\kappa)/I$ is generic, then one can form the generic ultrapower embedding $j : V \to M \subseteq V[G]$, where κ is the critical point of the elementary embedding j, and M is a transitive $<j(\kappa)$-closed subclass of $V[G]$. Thus it is natural to say that κ is "generically almost-huge", since it satisfies the definition of almost-hugeness except that the target model is constructed in a generic extension. For this reason, Kunen [14] suggested that the consistency strength of the existence of such ideals might be that of an almost-huge cardinal. This was refuted by Foreman-Magidor-Shelah [5], who showed that one can force the existence of a saturated ideal on ω_1 from a supercompact cardinal. This was later reduced by Shelah-Woodin [18] to a Woodin cardinal, and Jensen-Steel [12] proved that a Woodin cardinal is necessary.

However, at the time of this writing, the upper bound for the consistency strength of saturated ideals on successor cardinals above ω_1 has still not been reduced below an almost-huge cardinal. Magidor modified Kunen's approach to reduce the bound from huge to almost-huge (see [8, Section 7.11]). Several variations on Kunen's forcing, most notably by Laver [15], Foreman [7], and Shioya [19], have been presented, which obtain some combination of stronger combinatorial properties of the ideal, simultaneous occurrence of saturated ideals on many cardinals, and a simplification of the forcing.

A kind of "ultimate" saturation property is density. We say a κ-complete ideal I on a successor cardinal κ is κ-*dense*, or simply *dense* for brevity, if $\mathcal{P}(\kappa)/I$ has a dense subset of size κ. Most of the known variations on Kunen's approach do not obtain this property. In unpublished work in the late 70s, Woodin gave a forcing argument for obtaining a dense ideal on ω_1 from an almost-huge cardinal. Although it shared some general features with Kunen-style constructions, it differed in a few important ways, one of which was the involvement of a choiceless model as an intermediate step in the argument. A version of this argument was presented by

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the author in his thesis [4] and in [2], in a way that generalizes to get dense ideals on arbitrary successors of regular cardinals, as well as analogous ideals over $\mathcal{P}_\kappa(\lambda)$.

For dense ideals on ω_1, quite different approaches have been developed using weaker large cardinal assumptions. Woodin [21] showed that NS_{ω_1} being dense is equiconsistent with $\mathsf{ZF} + \mathsf{AD}$, which is equiconsistent with ZFC plus infinitely many Woodin cardinals. For one direction, Woodin forced over a model of $\mathsf{ZF} + \mathsf{AD} + V = L(\mathbb{R})$, obtaining a model of $\mathsf{ZFC} +$ "NS_{ω_1} is dense" $+ 2^\omega = \omega_2$. By Shelah [17], NS_{ω_1} being dense requires $\neg\mathsf{CH}$. To obtain a dense ideal on ω_1 along with CH, Woodin forced over a model of $\mathsf{ZF} + \mathsf{AD}_\mathbb{R} +$ "θ is regular" with a different forcing. This was later shown to be optimal by Adolf-Sargsyan-Trang-Wilson-Zeman [1]. Recently, Lietz [16] showed how to force NS_{ω_1} to be dense from a model of ZFC with an inaccessible limit of supercompact cardinals, answering a question of Woodin. Like in the case of saturation, all of these arguments are specific to ω_1.

Regarding dense ideals on higher cardinals, a big mystery discussed in [2] was how to get dense ideals on adjacent cardinals simultaneously, for example at ω_1 and ω_2. The problem is that the density property is so sensitive that it doesn't seem to leave much room to vary the forcing (in contrast to Kunen's approach), and the forcing lacked a certain uniformity that would be helpful for iterating it to obtain dense ideals at several cardinals.

The first progress in a long time on this problem came in a recent paper of Shioya [20]. He replaced the abstract arguments about complete Boolean algebras of [2] with a more structured partial order, and he eliminated the use of choiceless models. Sensing the promise of this approach, the author worked with Hayut on a modification, adding a bit more combinatorial structure, that ultimately resulted in the solution to the problem of dense ideals on adjacent cardinals [3]. Indeed, we obtain a model of ZFC in which for every regular cardinal κ, there is a κ^+-complete ideal I on κ^+ such that $\mathcal{P}(\kappa)/I$ has a dense set isomorphic to the simple Levy collapse $\mathrm{Col}(\kappa, \kappa^+)$. This answered several questions of Foreman [8].

The purpose of this note is to compare the main forcings of [2] and [3], which are respectively called the Anonymous Collapse $\mathbb{A}(\mu, \kappa)$ and the Dual Shioya Collapse $\mathsf{L}(\mu, \kappa)$. For technical reasons, $\mathsf{L}(\mu, \kappa)$ is well-defined only when μ is a regular uncountable cardinal, while $\mathbb{A}(\mu, \kappa)$ makes sense for $\mu = \omega$. In each case, a projection is defined from $\mathrm{Col}(\mu, <\kappa) * \dot{\mathrm{Add}}(\kappa)$ to the respective forcing. $\mathbb{A}(\mu, \kappa)$ has a certain minimality property that ensures it is a subforcing of $\mathsf{L}(\mu, \kappa)$. It is natural to ask whether these two forcings are actually equivalent. We show:

Theorem 1. *For inaccessible κ, $\mathbb{A}(\omega_1, \kappa)$ and $\mathsf{L}(\omega_1, \kappa)$ are forcing-equivalent.*

In the construction of a dense ideal using $\mathbb{A}(\mu, \kappa)$, the quotient forcing between $\mathbb{A}(\mu, \kappa)$ and $\mathrm{Col}(\mu, <\kappa) * \dot{\mathrm{Add}}(\kappa)$ plays a role. This is somewhat unfortunate, as the combinatorial structure of this quotient is unclear. In [3], this issue is avoided through the use of a certain "uniformization" forcing $\mathbb{U}$, which extracts a useful subset of the generic for $\mathsf{L}(\mu, \kappa)$. It turns out that the analysis behind Theorem 1 leads naturally to an elimination of these devices in an alternative argument for a dense ideal on ω_2, which we present here.

Theorem 2. *Suppose κ is almost-huge with target λ. Then $\mathsf{L}(\omega_1, \kappa) * \dot{\mathrm{Col}}(\kappa, <\lambda)$ forces that there is a normal ideal I on ω_2 such that $\mathcal{P}(\omega_2)/I$ is forcing-equivalent to $\mathrm{Col}(\omega_1, \omega_2)$.*

However, we don't know how to get dense ideals on ω_1 and ω_2 simultaneously without the uniformization forcing $\mathbb{U}$.

2. CANONICITY OF COLLAPSING WITH σ-STRATEGICALLY CLOSED POSETS

In this section, we present a generalization of a result of Foreman [6]. It is a well-known result of McAloon (see [9, Section 4, Theorem 1]) that for every regular cardinal κ, every κ-closed poset $\mathbb{P}$ that collapses its own size to κ is forcing-equivalent to the canonical Levy collapse $\mathrm{Col}(\kappa, |\mathbb{P}|)$. For $\kappa = \omega_1$, we can weaken the closure assumption to strategic closure. This will be useful because, unlike in the case of countable closure, σ-strategic closure is always inherited by subforcings.

For a poset $\mathbb{P}$ and an ordinal δ, we define two games $\mathcal{G}_\delta^{\mathrm{I}}(\mathbb{P})$ and $\mathcal{G}_\delta^{\mathrm{II}}(\mathbb{P})$. Two players alternate playing elements of $\mathbb{P}$ in a descending sequence, with Player I making the first move. At limit stages, Player I plays first in $\mathcal{G}_\delta^{\mathrm{I}}(\mathbb{P})$, and Player II plays first in $\mathcal{G}_\delta^{\mathrm{II}}(\mathbb{P})$. Player II wins if the game lasts for δ-many rounds; otherwise, Player I wins. For a cardinal κ, we say that $\mathbb{P}$ is κ-*strategically closed* if Player II has a winning strategy for $\mathcal{G}_\kappa^{\mathrm{II}}(\mathbb{P})$, and *strongly κ-strategically closed* if Player II has a winning strategy for $\mathcal{G}_\kappa^{\mathrm{I}}(\mathbb{P})$. For $\alpha = \omega + 1$, Player II has a winning strategy in either game iff they have a strategy for the first ω moves that guarantees the existence of a lower bound to the play. In this case, we say that $\mathbb{P}$ is σ-*strategically closed*.

Theorem 3 (Ishiu-Yoshinobu [10]). *A poset is σ-strategically closed iff it is strongly ω_1-strategically closed.*

Let us briefly outline the argument for the nontrivial direction, using:

Lemma 4 (Ishiu-Yoshinobu [10]). *There exists a partial order $\prec$ on ω_1 with the following properties:*
  (1) *If $\alpha \prec \beta$, then $\alpha < \beta$.*
  (2) *Each $\alpha < \omega_1$, the set of $\beta \prec \alpha$ is finite and linearly ordered by $\prec$.*
  (3) *For each limit ordinal $\alpha < \omega_1$, there is a sequence $\beta_0 \prec \beta_1 \prec \beta_2 \prec \dots$ such that $\alpha = \sup_i \beta_i$.*

Suppose τ_0 is a winning strategy for II in $\mathcal{G}_{\omega+1}^{\mathrm{I}}(\mathbb{P})$. We convert this into a winning strategy τ_1 for II in $\mathcal{G}_{\omega_1}^{\mathrm{I}}(\mathbb{P})$ inductively as follows. Suppose that we have defined τ_1 for plays in which I has played $<\alpha$-many times. Assume that for each $\beta < \alpha$, if $\langle p_0, q_0, \dots, p_\beta, q_\beta \rangle$ is a run of the game where II follows τ_1, and $\beta_0 \prec \dots \prec \beta_{n-1} \prec \beta_n = \beta$ are the $\prec$-predecessors of β, then $\langle p_{\beta_0}, q_{\beta_0}, p_{\beta_1}, q_{\beta_1} \dots, p_{\beta_n}, q_{\beta_n} \rangle$ follows the strategy τ_0. If $\alpha = \alpha' + 1$, then define

$$\tau_1(\langle p_0, q_0, \dots, p_{\alpha'}, q_{\alpha'}, p_\alpha \rangle) = \tau_0(\langle p_{\alpha_0}, q_{\alpha_0}, \dots, p_{\alpha_{n-1}}, q_{\alpha_{n-1}}, p_\alpha \rangle),$$

where $\alpha_0 \prec \dots \prec \alpha_{n-1}$ are the $\prec$-predecessors of α. If α is a limit, then there is a sequence $\beta_0 \prec \beta_1 \prec \beta_2 \prec \dots$ such that $\alpha = \sup_i \beta_i$. By induction, the subsequence of the play $\langle p_{\beta_0}, q_{\beta_0}, p_{\beta_1}, q_{\beta_1}, \dots \rangle$ follows τ_0. Thus there is a lower bound p_α that Player I can play. For any such play, define II's next move according to τ_1 just as in the successor case. This completes the construction of τ_1.

To get the desired conclusion, we use the following generalization of McAloon's result:

Lemma 5. *If $\mathbb{P}$ is strongly κ-strategically closed and nowhere $|\mathbb{P}|$-c.c., and $\mathbb{P}$ collapses $|\mathbb{P}|$ to κ, then $\mathbb{P}$ is forcing-equivalent to $\mathrm{Col}(\kappa, |\mathbb{P}|)$.*

Proof. Let $\lambda = |\mathbb{P}|$. Let $\dot{f}$ be a $\mathbb{P}$-name for a surjection from κ to the generic filter $\dot{G}$ with $\dot{f}(0) = 1_\mathbb{P}$. We build a tree $T \subseteq \mathbb{P}$ of height κ with the following properties:

(1) $1_\mathbb{P}$ is the root of T.
(2) For each $t \in T$, there is a set $S(t) \subseteq T$ of size λ such that for each $s \in S(t)$ $s < t$, and there is no $x \in T$ such that $s < x < t$.
(3) For each descending chain $\langle t_i : i < \delta \rangle$ in T, where $\delta < \kappa$, there is a set of λ-many pairwise incompatible lower bounds in T to $\langle t_i : i < \delta \rangle$, all of which are maximal in T among such lower bounds.
(4) T is dense in $\mathbb{P}$.

This suffices to build a dense embedding from $\{q \in \mathrm{Col}(\kappa, \lambda) : \mathrm{dom}\, q$ is a successor ordinal$\}$ to $\mathbb{P}$. We build T inductively by levels, $\langle T_\alpha : \alpha < \kappa \rangle$, where T_α is the set of all $t \in T$ such that the chain in T above t has length α.

Fix a winning strategy τ for II in $\mathcal{G}^\mathrm{I}_\kappa(\mathbb{P})$. We inductively assume that:

(1) Each T_α is a maximal antichain in $\mathbb{P}$.
(2) For each $t \in T_\alpha$, if $\langle t_\beta : 0 < \beta \le \alpha \rangle$ enumerates the branch above $t = t_\alpha$, then there is a run of the game $\langle s_0, t_0, s_1, t_1, \dots, s_\alpha, t_\alpha \rangle$ following τ, with $s_0 = 1_\mathbb{P}$. We assume that these runs are chosen so that, for $\beta < \alpha$, $t_\beta \in T_\beta$ and $t_\alpha \in T_\alpha$ with $t_\beta > t_\alpha$, the chosen run of the game above t_β is an initial segment of that above t_α.
(3) For each $t \in T_\alpha$, there is $p \in \mathbb{P}$ such that $t \le p$ and $t \Vdash \dot{f}(\alpha) = p$.

Given $t_\alpha \in T_\alpha$ and the associated run of the game $\langle s_0, t_0, s_1, t_1, \dots, s_\alpha, t_\alpha = t \rangle$ as above, there is a dense set of $p < t$ that are the last play by II in a run of the game following τ of the form $\langle s_0, t_0, s_1, t_1, \dots, s_\alpha, t_\alpha, q, p \rangle$, and which have the property that for some $r \in \mathbb{P}$, $p \le r$ and $p \Vdash \dot{f}(\alpha + 1) = r$. We pick a maximal antichain A_t of such $p < t$, with $|A_t| = \lambda$, and let $T_{\alpha+1} = \bigcup_{t \in T_\alpha} A_t$.

Suppose we have constructed up to a limit α. Let $\vec{b} = \langle t_\beta : \beta < \alpha \rangle$ be a branch through $\bigcup_{\beta < \alpha} T_\beta$, with $t_\beta \in T_\beta$. There is a run of the game $\langle s_0, t_0, \dots, s_\beta, t_\beta, \dots \rangle$ of length α, where II plays according to τ. Thus there are lower bounds to the sequence. Among all such lower bounds, there is a dense set of p that are the last play by II in a run of the game following τ of the form $\langle s_0, t_0, \dots, s_\beta, t_\beta, \dots, q, p \rangle$, extending the above sequence by two points, and which have the property that for some $r \in \mathbb{P}$, $p \le r$ and $p \Vdash \dot{f}(\alpha) = r$. Let $A_{\vec{b}}$ be a maximal antichain of such p of size $|\lambda|$, and let T_α be the union of the $A_{\vec{b}}$, over all branches $\vec{b}$ through $\bigcup_{\beta < \alpha} T_\beta$. Since $\mathbb{P}$ is κ-distributive, the set of all $p \in \mathbb{P}$ that are below some $t \in T_\beta$ for each $\beta < \alpha$ is dense, and thus T_α is a maximal antichain in $\mathbb{P}$.

Let $T = \bigcup_{\alpha < \kappa} T_\alpha$. It is clear that T is κ-closed. To show that it is dense in $\mathbb{P}$, let $p_0 \in \mathbb{P}$, and let $p_1 \le p_0$ and $\alpha < \kappa$ be such that $p_1 \Vdash \dot{f}(\alpha) = p_0$. There is some $t \in T_\alpha$ that is compatible with p_1, and there is some $r \in \mathbb{P}$ such that $t \le r$ and $t \Vdash \dot{f}(\alpha) = r$. Thus $t \le r = p_0$. $\square$

3. COMPARING $\mathbb{A}(\omega_1, \kappa)$ AND $\mathbb{L}(\omega_1, \kappa)$

Let us first recall the definitions of the forcings $\mathbb{A}(\mu, \kappa)$ and $\mathbb{L}(\mu, \kappa)$. Suppose κ is inaccessible and $\mu < \kappa$ is regular and uncountable.

Let $\dot{X}$ be the canonical name for the Cohen subset of κ added by the second stage of $\mathrm{Col}(\mu, <\kappa) * \dot{\mathrm{Add}}(\kappa)$. Letting $\mathbb{B}$ be the Boolean completion of this forcing, we define $\mathbb{A}(\mu, \kappa)$ as the complete subalgebra of $\mathbb{B}$ generated by this name, i.e. the smallest complete subalgebra containing all Boolean values $||\check{\alpha} \in \dot{X}||$ for $\alpha < \kappa$.

If $G * X \subseteq \mathrm{Col}(\mu, <\kappa) * \dot{\mathrm{Add}}(\kappa)$ is generic over V, and H is the induced filter on $\mathbb{A}(\mu, \kappa)$, then $V[H] = V[X]$ (see [11, Lemma 15.40]). It is easy to see that all bounded subsets of κ added by G are coded in intervals of X, and moreover $V[H]$ and $V[G]$ have the same $<\kappa$-sequences of ordinals.

Let us recall a few preliminary notions before introducing the definition of $\llcorner(\mu, \kappa)$. We call a poset *completely μ-closed* if every descending sequence of length $<\mu$ has a greatest lower bound. A subset of a completely μ-closed poset is called *μ-closed* when it is closed under descending $<\mu$-sequences. A map from one μ-closed poset to another is called *μ-continuous* when it preserves limits of descending sequences of length $<\mu$. A crucial but easy fact is:

Lemma 6. *The intersection of $<\kappa$-many dense κ-closed subsets of a completely κ-closed poset is dense and κ-closed.*

Definition. $\llcorner(\mu, \kappa)$ *consists of all triples of the form $\langle \mathbb{P}, p, \tau \rangle$ where:*

(1) $\mathbb{P} \in V_\kappa$ *is a separative completely μ-closed poset.*
(2) $p \in \mathbb{P}$.
(3) *For some $\gamma < \kappa$, τ is a $\mathbb{P}$-name for a function from γ to 2.*

We put $\langle \mathbb{Q}, q, \sigma \rangle \leq \langle \mathbb{P}, p, \tau \rangle$ when there is a μ-continuous projection π from a μ-closed dense subset of $\mathbb{Q}$ to $\mathbb{P}$ such that $q \Vdash p \in \pi(\dot{G})$, where $\dot{G}$ is the canonical name for the $\mathbb{Q}$-generic filter, and $q \Vdash \pi^*(\tau) \unlhd \sigma$, where π^* is the canonical translation of $\mathbb{P}$-names into $\mathbb{Q}$-names via the projection π.

The key feature of both $\mathbb{A}(\mu, \kappa)$ and $\llcorner(\mu, \kappa)$ is that they have a kind of universal property with respect to a large class of "ordinary" collapsing posets.

Definition. *A poset $\mathbb{R}$ is called a* reasonable (μ, κ)-collapse *when:*

(1) $\mathbb{R}$ *is κ-c.c. and completely μ-closed.*
(2) *There is a $\subseteq$-increasing sequence of regular suborders $\langle \mathbb{R}_\alpha : \alpha < \kappa \rangle$, with $\mathbb{R} = \bigcup_\alpha \mathbb{R}_\alpha$.*
(3) *There is a sequence of maps $\langle \pi_\alpha : \alpha < \kappa \rangle$ such that for all α, $\pi_\alpha : \mathbb{R} \to \mathbb{R}_\alpha$ is a μ-continuous projection, $\pi_\alpha \upharpoonright \mathbb{R}_\alpha = \mathrm{id}$, and for $\alpha < \beta$, $\pi_\alpha = \pi_\alpha \circ \pi_\beta$.*
(4) *For unboundedly many regular $\alpha < \kappa$, $|\mathbb{R}_\alpha| = \alpha$ and $\Vdash_{\mathbb{R}_\alpha} |\alpha| = \mu$.*

Lemma 7. *Suppose $\mathbb{R}$ is a (μ, κ)-reasonable collapse.*

(1) *Let $\mathbb{A}(\mathbb{R})$ be the complete subalgebra of $\mathcal{B}(\mathbb{R} * \dot{\mathrm{Add}}(\kappa))$ generated by the canonical name for the generic subset of κ added by the second step. Then $\mathbb{A}(\mathbb{R}) \cong \mathbb{A}(\mu, \kappa)$.*
(2) *There is a dense subset $D \subseteq \mathbb{R} * \dot{\mathrm{Add}}(\kappa)$ and a projection $\psi : D \to \llcorner(\mu, \kappa)$.*
(3) *There is an $\mathbb{A}(\mathbb{R})$-name $\dot{X}_\mathbb{A}$ and a $\llcorner(\mu, \kappa)$ name $\dot{X}_\llcorner$, both for subsets of κ, such that whenever $G * X \subseteq \mathbb{R} * \dot{\mathrm{Add}}(\kappa)$ is generic, and H and K are the induced filters on $\mathbb{A}(\mathbb{R})$ and $\llcorner(\mu, \kappa)$ respectively, then $X = \dot{X}_\mathbb{A}^H = \dot{X}_\llcorner^K$.*

For a proof of (1), see [2, Theorem 2.12]; for (2), see [3, Lemma 25], and for (3), see [3, Lemma 28]. We note that $\dot{X}_\llcorner$ is defined as $\{\alpha < \kappa : (\exists \langle \mathbb{P}, p, \tau \rangle \in \dot{G}) p \Vdash_\mathbb{P} \tau(\alpha) = 1\}$, where $\dot{G}$ names the $\llcorner(\mu, \kappa)$-generic filter. (3) implies that there is a complete Boolean algebra $\mathbb{S} \cong \mathcal{B}(\llcorner(\mu, \kappa))$ such that $\mathbb{A}(\mathbb{R}) \subseteq \mathbb{S} \subseteq \mathcal{B}(\mathbb{R} * \dot{\mathrm{Add}}(\kappa))$.

The map ψ above is defined as follows. Fixing some sequence $\langle \mathbb{R}_\alpha : \alpha < \kappa \rangle$ witnessing that $\mathbb{R}$ is (μ, κ)-reasonable, we let D be the dense set of conditions $\langle r, \tau \rangle$ such that for some α, τ is forced to have domain α and $r \in \mathbb{R}_\alpha$. For such $\langle r, \tau \rangle$, we

define $\psi(r,\tau) = \langle \mathbb{R}_\alpha, r, \tau \rangle$. For such conditions, we have that $r \Vdash \alpha \in \dot{X}_{\mathrm{Add}(\kappa)}$ iff $r \Vdash \tau(\alpha) = 1$ iff $\psi(r,\tau) \Vdash \alpha \in \dot{X}_{\llcorner}$.

An important notion in the analysis of $\llcorner(\mu,\kappa)$ is that of a *coding condition*.

Definition. *We say* $\langle \mathbb{P}, p, \tau \rangle \in \llcorner(\mu,\kappa)$ *is a* coding condition *when τ is forced to have domain $\delta < \kappa$, and whenever $G \subseteq \mathbb{P}$ is generic,*

- $V[G] = V[\tau^G]$;
- *in all generic extensions of $V[G]$, G is the unique filter F that is $\mathbb{P}$-generic over V such that $\tau^F = \tau^G$.*

We say that $\langle \mathbb{P}, p, \tau \rangle$ is a strong coding condition *when τ is forced to have domain $\delta < \kappa$, and there is a dense $D \subseteq \mathbb{P}$, a set $X \subseteq \delta$, and a function $f : X \to D$ such that for all $\alpha \in X$,*

$$1_{\mathbb{P}} \Vdash f(\alpha) \in \dot{G} \leftrightarrow \tau(\alpha) = 1.$$

It is easy to show that strong coding conditions are coding, and that they are dense in $\llcorner(\mu,\kappa)$. Any condition can be extended to a coding condition by only lengthening the τ-part. The main fact about coding conditions is the "Projection Freezing Lemma":

Lemma 8. *Let $\langle \mathbb{P}, p, \tau \rangle$ be a coding condition and let $\langle \mathbb{Q}, q, \sigma \rangle \leq \langle \mathbb{P}, p, \tau \rangle$. For any dense $D \subseteq \mathbb{Q}$, if π_0, π_1 are projections from D to $\mathbb{P}$ such that $q \Vdash \pi_i^*(\tau) \trianglelefteq \sigma$ for $i = 0, 1$, then $\pi_0(r) = \pi_1(r)$ for all $r \leq q$ in D.*

Applying this lemma, we get that whenever $G \subseteq \llcorner(\mu,\kappa)$ is generic, and $\langle \mathbb{P}, p, \tau \rangle \in G$ is a coding condition, then we can define a local filter $G_{\mathbb{P}}^\tau$ as $\{ q \in \mathbb{P} : \langle \mathbb{P}, q, \tau \rangle \in G \}$, which will be $\mathbb{P}$-generic over V. (For non-coding conditions, this definition may not yield a filter.)

Let us begin the argument for Theorem 1. Per the discussion after Lemma 7, Let $\mathbb{S} \subseteq \mathbb{B} = \mathcal{B}(\mathrm{Col}(\mu, {<}\kappa) * \mathrm{Add}(\kappa))$ be the complete subalgebra corresponding to $\llcorner(\mu,\kappa)$, and let $\pi_{\mathbb{A}}, \pi_{\mathbb{S}}$ be the canonical projections from $\mathbb{B}$ to $\mathbb{A}(\mu,\kappa), \mathbb{S}$ respectively, so that $\pi_{\mathbb{A}} = \pi_{\mathbb{A}} \circ \pi_{\mathbb{S}}$. Let $H \subseteq \mathbb{A}(\mu,\kappa)$ be a generic filter over V, and let $X = \dot{X}_{\mathbb{A}}^H \subseteq \kappa$ be the corresponding set of ordinals that generates H. Let $\llcorner(\mu,\kappa)/H$ be the quotient forcing via $\pi_{\mathbb{A}} \restriction \mathbb{S}$.

Claim 9. *There is a dense set of $\langle \mathbb{P}, p, \tau \rangle \in \llcorner(\mu,\kappa)/H$ for which there exists $\alpha < \kappa$ with the following properties:*

(1) *α is regular in V and of cardinality μ in $V[X \cap \alpha]$.*
(2) *$V[X]$ is a κ-strategically closed forcing extension of $V[X \cap \alpha]$.*
(3) *$\langle \mathbb{P}, p, \tau \rangle$ is a coding condition such that $\Vdash_{\mathbb{P}} \mathrm{dom}\,\tau = \alpha$.*

Furthermore, every two conditions $\langle \mathbb{P}, p, \tau \rangle, \langle \mathbb{Q}, q, \sigma \rangle$ as in (3) are equivalent in the separative quotient of $\llcorner(\mu,\kappa)/H$.

Proof. Let $a \in \mathbb{A}(\mu,\kappa)$ and $\langle \mathbb{Q}, q, \sigma \rangle \in \llcorner(\mu,\kappa)$ be arbitrary such that $a \Vdash \langle \mathbb{Q}, q, \sigma \rangle \in \llcorner(\mu,\kappa)/\dot{H}$, so that if $s \in \mathbb{S}$ corresponds to $\langle \mathbb{Q}, q, \sigma \rangle$, then $a \leq \pi_{\mathbb{A}}(s)$. We can find $\langle p, \tau \rangle \in \mathrm{Col}(\mu, {<}\kappa) * \mathrm{Add}(\kappa)$ such that:

(1) $\pi_{\mathbb{A}}(p,\tau) \leq a$.
(2) There is $\alpha < \kappa$ such that:
  (a) α is regular.
  (b) $p \in \mathrm{Col}(\mu, {\leq}\alpha)$.
  (c) τ is a $\mathrm{Col}(\mu, {\leq}\alpha)$-name forced to be a condition of length α.

(3) $\psi(p,\tau)$ is a coding condition below $\langle \mathbb{Q}, q, \sigma \rangle$.

Let $G * X \subseteq \mathrm{Col}(\mu, <\kappa) * \dot{\mathrm{Add}}(\kappa)$ be a generic possessing $\langle p, \tau \rangle$. Let $G' = G \upharpoonright (\alpha + 1)$. Then $V[X \cap \alpha] = V[G']$, and $V[G * X]$ is a μ-closed forcing extension of $V[G']$. Since μ-strategic closure is inherited by subforcings, $V[X]$ is a μ-strategically closed extension of $V[X \cap \alpha]$. If $H \subseteq \mathbb{A}(\mu, \kappa)$ and $K \subseteq \mathsf{L}(\mu, \kappa)$ are the induced generic filters, then $\psi(p, \tau) \in K$, and thus $\langle \mathrm{Col}(\mu, \leq \alpha), p, \tau \rangle \in \mathsf{L}(\mu, \kappa)/H$, and it extends $\langle \mathbb{Q}, q, \sigma \rangle$. By the arbitrariness of a and $\langle \mathbb{Q}, q, \sigma \rangle$, the set of conditions satisfying properties (1)–(3) with respect to some α is forced to be dense.

For the last claim, suppose $\langle \mathbb{P}, p, \tau \rangle \in \mathsf{L}(\mu, \kappa)/H$ is coding and of length α. Then it is possible to force further to obtain a generic $G \subseteq \mathsf{L}(\mu, \kappa)$ such that $\dot{X}_{\mathsf{L}}^G = X$. Since $\langle \mathbb{P}, p, \tau \rangle$ is coding, there is in $V[X \cap \alpha]$ a unique filter $F \subseteq \mathbb{P}$ that is generic over V and such that τ^F is the characteristic function of $X \cap \alpha$; namely $F = G_{\mathbb{P}}^\tau$. Likewise, if $\langle \mathbb{Q}, q, \sigma \rangle \in \mathsf{L}(\mu, \kappa)/H$ is coding and of length α, then $G_{\mathbb{Q}}^\sigma$ can be defined in $V[X \cap \alpha]$. There is a $\mathbb{P}$-name $\dot{g}_{\mathbb{Q}}$ for a filter on $\mathbb{Q}$ and some $p_0 \leq p$ in $G_{\mathbb{P}}^\tau$ such that $p_0 \Vdash_{\mathbb{P}}$ "$\dot{g}_{\mathbb{Q}}$ is the unique V-generic filter F on $\mathbb{Q}$ such that $\sigma^F = \tau^{\dot{G}_{\mathbb{P}}}$." Likewise, there is $\mathbb{Q}$-name $\dot{g}_{\mathbb{P}}$ for a filter on $\mathbb{P}$ and some $q_0 \leq q$ in $G_{\mathbb{Q}}^\sigma$ such that $q_0 \Vdash_{\mathbb{Q}}$ "$\dot{g}_{\mathbb{P}}$ is the unique V-generic filter F on $\mathbb{P}$ such that $\tau^F = \sigma^{\dot{G}_{\mathbb{Q}}}$." Note that $(\dot{g}_{\mathbb{Q}})^{G_{\mathbb{P}}^\tau} = G_{\mathbb{Q}}^\sigma$, and $(\dot{g}_{\mathbb{P}})^{G_{\mathbb{Q}}^\sigma} = G_{\mathbb{P}}^\tau$. Let $p_1 \leq p_0$ be in $G_{\mathbb{P}}^\tau$ and force that whenever $g \subseteq \mathbb{P}$ is generic, and $h = (\dot{g}_{\mathbb{Q}})^g$, then $g = (\dot{g}_{\mathbb{P}})^h$. Likewise, let $q_1 \leq q_0$ be in $G_{\mathbb{Q}}^\sigma$ and force that whenever $h \subseteq \mathbb{Q}$ is generic, and $g = (\dot{g}_{\mathbb{P}})^h$, then $h = (\dot{g}_{\mathbb{Q}})^g$.

There is $p_2 \leq p_1$ in $G_{\mathbb{P}}^\tau$ such that for all $p' \leq p_2$, $||p' \in \dot{g}_{\mathbb{P}}|| \wedge q_1 \neq 0$, since otherwise, the set of $p' \leq p_1$ such that $q_1 \Vdash p' \notin \dot{g}_{\mathbb{P}}$ would be dense, contradicting the forced genericity of $\dot{g}_{\mathbb{P}}$. Thus $e : p' \mapsto ||p' \in \dot{g}_{\mathbb{P}}|| \wedge q_1$ is a complete embedding of $\mathbb{P} \upharpoonright p_2$ into $\mathcal{B}(\mathbb{Q}) \upharpoonright q^*$, where $q^* = ||p_2 \in \dot{g}_{\mathbb{P}}|| \wedge q_1$. Furthermore, the range of e is dense in the codomain, since if $q' \leq q^*$, then we can take $h \subseteq \mathbb{Q}$ generic with $q' \in h$, and there will be some $p' \leq p_2$ that forces $q' \in \dot{g}_{\mathbb{Q}}$. Thus if $h' \subseteq \mathbb{Q}$ is any generic with $e(p') \in h'$, then $p' \in g = (\dot{g}_{\mathbb{Q}})^{h'}$, and $q' \in (\dot{g}_{\mathbb{Q}})^g = h'$. Thus $e(p') \leq q'$.

If $e^*(\sigma)$ is the translation of σ into a $\mathbb{P}$-name via e, we want to show that $p_2 \Vdash e^*(\sigma) = \tau$. Let $g \subseteq \mathbb{P}$ be generic with $p_2 \in g$, and let $h \subseteq \mathbb{Q}$ be the generic generated by $e[g]$. Then $g = (\dot{g}_{\mathbb{P}})^h$ by the definition of e, and thus $h = (\dot{g}_{\mathbb{Q}})^g$. It follows that $\tau^g = \sigma^h = e^*(\sigma)^g$.

e lifts to an isomorphism $\varphi : \mathcal{B}(\mathbb{P} \upharpoonright p_2) \to \mathcal{B}(\mathbb{Q}) \upharpoonright q^*$. Note that φ is continuous, and $\mathbb{P}$ and $\mathbb{Q}$ appear as μ-closed dense subsets of their respective Boolean completions. The set $D = (\mathbb{P} \upharpoonright p_2) \cap \varphi^{-1}[\mathbb{Q} \upharpoonright q^*]$ is a dense μ-closed subset of $\mathbb{P} \upharpoonright p_2$. Thus $\varphi \upharpoonright D$ witnesses that $\langle \mathbb{P}, p_2, \tau \rangle \leq \langle \mathbb{Q}, q, \sigma \rangle$ in $\mathsf{L}(\mu, \kappa)$. Since $p_2 \in G_{\mathbb{P}}^\tau$, $\langle \mathbb{P}, p_2, \tau \rangle \in G$ for any generic $G \subseteq \mathsf{L}(\mu, \kappa)$ with $\dot{X}_{\mathsf{L}}^G = X$. Therefore, whenever we take a generic $G \subseteq \mathsf{L}(\mu, \kappa)/H$ with $\langle \mathbb{P}, p, \tau \rangle \in G$, there will be some $p' \in \mathbb{P}$ such that $\langle \mathbb{P}, p', \tau \rangle \in G$ and $\langle \mathbb{P}, p', \tau \rangle \leq \langle \mathbb{Q}, q, \sigma \rangle$, showing the desired claim. $\square$

Claim 10. *Let $H \subseteq \mathbb{A}(\omega_1, \kappa)$ be generic over V. Then every two conditions in the dense subset of $\mathsf{L}(\omega_1, \kappa)/H$ from Claim 9 are compatible in $\mathsf{L}(\omega_1, \kappa)/H$.*

This claim suffices to prove Theorem 1, since it implies that the quotient forcing between $\mathbb{A}(\omega_1, \kappa)$ and $\mathsf{L}(\omega_1, \kappa)$ is always trivial.

Proof. Let $H \subseteq \mathbb{A}(\omega_1, \kappa)$ be generic, and let $\langle \mathbb{P}, p_0, \tau \rangle, \langle \mathbb{Q}, _0, \sigma \rangle$ be in the dense subset of $\mathsf{L}(\omega_1, \kappa)/H$ from Claim 9, with respective witnessing ordinals $\alpha \leq \beta$. Let $X \subseteq \kappa$ be the set of ordinals that generates H. As in the proof of Claim 9,

there are filters $G_{\mathbb{P}}^{\tau} \subseteq \mathbb{P}$ and $G_{\mathbb{Q}}^{\sigma} \subseteq \mathbb{Q}$ definable in $V[X \cap \alpha], V[X \cap \beta]$ respectively, with the property that $V[X \cap \alpha] = V[G_{\mathbb{P}}^{\tau}]$ and $V[X \cap \beta] = V[G_{\mathbb{Q}}^{\sigma}]$. These filters are the unique generic filters over V that evaluate the names τ, σ to the characteristic functions of $X \cap \alpha, X \cap \beta$ respectively.

Let $q_1 \leq q_0$ be in $G_{\mathbb{Q}}^{\sigma}$ and force that $\dot{g}_{\mathbb{P}}$ is the unique $\mathbb{P}$-generic filter over V that evaluates τ to $\sigma \upharpoonright \alpha$. Note that $(\dot{g}_{\mathbb{P}})^{G_{\mathbb{Q}}^{\sigma}} = G_{\mathbb{P}}^{\tau}$. As in the proof of Claim 9, there must be some $p_1 \leq p_0$ in $G_{\mathbb{P}}^{\tau}$ such that for all $p \leq p_1$, $q_1 \wedge ||p \in \dot{g}_{\mathbb{P}}|| \neq 0$ The map $p \mapsto q_1 \wedge ||p \in \dot{g}_{\mathbb{P}}||$ is a complete embedding of $\mathbb{P} \upharpoonright p_1$ into $\mathcal{B}(\mathbb{Q}) \upharpoonright (q_1 \wedge ||p_1 \in \dot{g}_{\mathbb{P}}||)$.

Since $V[X]$ is an ω_1-strategically closed extension of $V[X \cap \alpha]$, and $V[X \cap \beta]$ is an intermediate model, $V[X \cap \beta]$ is an ω_1-strategically closed extension of $V[X \cap \alpha]$. There is $q_2 \leq q_1$ in $G_{\mathbb{Q}}^{\sigma}$ that forces this, and there is $p_2 \leq p_1$ in $G_{\mathbb{P}}^{\tau}$ such that $e : p \mapsto q_2 \wedge ||p \in \dot{g}_{\mathbb{P}}||$ is a complete embedding of $\mathbb{P} \upharpoonright p_2$ into $\mathcal{B}(\mathbb{Q}) \upharpoonright (q_2 \wedge ||p_2 \in \dot{g}_{\mathbb{P}}||)$. By Theorem 3 and Lemma 5,

$$\mathcal{B}(\mathbb{Q}) \upharpoonright (q_2 \wedge ||p_2 \in \dot{g}_{\mathbb{P}}||) \cong \mathcal{B}(\mathbb{P} \upharpoonright p_2 * \dot{\mathrm{Col}}(\omega_1, \beta)).$$

It follows that there is a countably closed dense $D \subseteq \mathbb{Q}$ and a countably continuous projection $\pi : D \to \mathbb{P}$, with the property that $q_2 \wedge ||p_2 \in \dot{g}_{\mathbb{P}}||$ forces that $\pi[\dot{G}_{\mathbb{Q}}]$ generates $\dot{g}_{\mathbb{P}}$. If $q_3 \leq q_2 \wedge ||p_2 \in \dot{g}_{\mathbb{P}}||$ is in $G_{\mathbb{Q}}^{\sigma}$, then $\langle \mathbb{Q}, q_3, \sigma \rangle \leq \langle \mathbb{P}, p_0, \tau \rangle$, and $\langle \mathbb{Q}, q_3, \sigma \rangle \in \mathsf{L}(\omega_1, \kappa)/H$.   □

We remark that, for general $\mathsf{L}(\mu, \kappa)$, the above argument can be carried out up to the point of getting the complete embedding from $\mathbb{P}$ to $\mathcal{B}(\mathbb{Q})$ (restricted to some conditions) with μ-strategically closed quotient. But perhaps it is not strongly μ-strategically closed, and so Lemma 5 is not applicable.

Let $D \subseteq \mathsf{L}(\mu, \kappa)/H$ be the dense set from Claim 9, and let $f : D \to \kappa$ give the witnessing ordinal α to membership in D. For any generic $G \subseteq \mathsf{L}(\mu, \kappa)/H$, let $Y_G = f[D \cap G]$. Then Y_G is some unbounded subset of κ, and $V[H][Y_G] = V[G]$, by Claim 9. When $\mu = \omega_1$, we have that $Y_G = f[D]$, but perhaps for $\mu > \omega_1$, $Y_G \notin V[H]$.

4. An alternative argument for a dense ideal on ω_2

Theorem 3 and Lemma 5 imply that $\mathbb{A}(\omega_1, \kappa) \times \mathrm{Col}(\omega_1, \kappa)$ is forcing-equivalent to $\mathrm{Col}(\omega_1, \kappa)$. It follows that, for inaccessible $\lambda > \kappa$, $[\mathbb{A}(\omega_1, \kappa) \times \mathrm{Col}(\omega_1, \kappa)] * \dot{\mathrm{Col}}(\kappa, <\lambda)^{\mathbb{A}(\omega_1, \kappa)}$ is equivalent to a reasonable (ω_1, λ)-collapse. This observation leads to an adjustment of the proof of Theroem 2.20 in [2] to show that, if κ is almost-huge with target λ, then in $V^{\mathbb{A}(\omega_1, \kappa) * \dot{\mathrm{Col}}(\kappa, <\lambda)}$, we can lift the almost-hugeness embedding by forcing with $\mathrm{Col}(\omega_1, \kappa)$, without the need for a generic for the quotient between $\mathbb{A}(\omega_1, \kappa)$ and $\mathrm{Col}(\omega_1, <\kappa) * \dot{\mathrm{Add}}(\kappa)$. Choiceless submodels still make some appearance in the lifting procedure. We conclude that in $V^{\mathbb{A}(\omega_1, \kappa) * \dot{\mathrm{Col}}(\kappa, <\lambda)}$, there is a normal ideal I on $\omega_2 = \kappa$ such that $\mathcal{P}(\kappa)/I \cong \mathcal{B}(\mathrm{Col}(\omega_1, \omega_2))$.

We would like to present here a direct argument for the existence of such an ideal on ω_2 after forcing with $\mathsf{L}(\omega_1, \kappa) * \dot{\mathrm{Col}}(\kappa, <\lambda)$, using the idea of Lemma 5 as a key component. This will avoid the use of choiceless submodels. Rather than appeal to Theorem 3 abstractly, we will directly build a strategy witnessing the strong ω_1-strategic closure of $\mathsf{L}(\omega_1, \kappa)$ that will have some additional useful properties.

Let us recall the notion of an inverse limit of partial orders under a system of projections. Suppose $\vec{\mathbb{P}} = \langle \mathbb{P}_\alpha : \alpha < \delta \rangle$ is a sequence of posets and $\vec{\pi} = \langle \pi_{\beta\alpha} : \alpha <$

$\beta < \delta\rangle$ is a sequence such that $\pi_{\beta\alpha} : \mathbb{P}_\beta \to \mathbb{P}_\alpha$ is a projection. Suppose also that whenever $\alpha < \beta < \gamma < \delta$, then $\pi_{\gamma\alpha} = \pi_{\beta\alpha} \circ \pi_{\gamma\beta}$. Such a pair of sequences is called an *inverse system*, and δ is referred to as the *length* of the system. We define the *inverse limit* of the system, $\varprojlim(\vec{\mathbb{P}}, \vec{\pi})$, to be the set of all sequences $\langle p_i : i < \delta\rangle$ such that for $\alpha < \beta < \delta$, $p_\alpha = \pi_{\beta\alpha}(p_\beta)$. The ordering on $\varprojlim(\vec{\mathbb{P}}, \vec{\pi})$ is pointwise.

We also allow inverse systems on which the maps $\pi_{\beta\alpha}$ are not necessarily defined at every point of $\mathbb{P}_\beta$, and the maps are not assumed to commute everywhere. In this case, we take $\varprojlim(\vec{\mathbb{P}}, \vec{\pi})$, to be the set of all sequences $\langle p_i : i < \delta\rangle$ such that for all $\alpha < \beta < \delta$, $p_\beta \in \mathrm{dom}\,\pi_{\beta\alpha}$ and $p_\alpha = \pi_{\beta\alpha}(p_\beta)$.

Lemma 15 of [3] shows that, if we have a commuting inverse system $\langle \vec{\mathbb{P}}, \vec{\pi}\rangle$ of length ω, where each $\mathbb{P}_i$ is assumed to be completely ω_1-closed, and the projections π_{ji} are ω_1-continuous and defined on ω_1-closed dense sets, then the inverse limit is a completely ω_1-closed poset that comes with a natural system of commuting ω_1-continuous projections to each $\mathbb{P}_i$. For inverse systems of uncountable length, it seems that we need to assume that this behavior already occurs at many limit points in order to draw an analogous conclusion.

Let us now define our strategy for Player II in $\mathcal{G}^{\mathrm{I}}_{\omega_1}(\mathsf{L}(\omega_1, \kappa))$. We require II to play coding conditions at each round. We will also carry a record of projections between the posets of II's plays that witness their descending ordering, and we require each of these maps to be defined *everywhere*. This last requirement is important because, per Remark 34 in [3], failure to do this might produce a descending ω-sequence of conditions in $\mathsf{L}(\omega_1, \kappa)$ without a lower bound.

Assume $\langle a_0, b_0, a_1, b_1, \ldots\rangle$ is a run of the game of length $\alpha < \omega_1$ where II plays according to a strategy that satisfies the above requirements. So each b_i is a coding condition of the form $\langle \mathbb{P}_i, p_i, \tau_i\rangle$, and for $i < j < \alpha$, we have an ω_1-continuous projection $\pi_{ji} : \mathbb{P}_j \to \mathbb{P}_i$. First assume $\alpha = \beta + 1$, let $b_\beta = \langle \mathbb{P}, p, \tau\rangle$ be the last move of II, and let $\langle \mathbb{Q}, q, \sigma\rangle$ be the next play of I. First choose a coding condition $\langle \mathbb{P}', p', \tau'\rangle \leq \langle \mathbb{Q}, q, \sigma\rangle$, and let π' witness the ordering $\langle \mathbb{P}', p', \tau'\rangle \leq \langle \mathbb{P}, p, \tau\rangle$. Then let $\mathbb{P}'' = \mathbb{P}' \cap \mathrm{dom}\,\pi'$, which is an ω_1-closed dense suborder, and let $\pi_{\beta\alpha} = \pi' \restriction \mathbb{P}''$. For $\gamma < \beta$, let $\pi_{\alpha\gamma} = \pi_{\beta\gamma} \circ \pi_{\alpha\beta}$. Let $p'' \in \mathbb{P}''$ be below p', and let II's next play be $\langle \mathbb{P}'', p'', \tau'\rangle$.

Next suppose α is a limit. Let $\langle \alpha_i : i < \omega\rangle$ be an increasing sequence converging to α. Let $\langle \langle \mathbb{P}_{\alpha_i}, p_{\alpha_i}, \tau_{\alpha_i}\rangle : i < \omega\rangle$ list the plays of II at rounds α_i. Since each play of II is coding, we have that whenever $i < j < k < \omega$ and $p \leq p_{\alpha_k}$, then $\pi_{\alpha_k,\alpha_i}(p) = \pi_{\alpha_j,\alpha_i} \circ \pi_{\alpha_k,\alpha_j}(p)$. For each $i < \omega$, let $p^*_{\alpha_i} = \inf\{\pi_{\alpha_j,\alpha_i}(p_{\alpha_j}) : i < j < \omega\}$. Let $\mathbb{P}$ be the inverse limit of the system $\langle \langle \mathbb{P}_{\alpha_i} : i < \omega\rangle, \langle \pi_{\alpha_j,\alpha_i} : i < j < \omega\rangle\rangle$. We have that $\langle p^*_{\alpha_i} : i < \omega\rangle \in \mathbb{P}$, and below this condition, the system of restriction maps $p \mapsto p(i)$ is a commuting system of ω_1-continuous projections from $\mathbb{P}$ to the $\mathbb{P}_{\alpha_i}$'s. Therefore, a lower bound to the sequence of plays exists.

Player I responds with some lower bound $\langle \mathbb{Q}, q, \sigma\rangle$ at round α. As explained in [3, footnote 7], $\mathbb{Q}$ does *not* have to project to the inverse limit of the previous rounds. But whatever I plays, II can play a coding condition $\langle \mathbb{P}, p, \tau\rangle$ below it. II then chooses a system of projections $\langle \pi'_{\alpha\beta} : \beta < \alpha\rangle$ from $\mathbb{P}$ to the posets of II's previous plays witnessing that $\langle \mathbb{P}, p, \tau\rangle$ is below them. II then takes $\mathbb{P}_\alpha$ to be $\bigcap_{\beta < \alpha} \mathrm{dom}\,\pi'_{\alpha\beta}$, which is a dense ω_1-closed subset of $\mathbb{P}$, and puts $\pi_{\alpha\beta} = \pi'_{\alpha\beta} \restriction \mathbb{P}_\alpha$. II plays $\langle \mathbb{P}_\alpha, p_\alpha, \tau\rangle$, where $p_\alpha \in \mathbb{P}_\alpha$ is below p.

This completes the construction of the strategy for II; call it Σ. Now, the proof of Lemma 5 gives a way of building a dense copy of $\mathrm{Col}(\omega_1, \kappa)$ in $\mathsf{L}(\omega_1, \kappa) \times \mathrm{Col}(\omega_1, \kappa)$

using Σ. We use it to construct a certain master condition for $\mathsf{L}(\omega_1,\kappa)$ in $\mathsf{L}(\omega_1,\lambda)$, for inaccessible $\lambda > \kappa$. This is similar to Lemma 43 of [3], but simpler.

Let Σ' be a winning strategy for II in $\mathcal{G}^{\mathrm{I}}_{\omega_1}(\mathsf{L}(\omega_1,\kappa) \times \mathrm{Col}(\omega_1,\kappa))$, where the first coordinate follows Σ and the second coordinate just plays below the previous moves arbitrarily (which works by countable closure). Using the construction of Lemma 5, build a dense tree $T \subseteq \mathsf{L}(\omega_1,\kappa) \times \mathrm{Col}(\omega_1,\kappa)$ that is isomorphic to the set of $p \in \mathrm{Col}(\omega_1,\kappa)$ whose domain is a successor ordinal. By the construction, every (upward-closed) branch through the tree comes along with a run of the game where II plays according to Σ', and the plays of II are the nodes of the tree that are in the branch.

Since the posets appearing in $\mathsf{L}(\omega_1,\lambda)$ need to be completely ω_1-closed T is not quite suitable. But we can correct this by artificially adding infima to every bounded branch of T, obtaining $T' \supseteq T$ that is isomorphic to the set of conditions $p \in \mathrm{Col}(\omega_1,\kappa)$ whose domain is an ordinal.

As in the previous section, let $\dot{X}_{\mathsf{L}}$ be the name for the special subset of κ added by $\mathsf{L}(\omega_1,\kappa)$. We interpret this as a T'-name using the factorization above. Let τ_X be a T'-name for the characteristic function of $\dot{X}_{\mathsf{L}}$. Let $\langle \mathbb{P}, p, \tau \rangle$ be a node of T.

Claim 11. *For inaccessible $\lambda > \kappa$, $\langle T', \langle \mathbb{P}, p, \tau \rangle, \tau_X \rangle$ is below $\langle \mathbb{P}, p, \tau \rangle$ in $\mathsf{L}(\omega_1,\lambda)$.*

Proof. For nodes $\langle \mathbb{Q}, q, \sigma \rangle \in T$ below $\langle \mathbb{P}, p, \tau \rangle$, the strategy Σ carries an ω_1-continuous projection $\pi : \mathbb{Q} \to \mathbb{P}$ witnessing the ordering. We define a map $\rho : T' \upharpoonright \langle \mathbb{P}, p, \tau \rangle$ first on the successor nodes, where at such $\langle \mathbb{Q}, q, \sigma \rangle \in T$, we put $\rho(\langle \mathbb{Q}, q, \sigma \rangle) = \pi(q)$ where $\pi : \mathbb{Q} \to \mathbb{P}$ is the projection chosen by Σ. For limit nodes $t \in T'$ below $\langle \mathbb{P}, p, \tau \rangle$, first select a sequence $\langle \langle \mathbb{Q}_i, q_i, \sigma_i \rangle : i < \omega \rangle$ of nodes of T converging to t. Let $\pi_i : \mathbb{Q}_i \to \mathbb{P}$ be the projection chosen by Σ. Since the nodes of T are coding, these projections commute below the conditions q_i, and $\langle \pi_i(q_i) : i < \omega \rangle$ is a descending sequence in $\mathbb{P}$. We define $\rho(t) = \inf_i \pi_i(q_i)$. Again because the nodes in T are coding, it does not matter which ω-sequence converging to t we pick, since any two of them will project to interleaved descending sequences in $\mathbb{P}$.

By construction, ρ is an ω_1-continuous map. To show it is a projection, first suppose $\langle \mathbb{Q}, q, \sigma \rangle$ is a successor node below $\langle \mathbb{P}, p, \tau \rangle$, and $p' \leq \pi(q)$, where π is the projection chosen by Σ. There is $q' \in \mathbb{Q}$ below q, and since $\langle \mathbb{Q}, q', \sigma \rangle \leq \langle \mathbb{Q}, q, \sigma \rangle$ and T is dense, there is $\langle \mathbb{R}, r, \chi \rangle \in T$ below $\langle \mathbb{Q}, q', \sigma \rangle$. If $\pi' : \mathbb{R} \to \mathbb{Q}$ and $\pi'' : \mathbb{R} \to \mathbb{P}$ are the projections chosen by Σ, then $\pi''(r) = \pi \circ \pi'(r) \leq \pi(q') \leq p'$. If t is a limit node below $\langle \mathbb{P}, p, \tau \rangle$, and $p' \leq \rho(r)$, then there is an ω-sequence $\langle \langle \mathbb{Q}_i, q_i, \sigma_i \rangle : i < \omega \rangle$ of nodes of T converging to t, with $\langle \mathbb{Q}_0, q_0, \sigma_0 \rangle = \langle \mathbb{P}, p, \tau \rangle$. If $\mathbb{Q}^*$ is the inverse limit of the $\mathbb{Q}_i$, and q^* is defined by $q^*(i) = \inf_{i < j < \omega} \pi_{ji}(q_j)$, where π_{ji} is the projection chosen by Σ, and σ^* is forced to be the concatenation of the σ_i, then $\langle \mathbb{Q}^*, q^*, \sigma^* \rangle$ is one lower bound to the sequence. Note that $q^*(0) = \rho(t) \geq p'$. Since the restriction map is a projection, we can find another lower bound by strengthening q^* to q^{**} with $q^{**}(0) \leq p'$. Since T is dense, there is some $\langle \mathbb{R}, r, \chi \rangle \in T$ below it, and if $\pi : \mathbb{R} \to \mathbb{P}$ is the projection chosen by Σ, then we must have $\pi(r) \leq p'$, using the fact that $\langle \mathbb{P}, p, \tau \rangle$ is coding.

The claim now follows, since $\langle \mathbb{P}, p, \tau \rangle$ forces in $\mathsf{L}(\omega_1,\kappa)$ that τ is an initial segment of τ_X. $\qquad\square$

Now let us show Theorem 2. Suppose $j : V \to M$ is an elementary embedding with critical point κ, $j(\kappa) = \lambda$, and $M^{<\lambda} \subseteq M$. By [13, Theorem 24.11], we may assume that $j(\lambda) < (\lambda^+)^V$ and $j[\lambda]$ is cofinal in $j(\lambda)$.

Let $G * H \subseteq \mathsf{L}(\omega_1, \kappa) * \dot{\mathrm{Col}}(\kappa, {<}\lambda)$ be generic over V. We want to show that the embedding can be lifted to one with domain $V[G * H]$ by forcing further with $\mathrm{Col}(\omega_1, \kappa)$. Let $g \subseteq \mathrm{Col}(\omega_1, \kappa)$ be generic over $V[G * H]$. By the above discussion, $\mathsf{L}(\omega_1, \kappa) \times \mathrm{Col}(\omega_1, \kappa)$ is equivalent to a completely ω_1-closed tree T'. The forcing $T' * \dot{\mathrm{Col}}(\kappa, {<}\lambda)^{\mathsf{L}(\omega_1, \kappa)}$ is a reasonable (ω_1, λ)-collapse. Let $\langle \mathbb{R}_\alpha : \alpha < \lambda \rangle$ be an increasing sequence of regular suborders witnessing reasonableness, with $\mathbb{R}_\alpha = T'$ for $\alpha \leq \kappa$.

Since $M[G][H][g]$ is a λ-c.c. forcing extension of M, $M[G][H][g]$ is ${<}\lambda$-closed in $V[G][H][g]$. Since $j(\lambda) < (\lambda^+)^V$, $\mathcal{P}(\mathrm{Add}(\lambda))^{M[G][H][g]}$ has size λ in $V[G][H][g]$. Therefore, we can inductively build in $V[G][H][g]$ a filter $K \subseteq \mathrm{Add}(\lambda)$ that is $\mathrm{Add}(\lambda)$-generic over $M[G][H][g]$, with $K \restriction \kappa = \tau_X^G$, where τ_X is the name for the canonical binary sequence of length κ added by $\mathsf{L}(\omega_1, \kappa)$ as above.

Let $\psi : T' * \dot{\mathrm{Col}}(\kappa, {<}\lambda)^{\mathsf{L}(\omega_1, \kappa)} \to \mathsf{L}(\omega_1, \lambda)$ be as in Lemma 7. Then $\langle 1, \tau_X \rangle \in [(G \times g) * H] * K$, and $\psi(1, \tau_x) = \langle T', 1, \tau_X \rangle$. Let $G' \subseteq \mathsf{L}(\omega_1, \lambda)$ be the generic filter generated by $\psi[[(G \times g) * H] * K]$. Claim 11 implies that $G \subseteq G'$. Therefore, we may lift the embedding to $j : V[G] \to M[G']$.

The quotient forcing to get from $M[G']$ to $M[[(G \times g) * H] * K]$ is λ-distributive. Therefore, $M[G']$ is also ${<}\lambda$-closed in $V[G][H][g]$. For each $\alpha < \lambda$, let $H_\alpha = H \cap \mathrm{Col}(\kappa, {<}\alpha)^{V[G]}$. We have that $j[H_\alpha] \in M[G']$ for each $\alpha < \lambda$. Since $\mathrm{Col}(\lambda, {<}j(\lambda))$ is λ-closed in $M[G']$, the union of each $j[H_\alpha]$ is a condition $m_\alpha \in \mathrm{Col}(\lambda, {<}j(\lambda))^{M[G']}$. At this stage, we use the same argument as in [2] and other works to build in $V[G][H][g]$ a filter $H' \supseteq j[H]$ that is $\mathrm{Col}(\lambda, {<}j(\lambda))$-generic over $M[G']$. Namely, we enumerate all the dense open sets that live in $M[G']$ in ordertype λ and meet each one in a way that is compatible with all m_α. This allows us to lift the embedding once again to $j : V[G][H] \to M[G'][H']$.

In $V[G][H]$, we define an ideal I on κ as $\{A \subseteq \kappa : 1 \Vdash_{\mathrm{Col}(\omega_1, \kappa)} \kappa \notin j(A)\}$. It is easy to see that this is a normal ideal on κ. We define a map $e : \mathcal{P}(\kappa)/I \to \mathcal{B}(\mathrm{Col}(\omega_1, \kappa))$ by $e([A]_I) = ||\kappa \in j(A)||$. The argument for Claim 45 of [3] shows that e is actually an isomorphism. This completes the proof.

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