Extendible cardinals, and Laver-generic large cardinal axioms for extendibility

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Abstract

We introduce (super-$C^{(\infty)}$-)Laver-generic large cardinal axioms for extendibility ((super-$C^{(\infty)}$-)LgLCAs for extendible, for short), and show that most of the previously known consequences of the (super-$C^{(\infty)}$-)LgLCAs for ultrahuge already follow from members of this family of axioms.

The consistency of the LgLCAs for extendible (for transfinitely iterable classes of posets) follows from an extendible cardinal while the consistency of super-$C^{(\infty)}$-LgLCAs for extendible follows from a model with a super-$C^{(\infty)}$-extendible cardinal. If κ is an almost-huge cardinal, there are cofinally many $\kappa_0 < \kappa$ such that $V_\kappa \models$ "κ_0 is super-$C^{(\infty)}$ extendible".

In contrast, it is known that each of (super-$C^{(\infty)}$-)LgLCAs for hyperhugeness for a transfinitely iterable class of posets, axioms apparently stronger than the corresponding axioms for ultrahugeness, is equiconsistent with the existence of a genuine (super-$C^{(\infty)}$-)hyperhuge cardinal.

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1 Introduction

The present note is a short version of the more extensive [5] in preparation.

In Section 2, we begin with reviewing known characterizations of extendible cardinals (Proposition 2.4). We then look into the super-$C^{(n)}$ and super-$C^{(\infty)}$ large cardinal versions of extendibility, and give their characterizations (Proposition 2.6, Theorem 2.7).

In Section 3, we show that if V_κ satisfies the second order Vopěnka principle then there are cofinally many $\kappa_0 < \kappa$ such that $V_\kappa \models$ "κ_0 is super-$C^{(\infty)}$ extendible" (Proposition 3.2).

In Section 4, we introduce Laver-generic large cardinal versions of these large cardinals, and the axioms asserting the existence of a/the Laver-generic large cardinals — the (super-$C^{(\infty)}$-) $\mathcal{P}$-Laver-generic large cardinal axioms for extendibility ((super-$C^{(\infty)}$-)LgLCAs for extendible, for short) for various classes $\mathcal{P}$ of posets, and show that most of the previously known consequences of the (super-$C^{(\infty)}$-)LgLCAs for ultrahugeness already follow from members of this family of axioms.

The consistency of the LgLCAs for extendible (for transfinitely iterable classes of posets) follows from an extendible cardinal while the consistency of super-$C^{(\infty)}$-LgLCAs for extendible for such classes of posets follow from a model with super-$C^{(\infty)}$ extendible cardinal (see Theorem 5.2).

In contrast, it is known that (super-$C^{(\infty)}$-)LgLCAs for hyperhugeness for transfinitely iterable class $\mathcal{P}$ of posets, axioms apparently stronger than the corresponding axioms for ultrahugeness, are equiconsistent with the existence of a genuine (super-$C^{(\infty)}$-)hyperhuge ([9]).

Our notation is standard, and mostly compatible with that of [18], [19], and/or [20], but with the following slight deviations: "$j : M \xrightarrow{\prec}_\kappa V$" expresses the situation that M and N are transitive (sets or classes), j is an elementary embedding of M into N and κ is the critical point of j. We use letters with under-tilde to denote $\mathbb{P}$-names for a poset $\mathcal{P}$. Underline added to a symbol like $\underline{\alpha}$ emphasizes that the symbol is used to denote a variable in a language, mostly the language of ZFC which is denoted by $\mathcal{L}_\in$. A letter with under-bracket like $\underbracket{c}$ emphasizes that the

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The most recent and extended version of this paper possibly icluding more details and proofs than the present one is downloadable as:
`https://fuchino.ddo.jp/papers/RIMS2024-extendible-x.pdf`

letter denotes a new constant symbol added to the language.

We adopt the notation of [1] and denote $C^{(n)} := \{\alpha \in \mathrm{On} : V_\alpha \prec_{\Sigma_n} \mathsf{V}\}$" for $n \in \mathbb{N}$. Intuitively we put $C^{(\infty)} := \{\alpha \in \mathrm{On} : V_\alpha \prec \mathsf{V}\}$, though this is not a definable class in the language of ZFC, due to undefinability of the truth. For each (transitive) set model M, however, $(C^{(\infty)})^M$ is a(n existent) set except that the first order logic in this context is not the metamathematical one but rather the corresponding set in ZFC.

The following (almost trivial) lemma is often used without mention:

Lemma 1.1 (see e.g. Section 1 in Bagaria [1]) *For an uncountable cardinal α, $\mathcal{H}(\alpha) = V_\alpha$ if and only if $V_\alpha \prec_{\Sigma_1} \mathsf{V}$.*

If $V_\alpha \prec_{\Sigma_1} \mathsf{V}$ then α is an uncountable strong limit cardinal.

Thus, we have

$$C^{(1)} = \{\alpha : \alpha \text{ is an uncountable limit cardinal with } V_\alpha = \mathcal{H}(\alpha)\}. \qquad \square$$

The following are results of an examination of what was suggested by Gabriel Goldberg in a discussion with the author during his visit to Kobe after the RIMS Set Theory Workshop 2024. Toshimichi Usuba pointed out some elementary flows in early sketches of the note. The author is grateful for their comments. I would like to thank Andreas Leitz for giving me a permission to write an exposition of his proof of Theorem 2.7 in the extended version of the present article mentioned above.

Back in the summer of 2015, I enjoyed a pleasant walk around the port of Yokohama with Joel Hamkins when we were together on the way from Tokyo to Kyoto and made a short stop in Yokohama. On the walk, Joel told me about his then recent researches and research projects, and one of them was about the Resurrection Axioms.

Now that his Resurrection Axioms are shown to be restricted versions of the LgLCAs (see Theorem 4.1), I notice that what I learned from him on that walk may have influenced me subliminally when I introduced the LgLCAs in the late 2010s. In that case I have to to thank Joel again sincerely also for the nice conversation we had in Yokohama.

2 Extendible and super-$C^{(n)}$-extendible cardinals

In this section we summarize some well-known and some other less well-known facts about extendible cardinals and introduce the super-$C^{(n)}$-extendible cardinals. It appears that the notion of super-$C^{(n)}$-extendible cardinals is equivalent to some other known strong variants of extendibility, see Theorem 2.7. It is yet unknown if similar equivalence is also available for super-$C^{(n)}$-ultrahuge cardinals or super-$C^{(n)}$-hyperhuge cardinals.

It is easy to see that the definition of an extendible cardinal in Kanamori [19] is equivalent to the following modification: a cardinal κ is *extendible* if (2.1): for any $\alpha > \kappa$ there are $\beta \in \mathrm{On}$ and $j : V_\alpha \xrightarrow{\preceq}_\kappa V_\beta$ such that (2.2): $j(\kappa) > \alpha$.

An extendible cardinal is supercompact (see e.g. Proposition 23.6 in [19]). The following is easy to prove:

Lemma 2.1 *If κ is extendible then there are class many measurable cardinals.* ❑

Since existence of a supercompact cardinal does not imply existence of any large cardinal above it, Lemma 2.1 explains the transcendence of extendible cardinals above supercompact.

In Jech [18], extendibility is defined by (2.1) without (2.2). We say in the following that κ is *Jech-extendible* if it satisfies (2.1) but not necessarily (2.2). The two definitions of extendibility are equivalent. In Proposition 2.4 below, we show the equivalence of these two together with some other characterizations of extendibility.

The key fact to Proposition 2.4 is that the elementary embedding in (2.1) can be often lifted to an elementary embedding of the whole universe V.

We call a mapping $f : M \to N$ *cofinal* (in N) if, for all $b \in N$, there is $a \in M$ such that $b \in f(a)$.

Lemma 2.2 (A special case of Lemma 6 in Fuchino-Sakai [8]) *Suppose that θ is a cardinal and $j_0 : \mathcal{H}(\theta) \xrightarrow{\preceq} N$ for a transitive set N. Let $N_0 := \bigcup j_0{}''\mathcal{H}(\theta)$. Then $j_0 : \mathcal{H}(\theta) \xrightarrow{\preceq} N_0$ and j_0 is cofinal in N_0.* ❑

Lemma 2.3 (A special case of Lemma 7 in [8]) *For any regular cardinal θ and any cofinal $j_0 : \mathcal{H}(\theta) \xrightarrow{\preceq} N$, there are j, $M \subseteq \mathsf{V}$ such that $j : \mathsf{V} \xrightarrow{\preceq} M$, $N \subseteq M$ and $j_0 \subseteq j$.* ❑

Proposition 2.4 *For a cardinal κ the following are equivalent:*

(a) *κ is extendible.*

(b) *κ is Jech-extendible.*

(a′) *For all $\lambda > \kappa$, there are j, $M \subseteq \mathsf{V}$ such that $j : \mathsf{V} \xrightarrow{\preceq}_\kappa M$, $j(\kappa) > \lambda$ and $V_{j(\lambda)} \in M$.*

(b′) *For all $\lambda > \kappa$, there are j, $M \subseteq \mathsf{V}$ such that $j : \mathsf{V} \xrightarrow{\preceq}_\kappa M$, and $V_{j(\lambda)} \in M$.*

Proof. (a) $\Rightarrow$ (b): is clear by definition.

(b) $\Rightarrow$ (a): This can be proved by an argument similar to that of the proof of (b) $\Rightarrow$ (a) of Proposition 2.6 below.

(a) $\Rightarrow$ (a$'$): follows from Lemmas 2.2 and 2.3.

(a$'$) $\Rightarrow$ (b$'$): is trivial.

(b$'$) $\Rightarrow$ (b): is obtained by restricting elementary embeddings on V to V_λ's.

❏ (Proposition 2.4)

The notion of super-$C^{(n)}$-large cardinals was introduced in Fuchino-Usuba [9]. Proposition 2.4 in mind, we define the super-$C^{(n)}$-extendibility as follows: For a natural number n, we call a cardinal κ *super-$C^{(n)}$-extendible* if for any $\lambda_0 > \kappa$ there are $\lambda \geq \lambda_0$ with $V_\lambda \prec_{\Sigma_n} \mathsf{V}$, and j, $M \subseteq \mathsf{V}$ such that $j : \mathsf{V} \overset{\prec}{\rightarrow}_\kappa M$, $j(\kappa) > \lambda$, $V_{j(\lambda)} \in M$ and $V_{j(\lambda)} \prec_{\Sigma_n} \mathsf{V}$.

It is easy to see that the definition of the super-$C^{(n)}$-extendibility is equivalent to the following variation:

Lemma 2.5 *κ is super-$C^{(n)}$-extendible if and only if the following holds:*

(*) *for any $\lambda \geq \kappa$ with $V_\lambda \prec_{\Sigma_n} \mathsf{V}$, there are j, $M \subseteq \mathsf{V}$ such that $j : \mathsf{V} \overset{\prec}{\rightarrow}_\kappa M$, $j(\kappa) > \lambda$, $V_{j(\lambda)} \in M$ and $V_{j(\lambda)} \prec_{\Sigma_n} \mathsf{V}$.* ❏

We call a cardinal κ *super-$C^{(\infty)}$-extendible* if κ is super-$C^{(n)}$-extendible for all $n \in \omega$. In general, we cannot formulate the assertion "κ is super-$C^{(\infty)}$-extendible" in the language of ZF since we would need an infinitary logic to do this unless we are allowed to introduce a new constant symbol for the large cardinal to refer it across infinitely many formulas. However, there are certain situations where we can say that a cardinal is super-$C^{(\infty)}$-extendible. One of them is when we are talking about a cardinal in a set model. In this case, being "super-$C^{(\infty)}$-extendible" in the model is an $\mathcal{L}_{\omega_1,\omega}$ sentence which is satisfied by the cardinal in the model. Another situation is when we are talking about a cardinal in an inner model and the cardinal is definable in V (e.g. as $2^{\aleph_0}$ in the outer model). Note that in the latter case, we can formulate the super-$C^{(\infty)}$-extendibility of the cardinal in infinitely many formulas, and hence n in this case ranges only over metamathematical natural numbers.

Similarly to Proposition 2.4, we have the following equivalence:

Proposition 2.6 *For a cardinal κ and $n \geq 1$, the following are equivalent:*

(a) *For any $\lambda_0 > \kappa$ there are $\lambda > \lambda_0$ with $V_\lambda \prec_{\Sigma_n} \mathsf{V}$, j_0, and μ such that $j_0 : V_\lambda \overset{\prec}{\rightarrow}_\kappa V_\mu$, $j(\kappa) > \lambda$, and $V_\mu \prec_{\Sigma_n} \mathsf{V}$.*

(b) *For any $\lambda_0 > \kappa$ there are $\lambda > \lambda_0$ with $V_\lambda \prec_{\Sigma_n} \mathsf{V}$, j_0, and μ such that $j_0 : V_\lambda \overset{\prec}{\rightarrow}_\kappa V_\mu$, and $V_\mu \prec_{\Sigma_n} \mathsf{V}$ (without the condition "$j(\kappa) > \lambda$").*

(a′) *κ is super-$C^{(n)}$-extendible.*

(b′) *for any $\lambda_0 > \kappa$ there are $\lambda \geq \lambda_0$ with $V_\lambda \prec_{\Sigma_n} \mathsf{V}$, and j, $M \subseteq \mathsf{V}$ such that $j : \mathsf{V} \xrightarrow{\preceq}_\kappa M$, $V_{j(\lambda)} \in M$, and $V_{j(\lambda)} \prec_{\Sigma_n} \mathsf{V}$.* ❑

Proof. The proof is similar to that of Lemma 2.4. We only show (b) ⇒ (a). The following proof is a modification of the proof of Lemma 2.4, (b) ⇒ (a) given by Farmer S in [26].

Assume, toward a contradiction, that κ satisfies (b) but not (a). Then there is a γ such that

(2.3) for all sufficiently large $\lambda > \kappa$, if (2.4): $V_\lambda \prec_{\Sigma_n} \mathsf{V}$, and μ, j are such that (2.5): $j : V_\lambda \xrightarrow{\preceq}_\kappa V_\mu$ and (2.6): $V_\mu \prec_{\Sigma_n} \mathsf{V}$, then $j(\kappa) < \gamma$.

In the following, let γ be the least such γ.

Claim 2.6.1 *γ is a limit ordinal. For all sufficiently large λ with* (2.4) *and for all $\xi < \gamma$, there are μ, j with* (2.5), (2.6) *such that $j(\kappa) > \xi$.*

⊢ Suppose γ is not a limit ordinal, say $\gamma = \xi + 1$. Then there are cofinally many $\lambda \in$ On such that $V_\lambda \prec_{\Sigma_n} \mathsf{V}$ (actually $\lambda \in$ *Card*, see Lemma 1.1), and there are j and μ with (2.5), (2.6) and $j(\kappa) = \xi$. By restricting of j's as right above, it follows that, for all $\lambda > \xi$ with $V_\lambda \prec_{\Sigma_n} \mathsf{V}$, there are j and μ as above.

Let λ^* be a sufficiently large such λ where "sufficiently large" is meant in terms of (2.3). Let j^* and μ^* be such that $j^* : V_{\lambda^*} \xrightarrow{\preceq}_\kappa V_{\mu^*}$, $j^*(\kappa) = \xi$, and $V_{\mu^*} \prec_{\Sigma_n} \mathsf{V}$.

Since $\lambda^* \leq \mu^*$, there is also $k : V_{\mu^*} \xrightarrow{\preceq}_\kappa V_{\nu^*}$ such that $V_{\nu^*} \prec_{\Sigma_n} \mathsf{V}$ and $k(\kappa) = \xi$. But then we have $k \circ j^* : V_{\lambda^*} \xrightarrow{\preceq}_\kappa V_{\nu^*}$ and $k \circ j^*(\kappa) = k(\xi) > k(\kappa) = \xi$. This is a contradiction to (2.3).

The second assertion of the claim follows from this and the minimality of γ.

⊣ (Claim 2.6.1)

Claim 2.6.2 *For all sufficiently large $\mu > \kappa$ with $V_\mu \prec_{\Sigma_n} \mathsf{V}$, and k, ν with $V_\nu \prec_{\Sigma_n} \mathsf{V}$ and $k : V_\mu \xrightarrow{\preceq}_\kappa V_\nu$, we have $k''\gamma \subseteq \gamma$.*

⊢ Suppose otherwise. Then we find $\xi < \gamma$ such that, for cofinally many $\mu > \kappa$ with $V_\mu \prec_{\Sigma_n} \mathsf{V}$, there are ν, k such that $k : V_\mu \xrightarrow{\preceq}_\kappa V_\nu$, $V_\nu \prec_{\Sigma_n} \mathsf{V}$ and $k(\xi) \geq \gamma$. By considering restrictions of k's as above, we conclude that for all $\mu > \xi$ with $V_\mu \prec_{\Sigma_n} \mathsf{V}$, there are ν and k as above.

Let $\lambda > \xi$ and j (together with rechosen μ and k for this λ) be such that $V_\lambda \prec_{\Sigma_n} \mathsf{V}$, $j : V_\lambda \xrightarrow{\preceq}_\kappa V_\mu$ and $j(\kappa) > \xi$ (possible by the second half of Claim 2.6.1). Then we have $k \circ j : V_\lambda \xrightarrow{\preceq}_\kappa V_\nu$ and $k \circ j(\kappa) > k(\xi) \geq \gamma$.

Since λ, μ, ν, j, k can be chosen such that λ is sufficiently large (in terms of (2.3)), this is a contradiction. $\dashv$ (Claim 2.6.2)

Now, let $\lambda > \kappa$ be sufficiently large with $\lambda \geq \gamma + 2$, $V_\lambda \prec_{\Sigma_n} \mathsf{V}$, and $j : V_\lambda \overset{\preceq}{\rightarrow}_\kappa V_\mu$ with $V_\mu \prec_{\Sigma_n} \mathsf{V}$. By Claim 2.6.2, we have $j''\gamma \subseteq \gamma$.

Case 1. $cf(\gamma) = \omega$. Then $j(\gamma) = \gamma$ and hence $j \upharpoonright V_{\gamma+2} : V_{\gamma+2} \overset{\preceq}{\rightarrow}_\kappa V_{\gamma+2}$. This is a contradiction to Kunen's proof (see e.g. Kanamori [19], Corollary 23.14).

Case 2. $cf(\gamma) > \omega$. then, letting $\kappa_0 := \kappa$, $\kappa_{n+1} := j(\kappa_n)$ for $n \in \omega$ and $\kappa_\omega := \sup_{n\in\omega} \kappa_n$, we have $\kappa_\omega < \gamma$, and $j \upharpoonright V_{\kappa_\omega+2} : V_{\kappa_\omega+2} \overset{\preceq}{\rightarrow}_\kappa V_{\kappa_\omega+2}$. This is again a contradiction to Kunen's proof. ❑ (Proposition 2.6)

Note that variants of (a), (b) and (b') in Proposition 2.6 similar to (*) of Lemma 2.5 can also be proved to be equivalent to the super-$C^{(n)}$-extendibility of κ.

Super-$C^{(n)}$-extendibility is actually equivalent to $C^{(n)}$-extendibility of Bagaria [1]. Konstantinos Tsaprounis proved the equivalence for a variant of super-$C^{(n)}$-extendibility which he called $C^{(n)+}$-extendibility in [28].

A cardinal κ is *$C^{(n)}$-extendible* if, for any $\alpha > \kappa$, there is β and $j : V_\alpha \overset{\preceq}{\rightarrow}_\kappa V_\beta$ such that $j(\kappa) > \alpha$ and $V_{j(\kappa)} \prec_{\Sigma_n} \mathsf{V}$.

The following notion is introduced by Benjamin Goodman [12].

A cardinal κ is *supercompact for $C^{(n)}$* if, for any $\lambda > \kappa$ there is $j : \mathsf{V} \overset{\preceq}{\rightarrow}_\kappa M$ such that ${}^\lambda M \subseteq M$ and $C^{(n)} \cap \lambda = (C^{(n)})^M \cap \lambda$.

Andreas Lietz recently found a short proof of the following Theorem 2.7. Goodman apparently proved the equivalence of (a) and (c) in the theorem, but mentioned only the case of $n = 1$ in his [12]. His proof is given in the extended version of the present article.

Theorem 2.7 (Andreas Lietz) *For a cardinal κ and for all $n \geq 1$ the following are equivalent:* (a) *κ is $C^{(n)}$-extendible.*

(b) *κ is super-$C^{(n)}$-extendible.*

(b') *κ is $C^{(n)+}$-extendible.*

(c) *κ is supercompact for $C^{(n+1)}$.* ❑

3 Models with super-$C^{(\infty)}$-extendible cardinals

We prove that there are unboundedly many super-$C^{(\infty)}$-extendible cardinals in V_κ below an almost huge cardinal κ (Theorem 3.3).

For a cardinal κ, we say that V_κ satisfies the *second-order Vopěnka's principle* if for any set $C \subseteq V_\kappa$ of structures of the same signature with $C \notin V_\kappa$ (which is not necessarily a definable subset of V_κ), there are non-isomorphic $\mathfrak{A}$, $\mathfrak{B} \in C$ such that $i : \mathfrak{A} \xrightarrow{\preceq} \mathfrak{B}$ for an elementary embedding i.

The following is well-known (see e.g. Jech [18], Lemma 20.27), and attributed to William C. Powell.

Lemma 3.1 (W.C. Powell [24]) *If κ is an almost-huge cardinal then V_κ satisfies the second-order Vopěnka's principle.*

Proof. Suppose that $C \subseteq V_\kappa$ where C is a set of structures of the same signature with $C \notin V_\kappa$. Without loss of generality, we may assume that (3.1): C is closed with respect to isomorphism inside V_κ. Then it is enough to show that there are non-isomorphic $\mathfrak{A}$, $\mathfrak{B} \in C$ such that $\mathfrak{A} \prec \mathfrak{B}$.

Let $j : \mathsf{V} \xrightarrow{\preceq}_\kappa M$ be an almost-huge elementary embedding (i.e. M satisfies (3.2): ${}^{j(\kappa)>}M \subseteq M$). Let $\mathfrak{A} \in j(C) \setminus C$ — note that $j(C) \setminus C \neq \emptyset$ since $M \models$ "$rank(j(C)) = j(\kappa) > \kappa$". Let A be the underlying set of the structure $\mathfrak{A}$.

We have (3.3): $j(\mathfrak{A}) \not\cong \mathfrak{A}$ — otherwise $M \models j(\mathfrak{A}) \cong \mathfrak{A} \in j(C)$, and hence $\mathsf{V} \models \mathfrak{A} \in C$ by elementarity. This is a contradiction to the choice of $\mathfrak{A}$.

Let $\mathfrak{A}' := j(\mathfrak{A}) \restriction j''A$.

Claim 3.1.1 (1) $M \models \mathfrak{A}' \in j(C)$.

(2) $M \models \mathfrak{A}' \prec j(\mathfrak{A})$.

$\vdash$ (1): $\mathfrak{A}' \in M$ by (3.2). Since $\mathfrak{A}' \cong \mathfrak{A} \in j(C)$, (3.1) and elementarity imply $M \models \mathfrak{A}' \in j(C)$.

(2): Working in M, we check Vaught's criterion.

Suppose $a_0, ..., a_{n-1} \in j''A$, $a \in j(A)$ and $j(\mathfrak{A}) \models \varphi(a, a_0, ..., a_{n-1})$. Let a'_0, ..., $a'_{n-1} \in A$ be such that $a_0 = j(a'_0), ..., a_{n-1} = j(a'_{n-1})$. Since $M \models \exists \underline{a} \in j(A)\ j(\mathfrak{A}) \models \varphi(\underline{a}, j(a'_0), ..., j(a'_{n-1}))$, it follows that $\mathsf{V} \models \exists \underline{a} \in A\ \mathfrak{A} \models \varphi(\underline{a}, a'_0, ..., a'_{n-1})$. Let $a' \in A$ be such that $\mathsf{V} \models \mathfrak{A} \models \varphi(a', a'_0, ..., a'_{n-1})$. Then $j(a') \in j''A$, and $M \models j(\mathfrak{A}) \models \varphi(j(a'), j(a'_0), ..., j(a'_{n-1}))$ by elementarity, as desired. $\dashv$ (Claim 3.1.1)

Now by (3.3) and Claim 3.1.1, (2), $M \models$ "there are non-isomorphic $\mathfrak{A}, \mathfrak{B} \in j(C)$ such that $\mathfrak{A} \prec \mathfrak{B}$". By elementarity it follows that $\mathsf{V} \models$ "there are non-isomorphic $\mathfrak{A}, \mathfrak{B} \in C$ such that $\mathfrak{A} \prec \mathfrak{B}$". □ (Lemma 3.1)

Proposition 3.2 *Suppose that κ is a Mahlo cardinal, and* (3.4): *V_κ satisfies the second-order Vopěnka's principle. Then there are unboundedly many $\kappa_0 < \kappa$ such that $V_\kappa \models$ "κ_0 is super-$C^{(\infty)}$-extendible".*

Proof. Suppose $\beta < \kappa$. We want to show that there is $\beta < \kappa_0 < \kappa$ such that $V_\kappa \models$ "κ_0 is super-$C^{(\infty)}$-extendible".

Let

$$I := \{\alpha < \kappa \,:\, \alpha \text{ is an } \omega\text{-limit of inaccessible cardinals } \eta \text{ such that } V_\eta \prec \mathsf{V}_\kappa\}.$$

I is cofinal in κ since κ is a Mahlo cardinal.

For each $\alpha \in I$, let $C_\alpha \subseteq \alpha$ be a cofinal subset of α of order-type ω consisting of (increasing sequence of) inaccessible cardinals η^α_n, $n \in \omega$ with $V_{\eta^\alpha_n} \prec V_\kappa$.

Let

$$\mathcal{C} := \{(V_{\alpha+1}, \in, C_\alpha, \xi)_{\xi<\beta} \,:\, \alpha \in I\}.$$

By (3.4), there are $\mathcal{V}_{\alpha+1} = (V_{\alpha+1}, \in, C_\alpha, \xi)_{\xi<\beta}$, $\mathcal{V}_{\beta+1} = (V_{\beta+1}, \in, C_\beta, \xi)_{\xi<\beta} \in \mathcal{C}$ such that there is an elementary embedding $i : \mathcal{V}_{\alpha+1} \xrightarrow{\preceq} \mathcal{V}_{\beta+1}$. Let $n \in \omega$ be arbitrary and let $\kappa_0 := \mathit{crit}(i)$. Then $\alpha > \kappa_0 \geq \beta$. Let $k \in \omega$ be large enough so that we have $\eta^\alpha_k > \kappa_0$.

Since $i(\eta^\alpha_k) \in C_\beta$, we have:

$$V_\kappa \models \text{“}V_{i(\eta^\alpha_k)} \prec_{\Sigma_n} \mathsf{V}\text{”} \ \wedge\ i \restriction V_{\eta^\alpha_k} : V_{\eta^\alpha_k} \xrightarrow{\preceq}_{\kappa_0} V_{i(\eta^\alpha_k)}.$$

Thus $V_\kappa \models \exists \underline{\nu}\, \exists \underline{i}\ (\text{“}V_{\underline{\nu}} \prec_{\Sigma_n} \mathsf{V}\text{”} \ \wedge\ \underline{i} : V_{\eta^\alpha_k} \xrightarrow{\preceq}_{\kappa_0} V_{\underline{\nu}})$. By elementarity, it follows that

$$V_{\eta^\alpha_k} \models \exists \underline{\eta}\, \exists \underline{\nu}\, \exists \underline{i}\ (\text{“}V_{\underline{\eta}} \prec_{\Sigma_n} \mathsf{V}\text{”} \ \wedge\ \text{“}V_{\underline{\nu}} \prec_{\Sigma_n} \mathsf{V}\text{”} \ \wedge\ \underline{i} : V_{\eta^\alpha_k} \xrightarrow{\preceq}_{\kappa_0} V_{\underline{\nu}})$$

for all $n \in \omega$.

By Lemma 2.6, this implies $V_{\eta^\alpha_k} \models$ "κ_0 is super-$C^{(n)}$-extendible". By the elementarity $V_{\eta^\alpha_k} \prec V_\kappa$, it follows that $V_\kappa \models$ "κ_0 is super-$C^{(n)}$-extendible" for all $n \in \omega$. □ (Proposition 3.2)

Theorem 3.3 *Suppose that κ is almost-huge. Then there are unboundedly many $\kappa_0 < \kappa$ such that $V_\kappa \models$ "κ_0 is super-$C^{(\infty)}$-extendible".*

Proof. By Lemma 3.1 and Proposition 3.2. □ (Theorem 3.3)

4 LgLCAs and super-$C^{(\infty)}$-LgLCAs for extendibility imply (almost) everything

Laver-generic large cardinals were introduced in [7]. For a class $\mathcal{P}$ of posets and a notion L of large cardinal, a cardinal κ is said to be $\mathcal{P}$-Laver generic L if the statement about the existence of elementary embedding $j : \mathsf{V} \xrightarrow{\preceq}_\kappa M$ for j, $M \subseteq \mathsf{V}$ with the closedness condition C_L of M in the definition of the notion L of large cardinal are replaced with the statement:

(4.1) for any $\mathbb{P} \in \mathcal{P}$, there is a $\mathbb{P}$-name $\underset{\sim}{\mathbb{Q}}$ such that $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{Q}} \in \mathcal{P}$", and for $\mathbb{P} * \underset{\sim}{\mathbb{Q}}$-generic $\mathbb{H}$ there are j, $M \subseteq \mathsf{V}[\mathbb{H}]$ such that $\mathbb{P} * \underset{\sim}{\mathbb{Q}}$, $\mathbb{H} \in M$, $j : \mathsf{V} \xrightarrow{\preceq}_{\kappa} M$, and M satisfies C'_L which is the generic large cardinal variant of the closedness property C_L associated with the notion L of large cardinal.

We usually assume that the class $\mathcal{P}$ of posets satisfies some natural properties. A class $\mathcal{P}$ of posets is (two-step) *iterable* if

(4.2) $\mathcal{P}$ is closed with respect to forcing equivalence, and $\{\mathbb{1}\} \in \mathcal{P}$;

(4.3) $\mathcal{P}$ is closed with respect to restriction. That is, for $\mathbb{P} \in \mathcal{P}$ and $\mathbb{p} \in \mathbb{P}$, we always have $\mathbb{P} \restriction \mathbb{p} \in \mathcal{P}$; and

(4.4) For any $\mathbb{P} \in \mathcal{P}$, and any $\mathbb{P}$-name $\underset{\sim}{\mathbb{Q}}$ of a poset with $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{Q}} \in \mathcal{P}$", we have $\mathbb{P} * \underset{\sim}{\mathbb{Q}} \in \mathcal{P}$.

$\mathcal{P}$ is *transfinitely iterable* if it is iterable and it permits iteration of arbitrary length for an appropriate notion of support with reasonable iteration lemmas.

For supercompactness, the instance of (4.1) for an iterable $\mathcal{P}$ is as follows: a cardinal κ is *$\mathcal{P}$-Laver-generically supercompact* if,

(4.5) for any $\lambda > \kappa$, and for any $\mathbb{P} \in \mathcal{P}$, there is a $\mathbb{P}$-name $\underset{\sim}{\mathbb{Q}}$ such that $\Vdash_{\mathbb{P}}$ "$\mathbb{Q} \in \mathcal{P}$", and, for any $(\mathsf{V}, \mathbb{P} * \underset{\sim}{\mathbb{Q}})$-generic $\mathbb{H}$, there are j, $M \subseteq \mathsf{V}[\mathbb{H}]$ such that $j : \mathsf{V} \xrightarrow{\preceq}_{\kappa} M$, $j(\kappa) > \lambda$, $\mathbb{P}$, $\mathbb{P} * \underset{\sim}{\mathbb{Q}}$, $\mathbb{H} \in M$, $j''\lambda \in M$.

Note that, in (4.5), the closure property "${}^{\lambda}M \subseteq M$" in the usual definition of supercompactness is replaced with "$j''\lambda \in M$". For a genuine elementary embedding introduced by some ultrafilter, these two conditions are equivalent (see e.g. Kanamori [19], Proposition 22.4, (b)). This equivalence is no more valid in general for generic embeddings. Never the less, the condition "$j''\lambda \in M$" can be still considered as a certain closure property (see Lemma 3.5 in Fuchino-Rodrigues-Sakai [7]).

We say that a $\mathcal{P}$-Laver-generically supercompact cardinal κ is *tightly $\mathcal{P}$-Laver-generically supercompact* if additionally, we have $|\,RO(\mathbb{P} * \underset{\sim}{\mathbb{Q}})\,| \leq j(\kappa)$.

A (tightly) $\mathcal{P}$-Laver-generically supercompact cardinal is often decided uniquely as the cardinal $\kappa_{\mathfrak{refl}} := \sup(\{2^{\aleph_0}, \aleph_2\})$. This is the case, if $\mathcal{P}$ is the class of all σ-closed posets. Then CH holds under the existence of a $\mathcal{P}$-Laver-generically supercompact κ and $\kappa = \aleph_2$ $(= \kappa_{\mathfrak{refl}})$.

Similarly, if $\mathcal{P}$ is either the class of all proper posets or the class of all semiproper posets, the existence of a $\mathcal{P}$-generically supercompact κ implies $2^{\aleph_0} = \aleph_2$ and again $\kappa = \kappa_{\mathfrak{refl}}$.

For the case that $\mathcal{P}$ is the class of all ccc posets, it is open whether a $\mathbb{P}$-Laver-generically supercompact cardinal is decided to be $\kappa_{\mathfrak{refl}}$. However a tightly $\mathcal{P}$-Laver-generically supercompact cardinal under the present definition of tightness[1)] is decided to be the continuum ($= \kappa_{\mathfrak{refl}}$) and, in this case, the continuum is extremely large. There is a more general theorem which suggests that for a "natural" class $\mathcal{P}$ of posets, the existence of (tightly) $\mathcal{P}$-Laver generically supercompact cardinal implies that the continuum is either $\aleph_1$ or $\aleph_2$ or else extremely large (see [7], [3], [4]).

The naming "Laver-generic ..." comes from the fact that the standard models with this type of generic large cardinal is created by starting from a large cardinal, and then iterating along with a Laver function for the large cardinal with the support appropriate for the class of posets in consideration. This is exactly the way to create models of PFA and MM. Actually, for $\mathcal{P}$ being the class of all proper posets or that of semi-proper posets, the existence of a $\mathcal{P}$-Laver generically supercompact cardinal implies the strong plus version of the corresponding forcing axiom, and can be considered as an axiomatization of the standard models of them.

In the following, we call the axiom claiming the existence of a/the tightly $\mathcal{P}$-Laver generic L, the *$\mathcal{P}$-Laver-generic large cardinal axiom for the notion of large cardinal L* (*$\mathcal{P}$-LgLCA for L*, for short).

The instances of $\mathcal{P}$-LgLCAs for other notions of large cardinal are summarized in the following chart.

$\mathcal{P}$-LgLCA for	The condition "$j''\lambda \in M$" for "super-compact" is replaced by:
hyperhuge	$j''j(\lambda) \in M$
ultrahuge	$j''j(\kappa) \in M$ and $V_{j(\lambda)}{}^{\mathsf{V}[\mathbb{H}]} \in M$
superhuge	$j''j(\kappa) \in M$
super-almost-huge	$j''j(\mu) \in M$ for all $\mu < j(\kappa)$
extendible	$V_{j(\lambda)}{}^{\mathsf{V}[\mathbb{H}]} \in M$

In Fuchino [3], it is proved that a boldface variant of Resurrection Axiom by Hamkins and Johnstone, [14], [15] for $\mathcal{P}$ and parameters from $\mathcal{H}(\kappa_{\mathfrak{refl}})$ follows $\mathcal{P}$-LgLCA for ultrahuge.

In [9] and [6], it is proved that $\mathcal{P}$-LgLCA for ultrahuge implies a restricted form of Maximality Principle for $\mathcal{P}$ and $\mathcal{H}(\kappa_{\mathfrak{refl}})$. It is shown that LgLCA type axiom formulated in a single formula is incapable to cover the full Maximal Principle. In

[1)] In course of the development of the theory of Laver-genericity we strengthened the definition of tightness such that it still holds in the standard models of Laver genericity

[9], it is proved that the super-$C^{(\infty)}$ version of $\mathcal{P}$-LgLCA for ultrahuge implies the full Maximality Principle for $\mathcal{P}$ and $\mathcal{H}(\kappa_{\mathfrak{refl}})$ ([9], Theorem 4.10).

In the following we show that all the results under (super $C^{(\infty)}$-)LgLCA for ultrahuge mentioned above holds already under (super $C^{(\infty)}$-)LgLCA for extendible.

Let us first check the specific formulation of (super $C^{(\infty)}$-)LgLCA for extendible: a cardinal κ is *tightly $\mathcal{P}$-Laver generically extendible* if, for any $\lambda > \kappa$, and for any $\mathbb{P} \in \mathcal{P}$, there is a $\mathbb{P}$-name $\underset{\sim}{\mathbb{Q}}$ such that $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{Q}} \in \mathcal{P}$" and for any $(\mathsf{V}, \mathbb{P} * \underset{\sim}{\mathbb{Q}})$-generic $\mathbb{H}$, there are j, $M \subseteq \mathsf{V}[\mathbb{H}]$ such that $j : \mathsf{V} \xrightarrow{\prec}_{\kappa} M$, $j(\kappa) > \lambda$, ${V_{j(\lambda)}}^{\mathsf{V}[\mathbb{H}]} \in M$, and $|\, RO(\mathbb{P} * \underset{\sim}{\mathbb{Q}})\,| \leq j(\kappa)$.

The *$\mathcal{P}$-Laver-generic large cardinal axiom* for the notion of extendibility (*$\mathcal{P}$-LgLCA for extendible*, for short) is the assertion that $\kappa_{\mathfrak{refl}}$ is a/the tightly $\mathcal{P}$-Laver-generic extendible cardinal.

A cardinal κ is *tightly super-$C^{(\infty)}$-$\mathcal{P}$-Laver generically extendible* if, for any $n \in \mathbb{N}$, $\lambda_0 > \kappa$, and $\mathbb{P} \in \mathcal{P}$, there are $\lambda \geq \lambda_0$ and a $\mathbb{P}$-name $\underset{\sim}{\mathbb{Q}}$ such that $V_\lambda \prec_{\Sigma_n} \mathsf{V}$, $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{Q}} \in \mathcal{P}$" and for any $(\mathsf{V}, \mathbb{P} * \underset{\sim}{\mathbb{Q}})$-generic $\mathbb{H}$, there are j, $M \subseteq \mathsf{V}[\mathbb{H}]$ such that ${V_{j(\lambda)}}^{\mathsf{V}[\mathbb{H}]} \prec_{\Sigma_n} \mathsf{V}[\mathbb{H}]$, $j : \mathsf{V} \xrightarrow{\prec}_{\kappa} M$, $j(\kappa) > \lambda$, ${V_{j(\lambda)}}^{\mathsf{V}[\mathbb{H}]} \in M$, and $|\, RO(\mathbb{P} * \underset{\sim}{\mathbb{Q}})\,| \leq j(\kappa)$.

The *super-$C^{(\infty)}$-$\mathcal{P}$-Laver-generic large cardinal axiom* for the notion of extendibility (*super-$C^{(\infty)}$-$\mathcal{P}$-LgLCA for extendible*, for short) is the assertion that $\kappa_{\mathfrak{refl}}$ is a/the tightly super $C^{(\infty)}$-$\mathcal{P}$-Laver-generic extendible cardinal.

The following boldface version of the Resurrection Axioms was studied by Hamkins and Johnstone in [15]: For a class $\mathcal{P}$ of posets and a definition $\mu^\bullet$ of a cardinal (e.g. as $\aleph_1$, $\aleph_2$, $2^{\aleph_0}$, $(2^{\aleph_0})^+$. etc.) the *Resurrection Axiom in Boldface for $\mathcal{P}$ and $\mathcal{H}(\mu^\bullet)$* is defined by:

$\mathbb{RA}^{\mathcal{P}}_{\mathcal{H}(\mu^\bullet)}$: For any $A \subseteq \mathcal{H}(\mu^\bullet)$ and any $\mathbb{P} \in \mathcal{P}$, there is a $\mathbb{P}$-name $\underset{\sim}{\mathbb{Q}}$ of poset such that $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{Q}} \in \mathcal{P}$" and, for any $(\mathsf{V}, \mathbb{P} * \underset{\sim}{\mathbb{Q}})$-generic $\mathbb{H}$, there is $A^* \subseteq \mathcal{H}(\mu^\bullet)^{\mathsf{V}[\mathbb{H}]}$ such that $(\mathcal{H}(\mu^\bullet)^{\mathsf{V}}, A, \in) \prec (\mathcal{H}(\mu^\bullet)^{\mathsf{V}[\mathbb{H}]}, A^*, \in)$.

Theorem 4.1 (A slight improvement of Theorem 7.1 in [3]) *For an iterable class $\mathcal{P}$ of posets, assume that $\mathcal{P}$-LgLCA for extendible holds. Then $\mathbb{RA}^{\mathcal{P}}_{\mathcal{H}(\kappa_{\mathfrak{refl}})}$ holds.*

Proof. Suppose $A \subseteq \mathcal{H}(\kappa_{\mathfrak{refl}})$ and $\mathbb{P} \in \mathcal{P}$. By the tightly $\mathcal{P}$-Laver-generic extendibility of $\kappa_{\mathfrak{refl}}$, there is a $\mathbb{P}$-name $\underset{\sim}{\mathbb{Q}}$ of a poset with $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{Q}} \in \mathcal{P}$" such that, for $(\mathsf{V}, \mathbb{P} * \underset{\sim}{\mathbb{Q}})$-generic $\mathbb{H}$, there are j, $M \subseteq \mathsf{V}[\mathbb{H}]$ with

(4.6) $\quad j : \mathsf{V} \xrightarrow{\prec}_{\kappa_{\mathfrak{refl}}} M$,

(4.7) $\quad j(\kappa_{\mathfrak{refl}}) = |\, RO(\mathbb{P} * \underset{\sim}{\mathbb{Q}})\,|$,

(4.8) $\quad \mathbb{P}$, $\mathbb{H} \in M$, and

(4.9) $\quad {V_{j(\lambda)}}^{\mathsf{V}[\mathbb{H}]} \in M.$

Without loss of generality, we may assume that the underlying set of $\mathbb{P} * \underset{\sim}{\mathbb{Q}}$ is $j(\kappa_{\mathfrak{refl}})$. Since $crit(j) = \kappa_{\mathfrak{refl}}$, $j(a) = a$ for all $a \in (\mathcal{H}(\kappa_{\mathfrak{refl}}))^{\mathsf{V}}$.

By (4.9), we have $\mathcal{H}(j(\kappa_{\mathfrak{refl}}))^{\mathsf{V}[\mathbb{H}]} \subseteq M$, and hence $j(\mathcal{H}(\kappa_{\mathfrak{refl}})) = \mathcal{H}(j(\kappa_{\mathfrak{refl}}))^{M} = \mathcal{H}(j(\kappa_{\mathfrak{refl}}))^{\mathsf{V}[\mathbb{H}]}$.

Thus, we have

$$id_{\mathcal{H}(\kappa_{\mathfrak{refl}})^{\mathsf{V}}} = j \restriction \mathcal{H}(\kappa_{\mathfrak{refl}})^{\mathsf{V}} : \ (\mathcal{H}(\kappa_{\mathfrak{refl}})^{\mathsf{V}}, A, \in) \ \overset{\prec}{\to} \ (\mathcal{H}(j(\kappa_{\mathfrak{refl}}))^{\mathsf{V}[\mathbb{H}]}, j(A), \in).$$

□ (Theorem 4.1)

Recurrence Axioms are introduced in Fuchino-Usuba [9].

For an iterable class $\mathcal{P}$ of posets, a set A (of parameters), and a set Γ of $\mathcal{L}_{\in}$- formulas, *$\mathcal{P}$-Recurrence Axiom^{+} for formulas in Γ with parameters from A* ($(\mathcal{P}, A)_{\Gamma}$-$\mathsf{RcA}^{+}$, for short) is the following assertion expressed as an axiom scheme formulated in $\mathcal{L}_{\in}$:

$(\mathcal{P}, A)_{\Gamma}$-$\mathsf{RcA}^{+}$*:* For any $\varphi(\overline{x}) \in \Gamma$ and $\overline{a} \in A$, if $\Vdash_{\mathbb{P}}$ "$\varphi(\overline{a})$", then there is a $\mathcal{P}$-ground W of V such that $\overline{a} \in \mathsf{W}$ and $\mathsf{W} \models \varphi(\overline{a})$.

Here, an inner model M of V is said to be a *$\mathcal{P}$-ground of* V, if there are $\mathbb{P} \in M$ and $\mathbb{G} \in \mathsf{V}$ such that $M \models$ "$\mathbb{P} \in \mathcal{P}$", $\mathbb{G}$ is an $(M, \mathbb{P})$-generic filter, and $\mathsf{V} = Ms[\mathbb{G}]$.

If Γ is the set of all $\mathcal{L}_{\in}$-formulas, we drop the subscript Γ and say simply $(\mathcal{P}, A)$-RcA^{+}.

As it is noticed in [9], $(\mathcal{P}, A)$-RcA^{+} is equivalent to the Maximality Principle $\mathsf{MP}(\mathcal{P}, A)$ (see Proposition 2.2, (2) in [9]).

Theorem 4.2 *Assume $\mathcal{P}$-LgLCA for extendible. Then $(\mathcal{P}, \mathcal{H}(\kappa_{\mathfrak{refl}}))_{\Sigma_2}$-$\mathsf{RcA}^{+}$ holds.*

For the proof of Theorem 4.2, we use the following lemma which should be a well-known fact.

Lemma 4.3 *If α is a limit ordinal and V_α satisfies a sufficiently large finite fragment of* ZFC*, then for any $\mathbb{P} \in V_\alpha$ and* $(\mathsf{V}, \mathbb{P})$*-generic $\mathbb{G}$, we have $V_\alpha[\mathbb{G}] = {V_\alpha}^{\mathsf{V}[\mathbb{G}]}$.*

Proof of Theorem 4.2: Assume that $\kappa = \kappa_{\mathfrak{refl}}$ is tightly $\mathcal{P}$-Laver generically extendible for an iterable class $\mathcal{P}$ of posets.

Suppose that $\varphi = \varphi(\overline{x})$ is Σ_2 formula (in $\mathcal{L}_{\in}$), $\overline{a} \in \mathcal{H}(\kappa)$, and $\mathbb{P} \in \mathcal{P}$ is such that

(4.10) $\quad \mathsf{V} \models \ \Vdash_{\mathbb{P}}$ "$\varphi(\overline{a})$".

Let $\lambda > \kappa$ be such that $\mathbb{P} \in \mathsf{V}_\lambda$ and

(4.11) $\quad V_\lambda \prec_{\Sigma_n} \mathsf{V}$ for a sufficiently large n.

In particular, we may assume that we have chosen the n above so that a sufficiently large fragment of ZFC holds in V_λ ("sufficiently large fragment" means here, in particular, in terms of Lemma 4.3).

Let $\underset{\sim}{\mathbb{Q}}$ be a $\mathbb{P}$-name such that $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{Q}} \in \mathcal{P}$", and for $(\mathsf{V}, \mathbb{P} * \underset{\sim}{\mathbb{Q}})$-generic $\mathbb{H}$, there are j, $M \subseteq \mathsf{V}[\mathbb{H}]$ with

(4.12) $\quad j : \mathsf{V} \overset{\prec}{\rightarrow}_\kappa M$,

(4.13) $\quad j(\kappa) > \lambda$,

(4.14) $\quad \mathbb{P} * \underset{\sim}{\mathbb{Q}}$, $\mathbb{P}$, $\mathbb{H}$, ${V_{j(\lambda)}}^{\mathsf{V}[\mathbb{H}]} \in M$, and

(4.15) $\quad |\, RO(\mathbb{P} * \underset{\sim}{\mathbb{Q}})\,| \leq j(\kappa)$.

By (4.15), we may assume that the underlying set of $\mathbb{P}*\underset{\sim}{\mathbb{Q}}$ is $j(\kappa)$ and $\mathbb{P}*\underset{\sim}{\mathbb{Q}} \in {V_{j(\lambda)}}^{\mathsf{V}}$.

Let $\mathbb{G} := \mathbb{H} \cap \mathbb{P}$. Note that $\mathbb{G} \in M$ by (4.14) and we have

$$\text{(4.16)} \quad {V_{j(\lambda)}}^M \underbrace{=}_{\text{by (4.14)}} {V_{j(\lambda)}}^{\mathsf{V}[\mathbb{H}]} \overbrace{=}^{\substack{\text{Since } {V_{j(\lambda)}}^M \ (= V_{j(\lambda)}^{\mathsf{V}[\mathbb{H}]}) \text{ satisfies a sufficiently large fragment of } \mathsf{ZFC} \\ \text{by elementarity of } j\text{, and hence the equality follows by Lemma 4.3}}} {V_{j(\lambda)}}^{\mathsf{V}}[\mathbb{H}].$$

Thus, by the definability of grounds and by (4.14), we have ${V_{j(\lambda)}}^{\mathsf{V}} \in M$ and ${V_{j(\lambda)}}^{\mathsf{V}}[\mathbb{G}] \in M$.

Claim 4.2.1 ${V_{j(\lambda)}}^{\mathsf{V}}[\mathbb{G}] \models \varphi(\overline{a})$.

$\vdash$ By Lemma 4.3, ${V_\lambda}^{\mathsf{V}}[\mathbb{G}] = {V_\lambda}^{\mathsf{V}[\mathbb{G}]}$, and ${V_{j(\lambda)}}^{\mathsf{V}}[\mathbb{G}] = {V_{j(\lambda)}}^{\mathsf{V}[\mathbb{G}]}$ by (4.11) and (4.16). By (4.11), both ${V_\lambda}^{\mathsf{V}}[\mathbb{G}]$ and $V_{j(\lambda)}^{\mathsf{V}}[\mathbb{G}]$ satisfy large enough fragment of ZFC. In particular,

(4.17) $\quad {V_\lambda}^{\mathsf{V}}[\mathbb{G}] \prec_{\Sigma_1} {V_{j(\lambda)}}^{\mathsf{V}}[\mathbb{G}]$.

By (4.10) and (4.11), we have ${V_\lambda}^{\mathsf{V}}[\mathbb{G}] \models \varphi(\overline{a})$. By (4.17) and since φ is Σ_2, it follows that ${V_{j(\lambda)}}^{\mathsf{V}}[\mathbb{G}] \models \varphi(\overline{a})$. $\dashv$ (Claim 4.2.1)

Thus we have

(4.18) $\quad M \models$ "there is a $\mathcal{P}$-ground N of $V_{j(\lambda)}$ with $N \models \varphi(\overline{a})$".

By the elementarity (4.12), it follows that

(4.19) $\quad \mathsf{V} \models$ "there is a $\mathcal{P}$-ground N of V_λ with $N \models \varphi(\overline{a})$".

Now by (4.11), it follows that there is a $\mathcal{P}$-ground W of V such that $\mathsf{W} \models \varphi(\overline{a})$.

❑ (Theorem 4.2)

Theorem 4.4 *Suppose that $\mathcal{P}$ is an iterable class of posets and super-$C^{(\infty)}$-$\mathcal{P}$-LgLCA for extendible holds. Then* $\mathsf{MP}(\mathcal{P},\mathcal{H}(\kappa_{\mathfrak{refl}}))$ *holds.*

Proof. It is enough to show that $(\mathcal{P},\mathcal{H}(\kappa_{\mathfrak{refl}}))$-$\mathsf{RcA}^+$ holds. For this, a modification of the proof of Theorem 4.2 works.

Suppose that $\kappa=\kappa_{\mathfrak{refl}}$ is tightly super-$C^{(\infty)}$-$\mathcal{P}$-Laver-generically extendible, $\mathbb{P}\in\mathcal{P}$, and $\Vdash_{\mathbb{P}}$ "$\varphi(\overline{a})$" for an $\mathcal{L}_{\in}$-formula φ and $\overline{a}\in\mathcal{H}(\kappa)$. We want to show that $\varphi(\overline{a})$ holds in some $\mathcal{P}$-ground of V.

Let n be a sufficiently large natural number ≥ 1 such that the following arguments go through. In particular, we assume that $V_\alpha{}^{\mathsf{V}}\prec_{\Sigma_n}\mathsf{V}$ implies that "$\varphi(\overline{x})$" and "$\Vdash_\cdot$ "$\varphi(\overline{x})$"" are absolute between $V_\alpha{}^{\mathsf{V}}$ and V, and $V_\alpha{}^{\mathsf{V}}\prec_{\Sigma_n}\mathsf{V}$ also implies that a sufficiently large fragment of ZFC holds in V_α.

Let $\underset{\sim}{\mathbb{Q}}$ be a $\mathbb{P}$-name such that $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{Q}}\in\mathcal{P}$" and, for $(\mathsf{V},\mathbb{P}*\underset{\sim}{\mathbb{Q}})$-generic $\mathbb{H}$, there are a $\lambda>\kappa$ with

$$(4.20)\qquad V_\lambda\prec_{\Sigma_n}\mathsf{V},$$

and j, $M\subseteq\mathsf{V}[\mathbb{H}]$ such that $j:\mathsf{V}\overset{\prec}{\rightarrow}_\kappa M$, $j(\kappa)>\lambda$, $\mathbb{P}$, $\mathbb{H}$, $V_{j(\lambda)}{}^{\mathsf{V}[\mathbb{H}]}\in M$, $|\,RO(\mathbb{P}*\underset{\sim}{\mathbb{Q}})\,|\leq j(\kappa)$ $(<j(\lambda)$, and $V_{j(\lambda)}{}^{\mathsf{V}[\mathbb{H}]}\prec_{\Sigma_n}\mathsf{V}[\mathbb{H}]$.

Replacing $\mathbb{P}*\underset{\sim}{\mathbb{Q}}$ by an appropriate isomorphic poset (and replacing $\mathbb{H}$ by corresponding filter), we may assume that $\mathbb{P}*\underset{\sim}{\mathbb{Q}}\in V_{j(\lambda)}{}^{\mathsf{V}}$.

By the choice of n, we have $V_\lambda\models\ \Vdash_{\mathbb{P}}$ "$\varphi(\overline{a})$". $j(V_\lambda{}^{\mathsf{V}})=V_{j(\lambda)}{}^M\prec_{\Sigma_n}M$ by elementarity of j, and

$$(4.21)\qquad V_{j(\lambda)}{}^M=V_{j(\lambda)}{}^{\mathsf{V}[\mathbb{H}]}$$

by the closedness of M. Since $V_\lambda\prec_{\Sigma_n}\mathsf{V}$, we have $V_\lambda[\mathbb{H}]\prec_{\Sigma_{n_0}}\mathsf{V}[\mathbb{H}]$ for a still large enough $n_0\leq n$. Since $V_{j(\lambda)}{}^{\mathsf{V}[\mathbb{H}]}\prec_{\Sigma_n}\mathsf{V}[\mathbb{H}]$, it follows that $V_\lambda{}^{\mathsf{V}[\mathbb{H}]}\prec_{\Sigma_{n_0}}V_{j(\lambda)}{}^{\mathsf{V}[\mathbb{H}]}$. Thus

$$(4.22)\qquad V_\lambda{}^{\mathsf{V}}\prec_{\Sigma_{n_1}}V_{j(\lambda)}{}^{\mathsf{V}}$$

for a still large enough $n_1\leq n_0$.

In particular, we have $V_{j(\lambda)}{}^{\mathsf{V}}\models\ \Vdash_{\mathbb{P}}$ "$\varphi(\overline{a})$", and hence $V_{j(\lambda)}[\mathbb{G}]\models\varphi(\overline{a})$ where $\mathbb{G}$ is the $\mathbb{P}$-part of $\mathbb{H}$. Note that by (4.20) and (4.22), $V_{j(\lambda)}$ satisfies a sufficiently large fragment of ZFC.

Thus we have $V_{j(\lambda)}[\mathbb{H}]\models$ " there is a $\mathcal{P}$-ground satisfying $\varphi(\overline{a})$", and hence

$$V_{j(\lambda)}{}^{\mathsf{V}[\mathbb{H}]}\models\text{" there is a }\mathcal{P}\text{-ground satisfying }\varphi(\overline{a})\text{"}$$

by Lemma 4.3. By (4.21) and elementarity, it follows that

$$V_\lambda\models\text{" there is a }\mathcal{P}\text{-ground satisfying }\varphi(\overline{a})\text{"}.$$

Finally, this implies $\mathsf{V}\models$ " there is a $\mathcal{P}$-ground satisfying $\varphi(\overline{a})$" by (4.20).

□ (Theorem 4.4)

5 Consistency of LgLCAs and super-$C^{(\infty)}$-LgLCAs for extendibility

Lemma 5.1 (1) *Suppose that κ is extendible. Then there is a Laver function $f : \kappa \to V_\kappa$ for extendibility.*

(2) *Suppose that κ is super $C^{(n)}$-extendible for an $n \in \mathbb{N} \setminus 1$. Then there is a Laver function $f : \kappa \to V_\kappa$ for super $C^{(n)}$-extendibility.*

(3) *Suppose that, for an inaccessible cardinals κ^* and a cardinal $\kappa < \kappa^*$ we have $V_{\kappa^*} \models$ "κ is super-$C^{(\infty)}$-extendible". Then there is a Laver function $f : \kappa \to V_\kappa$ for super-$C^{(\infty)}$-extendible κ in V_{κ^*}.*

Proof. (1) has been known previously (see e.g. [2]). The proof of (2) below can be easily modified to a proof of (1).

(2): Assume, toward a contradiction, that there is no Laver function $f : \kappa \to V_\kappa$ for super $C^{(n)}$-extendible κ.

Let $\varphi_n(f)$ be the formula

$$\exists\underline{\alpha}\exists\underline{x}\,\forall\underline{\delta}\,\forall\underline{\delta}'\,\forall\underline{j}\,((\,f : \underline{\alpha} \to V_{\underline{\alpha}} \ \wedge\ \underline{\alpha} < \underline{\delta}\ \wedge\ \ V_{\underline{\delta}} \prec_{\Sigma_n} \mathsf{V}\ \wedge\ \ \underline{x} \in V_{\underline{\delta}}\ \wedge\ V_{\underline{\delta}'} \prec_{\Sigma_n} \mathsf{V}$$
$$\wedge\ \ \underline{j} : V_{\underline{\delta}} \overset{\prec}{\to}_{\underline{\alpha}} V_{\underline{\delta}'} \ \wedge\ \underline{j}(dom(f)) > \underline{\delta} \wedge \underline{j} \text{ is cofinal in } V_{\underline{\delta}'}) \ \ \to\ \ \underline{j}(f)(\underline{\alpha}) \neq \underline{x}\,).$$

If $\varphi_n(f)$ holds then the witness of $\underline{\alpha}$ in $\varphi_n(f)$ is uniquely determined. In this case, let x_f be a witnesses for $\underline{x}$ in $\varphi_n(x)$, and let $\mu_f := rank\,(x_f)$. x_f might not be determined uniquely. However, we will choose x_f such that μ_f is minimal and thus μ_f is determined uniquely to each f. If $\varphi_n(f)$ does not hold, we let $\mu_f := 0$.

By assumption, we have

(5.1) $\quad \varphi_n(f)$ for all $f : \kappa \to V_\kappa$.

Let ν, ν_0 be be cardinals such that $\nu \geq \nu_0 \geq \max\{\mu_f \,:\, f : \alpha \to V_\alpha$ for an inaccessible $\alpha \leq \kappa\}$, (5.2) : $V_{\nu_0} \prec_{\Sigma_m} \mathsf{V}$ for sufficiently large $m \in \mathbb{N}$, $V_\nu \prec_{\Sigma_n} \mathsf{V}$, and there is $j^* : \mathsf{V} \overset{\prec}{\to}_\kappa M$ with (5.3) : $j^*(\kappa) > \nu$, (5.4) : $V_{j(\nu)} \prec_{\Sigma_n} \mathsf{V}$, and (5.5) : $V_{j(\nu)} \in M$.

Let $A := \{\alpha < \kappa \,:\, \alpha$ is inaccessible, and $\forall\underline{f}\,((\underline{f} : \alpha \to V_\alpha) \ \to\ \varphi_n(\underline{f}))\}$.

By assumption, $\mathsf{V} \models$ "$\forall\underline{f}\,((\underline{f} : \kappa \to V_\kappa) \ \to\ \varphi_n(\underline{f}))$". By (5.2) and (5.5), it follows that $M \models$ "$\forall\underline{f}\,((\underline{f} : \kappa \to V_\kappa) \ \to\ \varphi_n(\underline{f}))$".[2)] Thus we have $M \models j^*(A) \ni \kappa$.

Let $f^* : \kappa \to V_\kappa$ be defined by

[2)] Actually, by analyzing the statement of $\varphi_n(f)$ carefully we see that (5.2) and the details connected to it are redundant here.

$$f^*(\alpha) := \begin{cases} x_{f^* \restriction \alpha}, & \text{if } \alpha \in A; \\ \emptyset, & \text{otherwise.} \end{cases}$$

Let $x^* := j^*(f^*)(\kappa)$. By definition of f^*, by (5.1), and since $j(f^*) \restriction \kappa = f^*$, x^* witnesses $\varphi_n(f^*)$.
(μ_{f^*} is just as chosen before since it is uniquely determined. x^* may be different from x_{f^*} but this does not matter.)

In particular, $x^* \neq (j^* \restriction V_{\delta_{f^*}})(f^*)(\kappa) = j(f^*)(\kappa)$. This is a contradiction.

(3): Let $\varphi_n(f)$ be as in (1) where n runs now over ω, and let $\varphi(f) := \bigwedge\!\!\!\bigwedge_{n\in\omega}\varphi_n(f)$. As we already have noticed in Section 2, we cannot discuss about the validity of $\varphi(f)$ in V (at least not in the framework of ZFC) while $V_{\kappa^*} \models \varphi(f)$ is a well-defined notion. The conclusion of (2) is obtained by arguing analogously to (1) in V_{κ^*} with $\varphi_n(f)$ replaced by $\varphi(f)$. ❑ (Lemma 5.1)

Theorem 5.2 (1) *For an extendible κ, and for a Σ_2-definable transfinitely iterable class $\mathcal{P}$ of posets, there is a poset $\mathbb{P}_\kappa$ such that $\Vdash_{\mathbb{P}_\kappa}$ "$\kappa = \kappa_{\mathfrak{refl}}$ and κ is tightly $\mathcal{P}$-Laver generic extendible".*

(2) *Suppose that κ is super-$C^{(n^*)}$-extendible for $n \in \mathbb{N}$ and $n^* = \max\{n, 2\}$. Then for any Σ_{n+1}-definable transfinitely iterable class $\mathcal{P}$ of posets, there is a poset $\mathbb{P}_\kappa$ such that*
$\Vdash_{\mathbb{P}_\kappa}$ "$\kappa = \kappa_{\mathfrak{refl}}$ and κ is tightly super-$C^{(n)}$-$\mathcal{P}$-Laver generic extendible".

(3) *$V_\mu \models$ "κ is super-$C^{(\infty)}$ extendible" for an inaccessible μ. Then for any transfinitely iterable class $\mathcal{P}$ of posets there is a poset $\mathbb{P}_\kappa \in V_\mu$ such that $V_\mu \models$ " $\Vdash_{\mathbb{P}_\kappa}$ "$\kappa = \kappa_{\mathfrak{refl}}$ and κ is tightly super-$C^{(\infty)}$-$\mathcal{P}$-Laver generic extendible" ".*

Note that most of the natural classes of posets including the classes of all ccc posets, all σ-closed posets, all proper posets, all semi-proper posets, etc. are Σ_2.

Proof of Theorem 5.2: (1): We show the assertion for the case that $\mathcal{P}$ is the class of all proper posets. The proof for the general case can be done by replacing the CS-iteration in the proof by the iteration for which the class $\mathcal{P}$ is transfinitely iterable. Let f be a Laver function for extendible κ (f exists by Lemma 5.1, (1)).

Let $\langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\beta : \alpha \leq \kappa, \beta < \kappa \rangle$ be an CS-iteration of elements of $\mathcal{P}$ such that

$$\underset{\sim}{\mathbb{Q}}_\beta := \begin{cases} f(\beta), & \text{if } f(\beta) \text{ is a } \mathbb{P}_\beta\text{-name} \\ & \text{and } \Vdash_{\mathbb{P}_\beta} \text{“} f(\beta) \in \mathcal{P} \text{”}; \\ \mathbb{P}_\beta\text{-name of the trivial forcing,} & \text{otherwise.} \end{cases}$$

We show that $\Vdash_{\mathbb{P}_\kappa}$ "$\mathcal{P}$-LgLCA for extendibility".

First, note that $\Vdash_{\mathbb{P}_\kappa}$ "$\kappa = 2^{\aleph_0} = \kappa_{\mathfrak{refl}}$" by definition of $\mathbb{P}_\kappa$.

Let $\mathbb{G}_\kappa$ be a $(\mathsf{V}, \mathbb{P}_\kappa)$-generic filter. In $\mathsf{V}[\mathbb{G}_\kappa]$, suppose that $\mathbb{P} \in \mathcal{P}$ and let $\underset{\sim}{\mathbb{P}}$ be a $\mathbb{P}_\kappa$-name for $\mathbb{P}$.

Suppose that $\lambda > \kappa$. By Lemma 2.1, there is an inaccessible $\lambda^* > \lambda$. Let $j : \mathsf{V} \overset{\prec}{\to}_\kappa M$ be such that (5.6): $j(\kappa) > \lambda^*$, (5.7): $V_{j(\lambda^*)} \in M$, and (5.8): $j(f)(\kappa) = \underset{\sim}{\mathbb{P}}$. This is possible since f is a Laver function for the extendible κ.

In M, there is a $\mathbb{P}_\kappa * \underset{\sim}{\mathbb{P}}$-name $\underset{\sim}{\mathbb{Q}}$ such that $\Vdash_{\mathbb{P}_\kappa * \underset{\sim}{\mathbb{P}}}$ "$\underset{\sim}{\mathbb{Q}} \in \mathcal{P}$ and $\underset{\sim}{\mathbb{Q}}$ is the direct limit of CS-iteration of small posets in $\mathcal{P}$ of length $j(\kappa)$, and $\mathbb{P}_\kappa * \underset{\sim}{\mathbb{P}} * \underset{\sim}{\mathbb{Q}} \sim j(\mathbb{P}_\kappa)$", By (5.7), and since "$\underset{\sim}{\mathbb{P}} \in \mathcal{P}$" is Σ_2, the same situation holds in V.

We have $j(\mathbb{P}_\kappa)/\mathbb{G}_\kappa \sim \mathbb{P} * \underset{\sim}{\mathbb{Q}}$ where we identify $\underset{\sim}{\mathbb{Q}}$ with a corresponding $\mathbb{P}$-name.

Let $\mathbb{H}$ be $(\mathsf{V}, j(\mathbb{P}_\kappa))$-generic filter with $\mathbb{G}_\kappa \subseteq \mathbb{H}$.

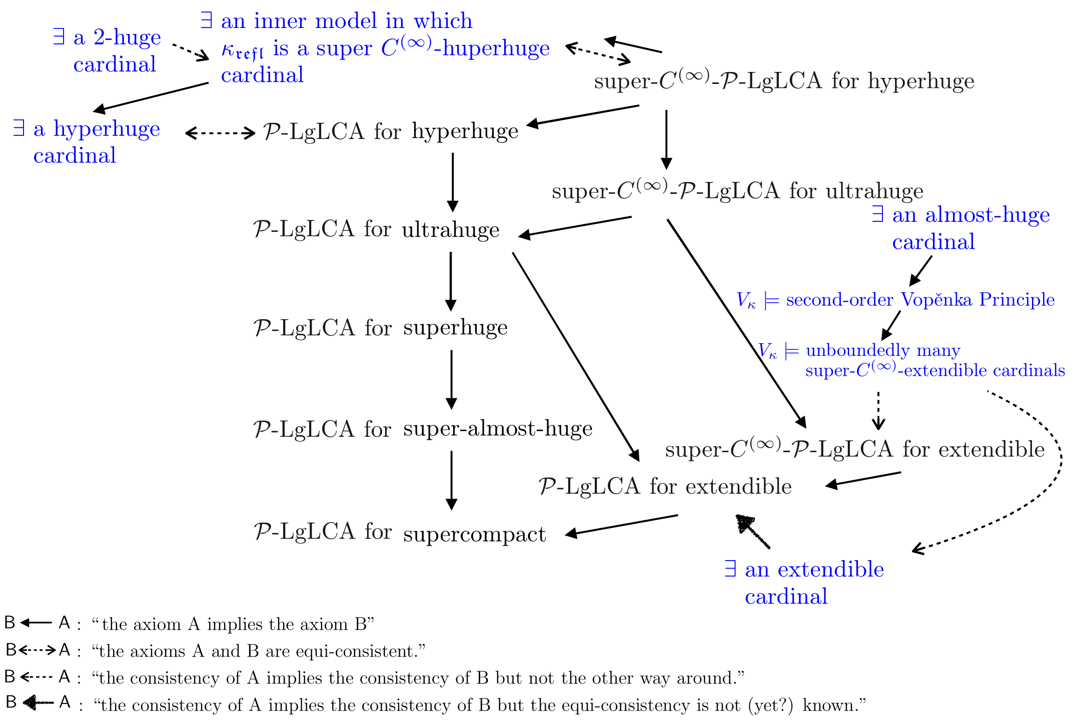
The lifting $\tilde{j} : \mathsf{V}[\mathbb{G}_\kappa] \overset{\prec}{\to}_\kappa M[\mathbb{H}]$; $\underset{\sim}{a}[\mathbb{G}_\kappa] \mapsto j(\underset{\sim}{a})[\mathbb{H}]$ witnesses that $\kappa = (\kappa_{\mathfrak{refl}})^{\mathsf{V}[\mathbb{G}_\kappa]}$ is tightly $\mathcal{P}$-Laver generic extendible. For this, it suffices to show:

Claim 5.2.1 *$V_\alpha{}^{\mathsf{V}[\mathbb{H}]} \in M[\mathbb{H}]$ for all $\alpha \leq j(\lambda)$.*

$\vdash$ By induction on $\alpha \leq j(\lambda)$. The successor step from $\alpha < j(\lambda)$ to $\alpha + 1$ can be proved by showing that $\mathbb{P}_\kappa$-names of subsets of $V_\alpha{}^{\mathsf{V}[\mathbb{H}]}$ can be chosen as elements of M. This is possible because of (5.7). $\dashv$ (Claim 5.2.1)

(2) and (3) can be proved similarly. □ (Theorem 5.2)

The following chart summarizes our view of the landscape with LgLCAs.



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