REDUCING PROJECTIVE COMPLEXITY: AN OVERVIEW

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Abstract. We survey a series of proofs appearing within the last 10 years exhibiting some structural similarities and which serve the purpose of constructing a coanalytic (Π^1_1) collection of reals from a Σ^1_2 such collection, in such a way that the combinatorial properties of the latter are also had by the former. The principal application of these results is to obtain coanalytic examples of sets which are known to have no analytic examples. We address the cases of maximal almost disjoint families, maximal independent families, maximal eventually different families, maximal sets of orthogonal Borel probability measures, ultrafilters, and Hausdorff gaps.

1. Introduction

The interest of this paper is in projective definability of certain subsets of Polish spaces, and methods by which to obtain examples whose projective definitions are of lowest possible complexity. Specifically, we focus on collections of subsets which cannot be analytic - such as maximal almost disjoint families, ultrafilter bases, and Hausdorff gaps - and whose existence usually follows from an application of the Axiom of Choice. However, a recursive construction given a projectively definable well-order of the reals will produce a definable family, such as is the case for Gödel's constructible universe L; the family thus obtained inherits the definability of the well-order. To reduce this complexity to the level Π^1_1 is a nontrivial task and, given the lower bound of Σ^1_1, yields a family of optimal complexity. Advancing a method originally introduced by Erdös, Kunen, and Mauldin ([Erd81]), Miller ([Mil89]) showed that in L there exist Π^1_1 witnesses to independent families, mad families, and Hamel bases. This robust technique was used by other authors to produce (Π^1_1, Π^1_1)-Hausdorff gaps, towers, and ultrafilter bases; see [FKK14, Section 4], [FS22], and [Sch19], respectively. The robustness of this techique is witnessed by the variety of applications in the subsequent literature, giving coanalytic examples of objects such as Hausdorff gaps, towers, and ultrafilters.

The deeper set theoretic reasons for the success of this technique were extracted and compiled into an efficient, formal theorem by Vidnyánszky [Vid14], allowing for shorter proofs of the above Π^1_1 existence results via a sort of "black-box" condition.

In 2014, Törnquist [TÖ9] provided a concise construction of a Π^1_1 mad family from one which is Σ^1_2. Similar such constructions have then been done for the other combinatorial sets under consideration, including maximal independent families and maximal families of orthogonal measures. The principal application of such results has been to obtain Π^1_1 examples of sets which are known to have no analytic examples, showing that Σ^1_1 is an optimal lower bound on the complexity of the type of set in question. The purpose of this survey is to collect such results in a single place so to reveal their similarities and differences, as well as to present the advantages of various coding techniques. We believe the constructions also offer an interesting comparison of the combinatorics particular to each of the families. In all but the case of Hausdorff gaps in $(\mathcal{P}(\omega), \subseteq^*)$, the constructions presented are due to the original authors, with slight modifications of notation so to give a uniform presentation.

In Section 2 we give preliminaries, and in Sections 3 and 4 we consider the cases of maximal almost disjoint families and maximal independent families. In Sections 5,6, and 7 we look at towers, maximal eventually different families, and maximal orthogonal families of Borel probability measures. Lastly we cover ultrafilter bases and Hausdorff gaps in Sections 8 and 9, respectively. Section 10 includes some open questions.

2. Preliminaries

Our notation is fairly standard; we will use ω to denote the set $\mathbb{N}$ of natural numbers, and $\mathcal{P}(X)$ denotes the set of subsets of a set X. The *Baire space* is denoted ω^ω and consists of infinite sequences of elements of ω, equivalently, functions $f\colon \omega \to \omega$. By $[\omega]^\omega$ we mean the set of the infinite subsets of ω. If $f\colon X \to Y$ is a function and $A \subseteq X$, we let $f[A]$ denote the pointwise image of A under f, that is, the set $\{y \in Y \mid \exists x \in A\ f(x) = y\}$.

The standard references for descriptive set theory are [Kec95] and [Mos80]. In the definitions below we fix X an uncountable Polish (i.e., a separable, completely metrizable topological space), such as ω^ω, 2^ω, or $[\omega]^\omega$. Elements of Polish spaces will often be referred to in the subsequent sections as *reals*.

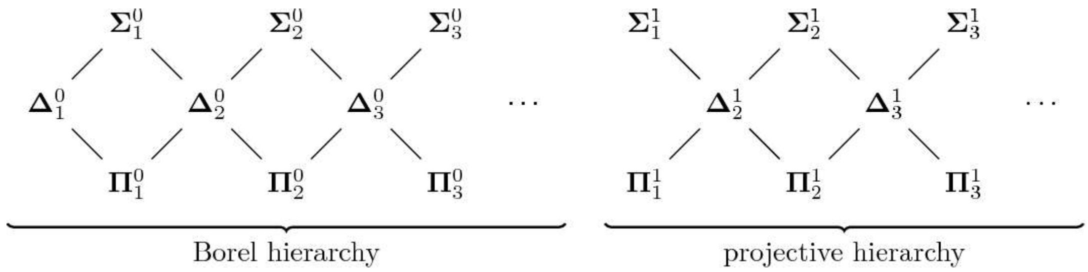
Recall that Σ^1_1 denotes the pointclass of (effectively) analytic sets $A \subseteq X$; these are sets A defined by a Σ^1_1 formula, or equivalently, such that there exists a Polish space Y and a closed (Π^0_1) set $C \subseteq X \times Y$ such that $A = \{x \in \omega^\omega \mid \exists y \in \omega^\omega (x, y) \in C\}$. Via this characterization, analytic sets admit a sort of *tree representation*: first recall that T is a *tree* on a set X if $T \subseteq X^{<\omega}$, and if $s \in T$ and t is an initial segment of s, then also $t \in T$. The set of *branches* of T is the set of functions $f\colon X \to \omega$ such that $f \restriction n \in T$ for all $n \in \omega$. Since the closed subsets of Polish spaces are in one-to-one correspondance with branches of trees, we have A is Σ^1_1 if and only if there is a tree T on $\omega \times \omega$ such that $x \in A$ iff $T_x = \{t \in \omega^{<\omega} \mid (x \restriction \mathrm{lh}(t), t) \in T\}$ is ill-founded, meaning T_x has a branch. The class is closed under countable union, countable intersection, and projections.

The class Π^1_1 denotes the class of *coanalytic* sets, these being the complements of analytic sets; equivalently a set $B \subseteq X$ is Π^1_1 if and only if there is a tree $T \subseteq \omega^{<\omega}$ such that $x \in B$ iff T_x is well-founded, i.e. T_x has a branch. The Π^1_1 sets are closed under countable union, countable intersection, and quantification over ω^ω. The class of *arithmetical* sets is $\Delta^1_1 = \Sigma^1_1 \cap \Pi^1_1$; by Kleene's theorem, Δ^1_1 is equal to the class HYP of hyperarithmetic sets.

The class of $\Sigma^1_2(X)$ sets consists of projections of Π^1_1 sets, so for $A \subseteq X$ is Σ^1_2 if and only if there exists a Polish space Y and a Π^1_1 set $F \subseteq X \times Y$ such that $x \in A$ iff $\exists y (x, y) \in F$.

In general, A is Σ^1_{n+1} iff $A = \{x \in X \mid \exists y\ (x, y) \in F\}$, where F is a Π^1_n subset of $X \times Y$, and the Π^1_n sets are the complements of Σ^1_n sets.

For Γ a pointclass as above and y is a real, so, $y \in Y$ for some Polish space Y, we denote by $\Gamma(y)$ the relativized poinclass; these are the sets $A \subseteq X$ such that there is some $B \in Y \times X$, $B \in \Gamma$, such that $x \in A \Leftrightarrow (y, x) \in B$. While we will restrict our attention to the lightface pointclasses, the constructions apply to the Borel and Projective sets, or the bold face pointclasses $\boldsymbol{\Delta^1_1}$, $\boldsymbol{\Sigma^1_n}$, etc., which are the context of the material found in [Kec95]. Indeed, if $\boldsymbol{\Gamma} \in \{\boldsymbol{\Sigma^0_n}, \boldsymbol{\Pi^0_n}, \boldsymbol{\Sigma^1_n}, \boldsymbol{\Pi^1_n}\}$ and Γ is the corresponding lightface pointclass, we have $A \in \boldsymbol{\Gamma}$ if and only if there is some $x \in \omega^\omega$ such that $A \in \Gamma(x)$. When y, x are reals and $y \in \Delta^1_1(x)$, we will often say y is *constructible from* x. In the proofs below we will suppress the parameter x.



An important property of the Π^1_1 sets is that they enjoy a modest choice principle. If $B \subseteq X \times Y$ is a such that for all $x \in X$ there is $y \in Y$ with $(x, y) \in B$, then a *uniformization* of a B is a subset $B' \subseteq B$ such that for each $x \in X$, there is a unique y with $(x, y) \in B'$. In other words, B' is the graph of a choice function for the collection $\bigcup_{x \in X}\{y \in Y \mid (x, y) \in B\}$. A pointclass Γ is said to have the *uniformization property* if for every Γ set $B \subseteq X \times Y$, B admits a uniformization which is also a member of Γ.

Theorem 2.1. *(Novikov-Kondô 1938, Addison 1950's; see* [Mos80, Theorem 4E.4]*) The classes* Π^1_1 *and* $\boldsymbol{\Pi^1_1}$ *have the uniformization property.*

Another deep result in descriptive set theory which is used in almost all of the constructions presented in this paper is the following theorem; essentially it says that the class of Π^1_1 sets is closed under existential quantification over the class Δ^1_1, i.e., the class of hyperarithmetic reals.

Theorem 2.2. *(Spector-Gandy 1960; see* [Mil95, Corollary 29.3], [Mac86, Corollary 4.19]*,* [Mos80, Theorem 4F.3]*) Let* $F \subseteq X \times Y$ *be a* Π^1_1 *set. Then, if* $A \subseteq X$ *defined by*

$$x \in A \Leftrightarrow \exists y \in \Delta^1_1(x)\ ((x, y) \in F),$$

then A *is also* Π^1_1.

In the proofs in the following sections, when we appeal to the definition of Σ^1_2 given above, we will take Y to be one of the sets $\omega^\omega, [\omega]^\omega$, or 2^ω. Note that there is no loss of generality in doing this, as 2^ω and $[\omega]^\omega$ are easily seen to be Δ^1_1-isomorphic, and ω^ω is Δ^1_1-isomorphic with the dense G_δ set $C := \{x \in 2^\omega \mid$ there exist infinitely many n $x(n) = 1\}$, where $x = (x(0), x(1), x(2), \dots)$ (see [Kec95, 3.12]). As C is a G_δ subset of a Polish space, C is also an uncountable Polish space. Then if A is Σ^1_2 and $F \subseteq X \times \omega^\omega$ is Π^1_1 such that $A = \{x \in X \mid \exists y(x, y) \in F\}$, letting $\iota \colon \omega^\omega \to C$ be a Δ^1_1 isomorphism, we have that $F' := \{(x, z) \in X \times C \mid (x, \iota^{-1}(z)) \in F\}$ is a Π^1_1 subset of $X \times C$ such that $x \in A$ iff $\exists y \in C(x, y) \in F'$. Therefore when we take our parameter set to be 2^ω or $[\omega]^\omega$ we will be using this convention.

Gödel's constructible universe is the proper class defined recursively on the ordinals as follows. We let $L_0 = \emptyset$, and for $\alpha \in \mathrm{Ord}$, $L_{\alpha+1}$ is the set of $A \subseteq L_\alpha$ such that A is definable by a formula (in the language of set theory) using parameters from L_α. L denotes $\bigcup_{\alpha \in \mathrm{Ord}} L_\alpha$, and L satisfies all the axioms of ZFC. In particular, L is a model of the Axiom of Choice, since L admits a Σ^1_2-definable global well-order: one defines a well-ordering on each $L_{\alpha+1}$

using the Gödel numbering of formulas with parameters from L_α (see [Kun13, Section II.6] [Mil95, Theorem 18.1]) The Axiom of Constructibility is the statement that the universe of all sets V is equal to L and is denoted $V = L$.

The proofs in the following sections will not require advanced knowledge of set theory of the reals and the associated forcing notions, though we will sometimes mention such topics so to situate the results in their original context. We try to include all relevant references, and for more general background we refer to [Hal11] or [Kun13].

3. Almost disjointedness

For sets $x, y \in [\omega]^\omega$, we say x and y are *almost disjoint* if $x \cap y$ is finite, and a collection $A \subseteq [\omega]^\omega$ is an *almost disjoint family* if all members of A are pairwise almost disjoint. An almost disjoint family A is *maximal almost disjoint (mad)* if it is maximal with respect to inclusion, that is, for all $y \in [\omega]^\omega$, there is $x \in A$ such that $x \cap y$ is infinite. Note that any partition of ω into finitely many pieces is a mad family, and so in the following "mad families" will always refer to mad families which are infinite.

Mad families have been of interest for their applications in topology, as well as in set theory of the reals, as they define the the cardinal invariant $\mathfrak{a} := \min\{|A| \mid A \subseteq [\omega]^\omega \text{ is mad}\}$. They become an object subejct to descriptive set theoretic inquiry beginning with Mathias's [Mat77], where he showed that no analytic almost disjoint family is maximal. A different proof of this fact is given by Törnquist [Tö18], which relied on the fact that if A is an infinite analytic almost disjoint family, one can use its tree representation A to produce a countable sequence $\{B_n \mid n \in \omega\}$ of subsets of ω such that

- for all $x \in A$, there exists $n < \omega$ such that $x \subseteq^* \bigcup_{i \leq n} B_i$, yet
- for all n, $\omega \setminus \bigcup_{i \leq n} B_i$ is infinite.

The existence of such a collection allows one to construct $y \in [\omega]^\omega$ contradicting the maximality of A.

In that same paper Törnquist answered a question of Mathias by showing there are no mad families in Solovay's model, this being the canonical model of ZF + Axiom of Dependent Choice (DC) in which every set is Lebesgue measurable and has the property of Baire, among other desirable regularity properties. Moreover each of the following implies the nonexistence of mad families:

- ZF+DC+ "All sets have the Ramsey property" ([ST19]);
- $V = L(\mathbb{R})$+ Axiom of Determinacy (AD) ([BHST21]);
- ZF+ AD^+ (Woodin's technical strengthening of the Axiom of Determinacy).

It is shown in [NN18] that under ZF+DC+Projective Determinacy (PD), there are no projectively definable mad families, and the more recent [FSW21] investigates the relationship between projective mad families and forcing axioms.

On the other hand, the Axiom of Consructibility $V = L$ does not fit into the above schema, as a recursive construction with respect to the Σ^1_2-definable well-order of $L \cap [\omega]^\omega$ yields a Σ^1_2 mad family. This was improved by Miller [Mil89], where he uses advanced coding techniques originating in work of Erdös, Kunen, and Mauldin to show that this construction can be done in a Π^1_1 way in L. Thus, it is consistent with $V = L$ to have a projectively definable mad family whose definition is at the lowest possible level in the projective hierarchy. Years later, Törnquist gave a short, combinatorial construction of a Π^1_1 mad family under a weaker set theoretic assumption; this is the starting point for proofs of this kind.

Theorem 3.1 (Törnquist; [T̈09]). *If there exists a Σ^1_2 mad family then there exists a Π^1_1 mad family.*

Proof. Let A be a Σ^1_2 mad family, and let $F \subseteq [\omega]^\omega \times [\omega]^\omega$ be Π^1_1 such that

$$x \in A \Leftrightarrow \exists y(F(x, y)).$$

By the Kondô-Addison theorem, we may assume F is the graph of a function. For $x \in [\omega]^\omega$, let $e_x : \omega \to x$ be the increasing enumeration of elements of x.

Define functions $g_i : [\omega]^\omega \times [\omega]^\omega \to \mathcal{P}(3 \times \omega)$ for $i < 2$, where

$$g_0(x, y) = (\{0\} \times x) \cup (\{2\} \times \{e_x(n) \mid n \in y\}),$$

and

$$g_1(x, y) = (\{1\} \times x) \cup (\{2\} \times \{e_x(n) \mid n \notin y\}).$$

Clearly g_0, g_1 are continuous functions. Let us see that $A' = g_0[F] \cup g_1[F]$ is a mad family of infinite subsets of $3 \times \omega$. That the family is almost disjoint is immediate, so it remains to show maximality. Let $z \subseteq 3 \times \omega$ be an infinite set not in A'. Denoting by $p_i(z)$ the projection of z onto the ith coordinate, there must be some some $i < 3$ such that $p_i(z) \in [\omega]^\omega$. Then the maximality of A gives some $x \in A$ such that $p_i(z) \cap x$ is infinite, and let $y \in [\omega]^\omega$ be such that $(x, y) \in F$. If $i \in \{0, 1\}$ then $g_i(x, y) \in A'$ and $\{(i, n) \mid n \in p_i(z) \cap x\} \subseteq g_i(x, y) \cap z$, so since the former is infinite, so is the latter. In the case $p_2(z)$ is infinite, since

$$p_2(g_0(x, y) \cup g_1(x, y)) = e_x[\omega] = x,$$

it must be that one of $z \cap g_0(x, y)$ or $z \cap g_1(x, y)$ is infinite.

Next, we turn to the definability of A'. We have that

$$z \in A' \Leftrightarrow \exists x, y[(x, y) \in F \wedge (g_0(x, y) = z \vee g_1(x, y) = z)].$$

Observe that the only obstruction of this being a Π^1_1 definition is the initial existential quantification. To overcome this the strategy is to bound the existential quantification over the hyperarithmetic sets and then apply the Spector-Gandy theorem. Indeed this can be done: given $z \in A'$, one can recursively recover the x, y witnessing membership of z in the set $g_0[F] \cup g_1[F]$: first one recovers x as $x = p_0(z) \cup p_1(z)$, and then to x apply the inverse $e_x^{-1} : x \to \omega$ to the set $p_2(z)$ to recover y. Therefore,

$$z \in A' \Leftrightarrow \exists x, y \in \Delta^1_1(z)[(x, y) \in F \wedge (g_0(x, y) = z \vee g_1(x, y) = z)],$$

so A' is Π^1_1. □

4. Independence

A family $A \subseteq [\omega]^\omega$ is *independent* if for all $k, \ell < \omega$ and distinct $x_0, \dots, x_k, y_0, \dots, y_\ell \in A$, the set $\bigcap_{i \le k} x_i \setminus \bigcup_{j \le \ell} y_j$ is infinite. Such a family is *maximal (mif)* if it is maximal with respect to inclusion. Equivalently, for all $z \in [\omega]^\omega$, there are $x_0, \dots, x_k, y_0, \dots, y_\ell \in A$ such that $z \cap \bigcap_{i \le k} x_i \setminus \bigcup_{j \le \ell} y_j$ is finite, or $(\bigcap_{i \le k} x_i \setminus \bigcup_{j \le \ell} y_j) \setminus z$ is finite.

Miller [Mil89, Theorem 10.28] shows that there are no analytic maximal independent families, and in fact the argument shows that maximal independent families cannot have the Baire property (see also [BFK19, Corollary 3.8]). Therefore there are no mifs in a model of AD, and no projective mifs in Solovay's model or in a model of ZF+PD. In fact [BFK19] shows that in a Cohen-generic extension, there is no projective mif, either.

Miller [Mil89, Theorem 10.31] constructed a Π^1_1 maximal independent family in L, similarly to the construction of the mad family mentioned above. Extracting implicit arguments from a construction of Shelah [She92], Brendle, Fischer, and Khomskii [BFK19] constructed a Π^1_1 mif in L which moreover remains maximal in any Sacks generic extension; this followed from a recursive construction of a Σ^1_2 such family, and a modification of Törnquist's proof for mad families:

Theorem 4.1. *(Brendle, Fischer, Khomskii;* [BFK19, Theorem 2.1]*) If there exists a Σ^1_2 maximal independent family then there exists a Π^1_1 maximal independent family.*

Proof. Let A be a Σ^1_2 maximal independent family, and let $F \subseteq ([\omega]^\omega)^2$ be a Π^1_1 set such that

$$x \in A \Leftrightarrow \exists y \in [\omega]^\omega((x, y) \in F).$$

Again by Π^1_1 uniformization we can assume F is the graph of a function.

Considering the space $\omega \cup 2^{<\omega}$ as a disjoint union, define $g\colon [\omega]^\omega \times [\omega]^\omega \to \mathcal{P}(\omega \cup 2^{<\omega})$ as follows:

$$g(x, y) = x \cup \{\chi_y \restriction n \mid n < \omega\},$$

where $\chi_y\colon \omega \to 2$ denotes the characteristic function of y. Clearly g is a continuous function.

Let $A' := g[F]$, and we claim this is a maximal independent family of subsets of $\omega\cup 2^{<\omega}$. To see A' is independent, let $x_0, \ldots, x_n$ and $w_0, \ldots, w_k$ be distinct elements of A'. Let $a_i = x_i \cap \omega$ and $b_j = w_j \cap \omega$; then a_i, b_j are distinct elements of A for all $i \le n$ and $j \le k$. Moreover,

$$\bigcap_{i\le n} a_i \setminus \bigcup_{j\le k} b_j \subseteq \bigcap_{i\le n} x_i \setminus \bigcup_{j\le k} w_j,$$

so since the former is infinite, so is the latter.

To see that A' is maximal, let z be an infinite subset of $\omega \cup 2^{<\omega}$. Then as $z' = z \cap \omega$ is a subset of ω, maximality of A gives $a_0, \ldots, a_k, b_0, \ldots, b_\ell \in A$ such that $z \cap (\bigcap_{i\le k} a_i \setminus \bigcup_{j\le \ell} b_j)$ is finite, or $(\bigcap_{i\le k} a_i \setminus \bigcup_{j\le\ell} b_j) \setminus z$ is finite. Supposing without loss of generality it is the former, let $y_i, t_j \in [\omega]^\omega$ be the unique sets such that $(a_i, y_i), (b_j, t_j) \in F$ for $i \le k$ and $j \le \ell$. If

$$z \cap \bigcap_{i\le k} g(a_i, y_i) \setminus \bigcup_{j\le\ell} g(b_j, t_j)$$

is not finite, it must be because

$$z \cap \bigcap\{\chi_{y_i} \restriction n\} \setminus \bigcup\{\chi_{t_j} \restriction n\}$$

is infinite, in which case we let $a \ne b$ be two members of A, distinct from each a_i, b_j, and let y, t be so that $(a, y), (b, t) \in F$. If $y = t$, then $(g(a, y) \setminus g(b, t)) \cap 2^{<\omega} = \emptyset$, and then

$$z \cap (\bigcap_{i\le k} g(a_i, y_i) \cap g(a, y)) \setminus (\bigcup_{j\le\ell} g(b_j, t_j) \cup g(b, t))$$

is finite. In the case $y \neq t$, then $\{\chi_y \restriction n \mid n \in \omega\} \cap \{\chi_t \restriction n \mid n \in \omega\}$ is finite, and therefore

$$z \cap (\bigcap_{i \leq k} g(a_i, y_i) \cap g(a, y) \cap g(b, t)) \setminus \bigcup_{j \leq \ell} g(b_j, t_j)$$

is finite.

It remains to show that A' is Π^1_1-definable. We have that for all infinite $z \subseteq \omega \cup 2^{<\omega}$,

$$z \in A' \Leftrightarrow \exists x, y((x, y) \in F \wedge g(x, y) = z),$$

and as before we want to bound the existential quantification to the set $\Delta^1_1(z)$. Indeed, given $z \in g[F]$, first recover x by considering $z \cap \omega$. Next, since the y such that $(x, y) \in F$ is unique, the characteristic function for y must be the unique branch through the tree $T = z \cap 2^{<\omega}$, and the property of being a unique branch can be defined arithmetically as follows:

(1) $\forall n(\chi_y \restriction n \cap z \neq \emptyset)$;
(2) $\forall n \forall s \in z \cap 2^n (s \neq \chi_y \restriction n \Rightarrow \exists m > n(\forall t \in z \cap 2^m \neg(t \sqsupseteq s)))$.

Therefore

$$z \in A' \Leftrightarrow \exists x, y \in \Delta^1_1(z)[(x, y) \in F \text{ and } g(x, y) = z],$$

which again by the Spector-Gandy theorem shows that A' is Π^1_1. □

5. Towers

A collection $A \subseteq [\omega]^\omega$ is said to be a *tower* if it is well-ordered by the relation $\supseteq^*$, where $y \supseteq^* x$ iff $x \setminus y$ is finite. If A is only linearly ordered by $\supseteq^*$ then we say A is a *linear tower*. A (linear) tower A is *maximal* if it cannot be properly extended by $\supseteq^*$, i.e. there is no $z \in [\omega]^\omega \setminus A$ such that for all $x \in A$, $x \leq z$. A set z exhibiting this property is called a *pseudointersection* of A.

It is well known that no countable tower can be maximal. Fischer and Schilhan ([FS22]) showed that no tower can be analytic; this follows from the fact that no tower can contain a perfect subset. Indeed, if $P \subseteq A$ is a nonempty perfect set, then P^2 is a perfect Polish space and $\supseteq^* \cap P^2$ is a well-ordering of P, so $\supseteq^* \cap P^2$ cannot have the Baire Property (see [Kec95, Theorem 8.48]). But $\supseteq^*$ is a Borel subset of P^2 and thus has the Baire property, a contradiction. Since all Σ^1_1 sets are either countable or contain a perfect subset, it follows that no maximal tower is analytic. [FS22, Theorem 2.5] also establishes the stronger fact, that no maximal linear tower can be analytic; as in the above sections, the proof relies on the tree representation for analytic sets to construct a contradiction of maximality.

Moreover, recall the Solovay-Mansfield theorem (see [Mil95, Theorem 21.1] or [Mac86, Corollary 6.9]) states that any $\Sigma^1_2(x)$ set is either countable or contains a perfect set of reals constructible from x; as a consequence, any $\Sigma^1_2(x)$ tower must be a subset of $L[x]$. The existence of a $\Sigma^1_2(x)$ maximal (linear) tower implies $\omega_1 = \omega_1^{L[x]}$ ([FS22, Corollary 2.4], [FS22, Theorem 5.3]). More generally,

- Under ZF+PD, there are no projectively definable towers;
- there are no maximal linear towers in Solovay's model ([FS22, Theorem 6.1]).

However, $V = L$ implies the existence of a Π^1_1 maximal tower in Gödel's universe L; this is shown in [FS22, Theorem 3.2] using the method of Miller. Another way to derive this theorem is by using the fact that any Π^1_1 maximal linear tower contains a Π^1_1 maximal tower ([FS22, Theorem 7.1]), constructing a Σ^1_2 maximal linear tower in L, and then applying the following theorem:

Theorem 5.1 ([FS22, Theorem 7.2])**.** *If there exists a Σ^1_2 maximal linear tower, then there exists a Π^1_1 maximal linear tower.*

Proof. Let A be a Σ^1_2 maximal linear tower, and let $F \subseteq [\omega]^\omega \times 2^\omega$ be Π^1_1 such that

$$x \in A \Leftrightarrow \exists y \in 2^\omega (x, y) \in F.$$

Again, we suppose F is the graph of a function. Define $g\colon [\omega]^\omega \times 2^\omega \to [\omega \times \omega]^\omega$ by letting

$$g(x, y) = (\{0\} \times x) \cup (\bigcup_{n\in\omega} \{n + 1\} \times (x \setminus x(n + y(n)))).$$

Clearly g is continuous.

Let $A' := g[F]$, and we claim this is a maximal linear tower in the space $[\omega \times \omega]^\omega$. To see A' is linearly ordered by $\supseteq^*$, let $a \neq b \in A'$, and let (x, y) and (z, t) be elements of F such that $g(x, y) = a$ and $g(w, z) = b$. Let a_n, b_n denote the nth column of a, b respectively, so note $a_0 = x$ and $b_0 = y$. As x, z are distinct members of A, without loss of generality suppose $x \subseteq^* z$ and $z \not\subseteq^* x$. Then $x \setminus z$ is finite by the former and $z \setminus x$ is infinite by the latter, so there is n sufficiently large so that for all $m \geq n$, $x(m) > z(m)$ and $x(m) \in z$. Then $(0, z(m)) <_{lex} (0, x(m))$ and $(0, x(m)) \in (\{0\} \times z)$ for all $m \geq n$. For the same n as above and all $m \geq n$,

$$x \setminus x(m + y(m)) \subseteq x \setminus x(m) \subseteq z \setminus z(m + t(m)).$$

In other words, for all $m \geq n$, the $a_{m+1} \subseteq b_{m+1}$. For $k \leq n$, $a_k \subseteq^* b_k$, as $a_k \setminus b_k \subseteq x \setminus z$. Altogether we have that $a \subseteq^* b$.

To see A' is maximal let b be an infinite subset of $\omega \times \omega$ with $b \notin A'$, and suppose towards contradiction that b is a pseudointersection of A'. If there exists $n \in \omega$ such that b_n is infinite, then $b_n \subseteq^* x$ for all $x \in A$, contradicting maximality of A. If no such n exists, let $c := \{\min b_n \mid n \in \omega \wedge b_n \neq \emptyset\}$. If c is finite then so is b, so c must be infinite. We will show that c is a pseudointersection of A. Since $b \subseteq^* a$ for all $a \in A'$, there is n sufficiently large so that for all $m \geq n$, if $b_m \neq \emptyset$, then $\min b_m \in a_m$. Letting $x, y \in F$ be such that $g(x, y) = a$, we have that for all $m \geq n$, $\min b_m \in x$, implying $c \subseteq^* x$, contradicting the maximality of A.

Last we show that A' is Π^1_1, again by looking to bound x, y by $\Delta^1_1(z)$, where $z \in A'$. Given such a z, easily x is constructible from z as $z_0 = x$. Then $y \in 2^\omega$ is the real such that

$$y(n) = \begin{cases} 0 & \text{if } \min z_{n+1} = x(n + 1); \\ 1 & \text{if } \min z_{n+1} = x(n + 2). \end{cases}$$

Therefore $(x, y) \in \Delta^1_1(z)$, and

$$z \in A' \Leftrightarrow \exists (x, y) \in \Delta^1_1(z)[(x, y) \in F \wedge g(x, y) = z],$$

showing that A' is Π^1_1.

□

6. Maximal Eventually Different Families

For functions $f, g \in \omega^\omega$, we say f and g are *eventually different* if there exists $n < \omega$ such that for all $m \geq n$, $f(m) \neq g(m)$. A family $A \subseteq \omega^\omega$ is *eventually different* if it consists of pairwise eventually different functions. Such a family A is *maximal (med)* if for any $g \in \omega^\omega$, there is $f \in A$ such that the set $\{n \in \omega \mid f(n) = g(n)\}$ is infinite.

Unlike other families considered thus far, there exist Borel and thus analytic med families. The Borel construction was given by Horowitz and Shelah [HS16], and recently Schrittesser [Sch17] has shown the existence of a closed (Π^0_1) such family. However, such families with nice definability must always be of size continuum, thus it is nontrivial to define a maximal eventually different family in L with a Π^1_1 definition which remains maximal after forcing over L which adds news reals. This was done by Fischer and Schrittesser in 2018 [FS21a]. More formally,

Definition 1. Let $A \subseteq \omega^\omega \cap V$ be such that $(A \text{ is med})^V$, and let $\mathbb{P} \in V$ be a forcing notion. We say A is $\mathbb{P}$*-indestructible* if $\Vdash_{\mathbb{P}}$"A is med".

Such "σ-strengthenings" of the maximality condition for these and other combinatorial families of reals will yield families which are $\mathbb{P}$-indestructible for a particular forcing $\mathbb{P}$, and these stronger combinatorial objects cannot be analytic; see, for instance, the notion of tight eventually different family introduced in Fischer and Switzer [FS21b] and the arguments of Theorem 3.3 therein. In 2008, Kastermans, Steprāns, and Zhang [KSZ08] introduce the notion of *strongly* maximal eventually different families, a notion previously investigated in the context of almost disjoint families by Malykhin in 1989 [Mal89], where the analagous notion is called ω-mad families. Similar work was done in 2001 by Kurilić [Kur01], and in 2003 by Hrušák, and Ferreira [HF03].

To define "strongly med", first say that $h \in \omega^\omega$ is *finitely covered* by an eventually different family $A \subseteq \omega^\omega$ if there is a nonempty finite $C \subseteq A$ such that for all but finitely many $n \in \omega$, $h(n) = f(n)$ for some $h \in C$; in other words, $\text{graph}(f) \subseteq^* \bigcup_{h \in C} \text{graph}(h)$. The *ideal generated by* A is the collection

$$\mathcal{I}(A) := \{h \in \omega^\omega \mid h \text{ is finitely covered by } A\}.$$

By $\mathcal{I}(A)^+$ we denote the set $\omega^\omega \setminus \mathcal{I}(A)$.

Definition 2. An eventually different family $A \subseteq \omega^\omega$ is said to be *strongly maximal (strongly med)* if for any countable collection $H \subseteq \mathcal{I}(A)^+$, there exists a single $f \in A$ such that for all $h \in H$, $f(n) = h(n)$ for all but finitely many $n \in \omega$

Raghavan [Rag09] in 2009 gives a proper hierarchy of combinatorial strengthenings of the maximality of eventually different families. Raghavan [Rag09] shows that strongly med families are $\mathbb{P}$-indestructible, when $\mathbb{P}$ is Sacks, Miller, or more generally any forcing which does not make the ground model reals meager; he moreover shows that the notions of med and strongly med can be separated, and gives a hierarchy of combinatorial strengthenings of maximality for eventually different families, as well as conditions under which the hierarchy is proper.

A proof that strongly med families cannot be analytic can be found in [KSZ08, Theorem 2.1]; as in the above cases, the strategy of the proof is to exploit the tree representation of an analytic family so to find a countable $H \subseteq \mathcal{I}(A)^+$ witnessing that A cannot have the stronger maximality condition.

Strongly cannot contain perfect sets (see [Rag09, Corollary 37, Corollary 38]), also [FS21b, Section 3]); this is deduced from their Cohen indestructibility. It follows that any $\Sigma^1_2(x)$ strongly maximal eventually different family is a subset of $L[x]$. It is worth noting that Π^1_2 strongly maximal families of size $\mathfrak{c} > \aleph_1$ have been constructed more recently, in Friedmanzdomskyy and [FFZ11].

Nevertheless there exists a Π^1_1 strongly med family in L, as first shown in [KSZ08, Theorem 3.1], following the techniques of Miller [Mil89]. Adapting the construction of the Sacks indestructible Π^1_1 maximal independent family from [BFK19], Fischer and Schrittesser construct a Π^1_1 Sacks indestructible med family, relying on the construction of a Σ^1_2 such family and then appealing to the following:

Theorem 6.1 (Fischer, Schrittesser; [FS21a, Theorem 8])**.** *If there exists a Σ^1_2 maximal eventually different family which is $\mathbb{P}$-indestructible for some forcing notion $\mathbb{P}$, then there exists a Π^1_1 maximal eventually different family which is also $\mathbb{P}$-indestructible.*

Proof. Let $A \subseteq \omega^\omega$ be a Σ^1_2 maximal eventually different family, and let $F \subseteq \omega^\omega \times \omega^\omega$ be Π^1_1 such that

$$f \in A \Leftrightarrow \exists y (f, y) \in F.$$

Again, we assume F is the graph of a function. First, fix a recursive bijection

$$c\colon \bigcup_{n<\omega} \omega^n \times \omega^n \to \omega,$$

and for $i < 2$ let $\pi_i\colon \omega \to \bigcup_{n<\omega} \omega^n \times \omega^n$ be the coordinate maps, i.e. $\pi_i(c(s_0, s_1)) = s_i$ for each $s_0, s_1 \in \bigcup_{n\in\omega} \omega^n \times \omega^n$. Next define a continuous function $g\colon \omega^\omega \times \omega^\omega \to \omega^\omega$ by letting

$$g(f, y)(n) = \begin{cases} f(\frac{n}{2}) & \text{if } n \text{ even;} \\ c(f \upharpoonright n, y \upharpoonright n) & \text{otherwise.} \end{cases}$$

Let $A' := g[F]$. Clearly A' is an eventually different family. To see it is maximal, let $h \in \omega^\omega$, and consider $h' \in \omega^\omega$, where $h'(n) := h(2n)$. By maximality of A, there is $f \in A$ such that $\{n \in \omega \mid f(n) = h'(n)\}$ is infinite, and therefore so is the set $\{n \in \omega \mid g(f, y)(2n) = h(2n)\}$, showing A' is maximal.

To see the definability of A', we will show

$$f \in A' \Leftrightarrow \exists (h, y) \in \Delta^1_1(f)((h, y) \in F \text{ and } f = g(h, y)) \tag{1}$$

For $i < 2$ and $n \in \omega$, let $p_i\colon \omega^n \times \omega^n \to \omega^n$ be the projection onto the ith coordinate. Given any $f \in \omega^\omega$, define $h \in \omega^\omega$ to be $h(n) = f(2n)$, and if it is the case that for all $i < 2$ and $n, m \in \omega$ with $n < m$, we have $p_i(c^{-1}(f(2n+1)) \subseteq p_i(c^{-1}(f(2m+1))$, then let $y \in \omega^\omega$ be defined as $y = \bigcup_{n\in\omega} p_1(c^{-1}(f(2n+1)))$. Then h, y are constructible from f, "$((h, y), f) \in \operatorname{graph}(g)$" is Borel, and "$(h, y) \in F$" is Π^1_1.

Lastly we verify that A' is $\mathbb{P}$-indestructible, assuming that A is. For $G \subseteq \mathbb{P}$ be $\mathbb{P}$-generic over V, denote by F^* the set of $(f, y) \in V[G]$ such that $V[G] \models \phi(f, y)$, where ϕ is the Π^1_1 formula defining the functional relation F. By Shoenfield absoluteness we have that the Σ^1_2 set $A^* = \{f \in V[G] \mid \exists! y (f, y) \in F^*\}$ contains the original family A, and since A remains maximal in $V[G]$, we must have that $A = A^*$, and so also $F = F^*$. Moreover by absoluteness, F remains a Π^1_1 functional relation in $V[G]$, and therefore $V[G]$ satisfies 1 above. So the above argument showing the maximality of A' can be run in $V[G]$. □

7. Maximal orthogonal families

Let X be a Polish space, $\mathcal{B}(X)$ its associated Borel σ-algebra, and let $P(X)$ be the set of Borel probability measures on X, that is, Borel measurable functions $\mu\colon \mathcal{B}(X) \to [0,1]$ such that $\mu(\emptyset) = 0$ and $\mu(\bigcup_{n\in\mathbb{N}} B_n) = \sum_{n\in\mathbb{N}} \mu(B_n)$ for pairwise disjoint collections $B_n \subseteq \mathcal{B}(X)$. For $\mu, \nu \in P(X)$, we say μ is *absolutely continuous with respect to* ν, denoted $\nu \ll \mu$, if $\nu(B) = 0$ implies $\mu(B) = 0$, and are *equivalent* if $\nu \ll \mu$ and $\mu \ll \nu$. We say μ, ν are *orthogonal*, written $\mu \perp \nu$, if there is $B \in \mathcal{B}(X)$ such that $\nu(B) = \mu(X \setminus B) = 0$. Equivalently, $\mu \perp \nu$ if and only if there does not exist $\eta \in P(X)$ such that $\eta \ll \mu$ and $\eta \ll \nu$. We also denote by $P_c(X)$ the set of $\mu \in P(X)$ which are non-atomic, meaning $\mu(\{x\}) = 0$ for all $x \in X$, and by $P_d(X)$ the set of Dirac point measures. Note that $P_c(X)$ and $P_d(X)$ are mutually orthogonal in the sense that for all $\mu \in P_c(X)$ and all $\delta \in P_d(X)$, $\mu \perp \delta$.

A family $A \subseteq P(X)$ is an *orthogonal family* if for all $\mu, \nu \in A$, $\mu \perp \nu$. An orthogonal family A is *maximal (mof)* if it is maximal with respect to inclusion, or equivalently, for all $\nu \in P(X) \setminus A$, there is $\mu \in A$ and $\eta \in P(X)$ such that $\eta \ll \mu$ and $\eta \ll \nu$.

Press and Rataj [PR85] originally showed in 1985 that if A is an analytic family of orthogonal measures, then it cannot be maximal; Kechris and Sofrondis gave another proof of this fact in 2000 [KS01], using Hjorth's theory of turbulence. A more recent proof is given in 2024, by Mejak [Mej24].

Unlike the families considered in previous sections, however, Σ^1_2 maximal sets of orthogonal measures are never indestructible with respect to Cohen or random forcing, and moreover must always be the size of the continuum. [Mej24] shows that PD precludes the existence of projectively definable mofs, while AD implies there are no such families. However, coanalytic mofs exist under $V = L$ by Fischer and Törnquist [FT10]; implicit in there construction is Theorem 7.1 below. We also note that it is consistent to have a Π^1_2 maximal set of orthogonal measures in the presence of $\mathfrak{c} \geq \aleph_2$; see [FFT11].

Theorem 7.1 ([FT10])**.** *If there exists a Σ^1_2 maximal set of orthogonal measures, then there exists a Π^1_1 maximal set of orthogonal measures.*

Proof. Let $A \subseteq P(2^\omega)$ be a Σ^1_2 maximal family of orthogonal measures. Since any $\mu \in A$ is a finite measure, μ can have at most countably many atoms $M \subseteq 2^\omega$, and we may define an atomless measure $\mu' \ll \mu$ by letting $\mu'(B) = \max(\mu(B \setminus M), 0)$. Therefore we may assume without loss of generality that $A \subseteq P_c(2^\omega)$. Take $F \subseteq P_c(2^\omega) \times 2^\omega$ to be the Π^1_1 graph of a function such that

$$\mu \in A \Leftrightarrow \exists y(\mu, y) \in F.$$

The idea is to define a continuous function

$$g\colon P_c(2^\omega) \times 2^\omega \to P_c(2^\omega)$$

so that $g(\mu, y)$ will be a measure equivalent to μ, which additionally codes the real y, implying $\mu, y \in \Delta^1_1(g(\mu, y))$. Because we want this procedure to remain recursive, we work not directly with the measure μ themselves, but rather with the set

$$p(2^\omega) := \{f\colon 2^{<\omega} \to [0,1] \mid f(\emptyset) = 1 \wedge \forall s (f(s) = f(s^\frown 0) + f(s^\frown 1))\}.$$

By [Kec95, 17.7], there is a unique probability measure μ on 2^ω with $f(s) = \mu(N_s)$[1], and conversely every Borel probability measure on 2^ω arises in this way. For $f \in p(2^\omega)$, denote by μ_f the corresponding measure; we think of $p(2^\omega)$ as the codes for elements of $P(2^\omega)$. Let $p_c(2^\omega)$ be the set of those $f \in p(2^\omega)$ such that μ_f is nonatomic. We will define a continuous function from $p(2^\omega) \times 2^\omega$ to $p(2^\omega)$.

For a measure $\mu \in P(2^\omega)$ and $s \in 2^{<\omega}$, let $t(s,\mu)$ denote the lexicographically minimal $t \in 2^{<\omega}$ extending s and such that $\mu(N_{s^\frown 0}) > 0$ and $\mu(N_{s^\frown 1}) > 0$, when such t exists. Otherwise we let $t(s,\mu) = \emptyset$. By induction on $n \in \omega$ we define a sequence $(t^f_n)_{n\in\omega} \subseteq 2^{<\omega}$. Let $t^f_0 := \emptyset$, and supposing t^f_n defined, let $t^f_{n+1} := t(t^{f\frown}_n 0, \mu_f)$. Note that when μ_f is nonatomic, $\text{length}(t^f_n) < \text{length}(t^f_m)$ for all $n < m$.

Next define inductively $g(f,y) \in p(2^\omega)$ on the length of $s \in 2^{<\omega}$. For $s \in 2^n$ such that $s = t^f_k$ for some k, let

$$g(f,y)(s^\frown 0) = \begin{cases} \frac{2}{3} g(f,y)(s) & \text{and } y(k) = 1; \\ \frac{1}{3} g(f,y)(s) & \text{and } y(k) = 0. \end{cases}$$

$$g(f,y)(s^\frown 1) = \begin{cases} \frac{1}{3} g(f,y)(s) & \text{and } y(k) = 1; \\ \frac{2}{3} g(f,y)(s) & \text{and } y(k) = 0. \end{cases}$$

In all other cases, let

$$g(f,y)(s^\frown i) = \begin{cases} 0 & \text{if } f(s) = 0; \\ \frac{g(f,y)(s)}{f(s)} f(s^\frown i) & \text{otherwise.} \end{cases}$$

We will show that the Borel probability measure $\mu_{g(f,y)}$ is equivalent to μ_f. First, define $\Theta\colon 2^{<\omega} \to [0,1]$ by letting

$$\Theta(s) = \frac{g(f,y)(s)}{f(s)}$$

when $f(s) \neq 0$, and let $\Theta(s) = 0$ otherwise. Let $\{s_i \mid i \in \omega\}$ enumerate the set

$$\{s \in 2^{<\omega} \mid \forall n (s \neq t^f_n \text{ and } \exists m (s \restriction (\text{lh}(s) - 1) \sqsubseteq t^f_m)\}.$$

Then for any Borel set $B \subseteq 2^\omega$,

$$\mu_{g(f,y)}(B) = \sum_{i=0}^{\infty} \Theta(s_i) \mu_f(B \cap N_{s_i}).$$

Therefore $\mu_f(B) = 0$ if and only if $f(s_i) = 0$, if and only if $\Theta(s_i) = 0$, if and only if $\mu_{g(f,y)}(B) = 0$. Since μ_f and $\mu_{g(f,y)}$ have the same null sets, they are measure equivalent.

Fix a recursive pairing function

$$\langle \cdot, \cdot \rangle \colon [0,1]^{(2^{<\omega})} \times 2^\omega \to 2^\omega,$$

and for $i < 2$ let $z \mapsto (z)_i$ be the maps such that $z = \langle (z)_0, (z)_1 \rangle$ for all $z \in 2^\omega$.

[1]Here, $N_s \subseteq 2^\omega$ denotes the basic open set for the Polish topology on 2^ω, that is, $N_s = \{t \in 2^{<\omega} \mid s \sqsubseteq t\}$.

We define $A' \subseteq P_c(2^\omega)$ to be the set

$$A' = \{\mu_{g(f,\langle f,y\rangle)} \mid (\mu_f, y) \in F\},$$

and as $\mu_{g(f,\langle f,y\rangle)}$ and μ_f are measure equivalent, A' is clearly a maximal family of atomless Borel probability measures on 2^ω. By adding to A' the Dirac point mass measures (a closed subset of $P(2^\omega)$), we obtain a maximal orthogonal family in $P(2^\omega)$.

Moreover, we claim that A' has the following Π^1_1 definition:

$$\mu \in A' \Leftrightarrow \exists (f,y) \in \Delta^1_1(\mu)[(\mu_f, y) \in F \text{ and } \mu_{g(f,\langle f,y\rangle)} = \mu].$$

Indeed, given $\mu \in P_c(2^\omega)$, there is a recursive relation $R \subseteq p_c(2^\omega) \times 2^\omega$ expressing that the $g \in p_c(2^\omega)$ such that $\mu = \mu_g$ codes a real $z \in 2^\omega$. Specifically, let

$$\begin{aligned} R(g,z) \Leftrightarrow & \forall n \in \omega[z(n) = 0 \Leftrightarrow (g(t^g_n {}^\frown 0) = \frac{1}{3} g(t^g_n) \wedge g(t^g_n {}^\frown 1) = \frac{2}{3} g(t^g_n)) \\ & \wedge z(n) = 1 \Leftrightarrow (g(t^g_n {}^\frown 0) = \frac{2}{3} g(t^g_n) \wedge g(t^g_n {}^\frown 1) = \frac{1}{3} g(t^g_n))]. \end{aligned}$$

So if $\mu = \mu_g$, one checks if there is a $z \in 2^\omega$ such that $R(g,z)$, and in this case one recovers $f \in p_c(2^\omega)$ and $y \in 2^\omega$ such that $z = (f,y)$. Therefore $(f,y) \in \Delta^1_1(\mu)$, finishing the proof. □

A shorter proof of the above was given later by Schrittesser and Törnquist in [ST18, Lemma 4.2], and we include this below for completeness.

Proof. (of Theorem 7.1)
Let $A \subseteq P(2^\omega)$ be a maximal orthogonal family, and again we may suppose $A \subseteq P_c(2^\omega)$. Fix $F \subseteq P_c(2^\omega) \times 2^\omega$ such that $\mu \in A$ if and only if there is y such that $(\mu, y) \in F$. Again we assume F is the graph of a function.

For each $n \in \omega$ and $s \in 2^n$, let N_s denote the basic open neighborhood of 2^ω given by s, that is $N_s = \{x \in 2^\omega \mid s \subseteq x\}$. For $\mu \in P_c(2^\omega)$, let y_μ be the left-most branch of $\{t \in 2^{<\omega} \mid \mu(N_t) > 0\}$. As μ is atomless we have $C = \{n \in \omega \mid \mu(N_{y_\mu^\frown(0)}) > 0 \text{ and } \mu(N_{y_\mu^\frown(1)}) > 0\}$ is infinite, so let $C(n)$ denote the nth element of C.

Define $h\colon P_c(2^\omega) \to 2^\omega$ by letting

$$h(\mu)(i) = \begin{cases} 0 & \text{if } \mu(N_{y_\ell \restriction C(i)^\frown(0)}) \geq \mu(N_{y_\ell \restriction C(i)^\frown(1)}); \\ 1 & \text{otherwise.} \end{cases}$$

Then we have $h(F(\mu, y)) = y$.

Let $\langle \cdot, \cdot \rangle\colon 2^\omega \times 2^\omega \to 2^\omega$ be a recursive bijection, and for $i < 2$ and $z \in 2^\omega$ let $z \mapsto (z)_i$ be the coordinate maps, so $z = \langle (z)_0, (z)_1 \rangle$. Finally, fix a recursive bijection $\phi\colon 2^\omega \to P_c(2^\omega)$.

Let $A' = \{g(\mu, \phi^{-1}(\mu) \oplus y) \mid (\mu, y) \in F\}$. Then since all measures in A' are measure equivalent to those in A, we have A' is also a maximal orthogonal family in $P_c(2^\omega)$. Moreover, A' is Π^1_1 since for all $\mu \in P_c(2^\omega)$,

$$\begin{aligned} \mu \in A' \Leftrightarrow \forall z, \nu, y[(z = h(\mu) \wedge \nu = \phi((z)_0) \wedge y = (z)_1) \\ \Rightarrow (\mu = g(\nu, z) \wedge (\nu, y) \in R)], \end{aligned}$$

and this is clearly a Π^1_1 formula. □

8. Ultrafilters

A collection $F \subseteq \mathcal{P}(X)$ is a *filter* on a set X if $X \in F$, $\emptyset \notin F$, $x, y \in F$ implies $x \cap y \in F$, and if $x \in F$ and $y \supseteq x$ then $y \in F$. A filter F is an *ultrafilter* if it is maximal in the sense that for all $x \subseteq X$, either $x \in F$ or $X \setminus x \in F$. That any filter is contained in an ultrafilter is a traditional application of Zorn's lemma. If U is an ultrafilter on X, $A \subseteq \mathcal{P}(X)$ is a *base* for U if A is closed under finite intersection and for every $x \in U$, there is $y \in A$ such that $y \subseteq x$.

Considering ultrafilters on ω, Sierpinski showed that ultrafilters cannot be Lebesgue measurable nor have the property of Baire; this follows from the fact that any filter on ω is either Lebesgue null or nonmeasurable, and similarly in terms of Baire category, any such filter is either meager or does not have the Baire property. Since an analytic ultrafilter base would give an analytic ultrafilter, there do not exist analytic ultrafilter bases.

However, Schilhan showed that in Gödel's L, there exists a Π^1_1 base for an ultrafilter, so the resulting ultrafilter is Σ^1_2. Since any Σ^1_2 ultrafilter is also Π^1_2, obtaining a Δ^1_2 ultrafilter is optimal.

One can likewise ask about definability of bases for ultrafilters satisfying stronger combinatorial properties: an ultrafilter U is called a *P-point* if for any countable collection $A \subseteq U$ admits a pseudointersection in U, i.e. $y \in U$ so that for all $x \in A$, $x \subseteq^* y$. An ultrafilter U is called a *Q-point* if for any countable partition $\omega = \bigcup_{n\in\omega} P_n$ with each P_n finite, there is $x \in U$ such that for all $n \in \omega$, $x \cap P_n$ contains at most one element. Schilhan gave constructions of Π^1_1 bases for a P-point and a Q-point ([Sch19, Theorem 1.1, Theorem 1.2]). In contrast, [Sch19, Theorem 5.1] shows that consistently every P-point which is Δ^1_2 has no Π^1_1 base .

For ultrafilters in general, there is equivalence between the existence of a $\Delta^1_{n+1}(r)$ ultrafilter and the existence of a $\Pi^1_n(r)$ ultrafilter base, for all $r \in \omega^\omega$ and $n \in \omega$. We give below the proof for the nontrivial direction of the simpler statement:

Theorem 8.1 ([Sch19, Theorem 1.4])**.** *If there exists a Δ^1_2 ultrafilter on ω, then there exists an ultrafilter on $\omega \times \omega$ with a Π^1_1 base.*

Proof. Let $U \subseteq \mathcal{P}(\omega)$ be a Δ^1_2 ultrafilter, and let $F \subseteq \mathcal{P}(\omega) \times 2^\omega$ be Π^1_1 such that

$$x \in U \leftrightarrow \exists w \in 2^\omega[(x, w) \in F].$$

The *Fubini product* of U is the collection of subsets of $\omega \times \omega$ defined by

$$U \otimes U := \{y \in \mathcal{P}(\omega) : \{n \in \omega \mid \{m \in \omega \mid (n, m) \in y\} \in U\} \in U\}.$$

One can check that if U is an ultrafilter on ω then $U \otimes U$ is an ultrafilter on $\omega \times \omega$, and moreover if U is Σ^1_2 definable then so is $U \otimes U$. We will construct a Π^1_1 base $A \subseteq U \otimes U$.

Fix a recursive function $r \colon \omega \times 2^\omega \to 2^\omega$ such that for any sequence $(w_n)_{n<\omega} \subseteq 2^\omega$, there exists $z \in 2^\omega$ which is not eventually constant such that $r(n, z) = w_n$. For $y \in [\omega \times \omega]^\omega$, let $y_n = \{m \in \omega \mid (n, m) \in y\}$ denote the nth vertical section of y, and let $z(y)$ denote the set of $n \in \omega$ such that $y_n \neq \emptyset$. In the case $z(y)$ is infinite, let y^n to denote the nth element of $z(n)$.

Define a function $O \colon [\omega \times \omega]^\omega \to 2^\omega$ by letting

$$O(y)(n) = \begin{cases} 0 & \text{if } |z(y)| < \omega; \\ 0 & \text{if } \min y^n \geq \min y^{n+1}; \\ 1 & \text{if } \min y^n < \min y^{n+1}. \end{cases}$$

This is clearly a recursive function. The idea is to code an infinite set $y \subseteq \omega \times \omega$ into an element of 2^ω via the sequence of integers determined by $(\min y^n)_{n\in\omega}$. We define the Π^1_1 base for $U \otimes U$ to be the subset $X \subseteq [\omega \times \omega]^\omega$, where

$$y \in X \Leftrightarrow |z(y)| = \omega \wedge (z(y), r(0, O(y))) \in F$$
$$\wedge \forall n \in \omega \exists s \in [\omega]^{<\omega}[(s \cup y^n, r(n+1, O(y))) \in F].$$

In other words, we let $y \in X$ iff y has infinitely many nonempty sections; the real coding y, $O(y)$ codes via the function r a countable sequence $(w_n)_{n\in\omega}$ such that $(z(y), w_0) \in F$, so $z(y) \in U$; and for some finite set $s \in [\omega]^{<\omega}$, $(s \cup y^n, w_{n+1}) \in F$, so $s \cup y^n$ and hence $y^n \in U$. This shows that $X \subseteq U \otimes U$.

Moreover we can show that X is indeed a base for $U \otimes U$: given any $x \in U \otimes U$, let $y_0 = \bigcup\{\{n\} \times x_n \mid n \in \omega, x_n \in U\}$, where x_n is the nth column of x. Then $z(y_0) = \{n \in \omega \mid x_n \in U\}$. As $x \in U \otimes U$, $z(y_0) \in U$, so there is $w_0 \in 2^\omega$ such that $(z(y_0), w_0) \in F$. Similarly for $n \in z(y_0)$, let w_{n+1} be such that $(x_n, w_{n+1}) \in F$. Then there exists $w \in 2^\omega$ which is not eventually constant such that for each n, $r(n, w) = w_n$. Let $e_y \colon \omega \to z(y_0)$ be the increasing enumeration of the vertical sections of y_0 which are in U. Recursively construct a sequence of integers $(m_n)_{n\in\omega}$ with $m_n \in (y_0)_{e_y(n)}$ coding w in the sense that we let $m_n < m_{n+1}$ iff $w(n) = 1$. Explicitly, whenever $w(n) = 1$, since $(y_0)_{e_y(n+1)} \in U$, there is some $m_{n+1} \in (y_0)_{e_y(n+1)}$ with $m_{n+1} > m_n$. Whenever there is some finite block of 0s $w(n+1), \dots, w(n+k)$, there is

$$m_{n+1} = \cdots = m_{n+k} \in \bigcap\{(y_0)_{e_y(i)} \mid n < i \leq n+k\},$$

as U is closed under finite intersections. Now define $y \bigcup\{\{e_y(n)\} \times ((y_0)_{e_y(n)} \setminus m_n) \mid n \in \omega\}$. Since for all n we have $(y_0)_{e_y(n)} = x_k$ for some $k \geq n$, we have $y \subseteq x$. Moreover, the set $z(y) = \{n \in \omega \mid x_n \in U\}$ is infinite, and that $O(y) = w$, and that there is some finite s, namely $((y_0)_{e_y(n)}) \cap m_n$, such that $(s \cup y^n, r(n+1, w)) \in F$. Therefore $y \in X$, finishing the proof. □

9. Hausdorff gaps

For $A, B \subseteq [\omega]^\omega$, the pair (A, B) is called a *pre-gap* if both A, B are well-ordered by $\subseteq^*$, and for all $x \in A$ and $y \in B$, $x \cap y$ is finite. Such a pair is a *gap* if there is no $z \in [\omega]^\omega$ such that $x \setminus z$ and $y \cap z$ are finite for all $x \in \mathcal{A}$ and $y \in \mathcal{B}$. Such a z is said to *separate* or *interpolate* (A, B). If (A, B) is a gap with $|A| = \kappa$ and $|B| = \lambda$, then (A, B) is called a (κ, λ)-gap. If (A, B) is a (κ, λ)-pre-gap, and $\kappa, \lambda < \omega_1$, then they can be separated . We call an (ω_1, ω_1)-gap (A, B) a *Hausdorff gap* if it satisfies the following condition:

$$\forall \alpha < \omega_1 \forall k < \omega \ \{\gamma < \alpha \mid a_\alpha \cap b_\gamma \subseteq k\} \text{ is finite.} \tag{2}$$

One can prove that (ω_1, ω_1)-pre-gaps satisfying the above condition are indeed gaps. In fact, Hausdorff gaps are very impervious to forcing extensions in the sense that a Hausdorff gap in the ground model will remain unseparated in any forcing extension preserving ω_1; see [Sch93]. However, it is consistent that not every (ω_1, ω_1)-gap satisfies the Hausdorff condition; see, for example, [BNC15].

We will restrict our attention to (ω_1, ω_1)-(pre-)gaps and for such pairs (A, B), we will say (A, B) is a $(\Gamma, \cdot)$ or (Γ, Γ) (pre-)gap if Γ is a pointclass and $A \in \Gamma$ or both $A, B \in \Gamma$.

A weaker assumption than being well-ordered by $\subseteq^*$ is assuming A, B are *σ-directed*, meaning that for every countable $\{a_n\} \subseteq A$, there is $a \in A$ such that for all n, $a_n \subseteq^* a$, and analogously for B. Todorcevic showed that even in this case, there is no $(\Sigma^1_1, \cdot)$ gap (see [Tod96], Corollary 1). It is interesting to note that dropping even the assumption of σ-directedness, there exist gaps (A, B) with A, B being perfect, hence closed, sets; see [Tod96, Theorem 2] or [Kho12, Section 4.6].

Beyond ZFC, we have that there are no gaps in Solovay's model, nor in a model of the theory ZF+($\mathrm{AD}_{\mathbb{R}}$).

- There are no gaps in Solovay's model ([Kho12, Section 4.4]);
- Under ZF+($\mathrm{AD}_{\mathbb{R}}$) there are no gaps ([Kho14]).

Via a Cantor-Bendixson derivative style argument, Khomskii [Kho12, Theorem 4.2.3, Corollary 4.2.4] shows that if for all $r \in \omega^\omega$, $\omega_1^{L[r]} < \omega_1$, then there are no $(\mathbf{\Pi^1_1}, \cdot)$ gaps. Nonetheless, a recursive construction in L gives a $(\mathbf{\Sigma^1_2}, \mathbf{\Sigma^1_2})$ Hausdorff gap, which by an application of Miller's method yields a $(\mathbf{\Pi^1_1}, \mathbf{\Pi^1_1})$ Hausdorff gap. A consequence of constructing a $\mathbf{\Pi^1_1}$-definable gap in L which moreover satisfies (2) above is that the inseperability of the gap is absolute, and thus the gap is indestructible by any forcing preserving ω_1. Thus, we have the converse implication of the above, namely that if there is r with $\omega_1^{L[r]} = \omega_1$, then there is a $(\Pi^1_1(r), \Pi^1_1(r))$ Hausdorff gap.

Below we show that there is a direct way of constructing a (Π^1_1, Π^1_1) Hausdorff gap given one which is (Σ^1_2, Σ^1_2). This will rely on showing that the analogue of condition (2) in $\mathcal{P}(\omega\times\omega)$ is sufficient for an (ω_1, ω_1) pre-gap in $(\mathcal{P}(\omega \times \omega), \subseteq^*)$ to be a gap.

Proposition 9.1. *Suppose (A, B) is an (ω_1, ω_1)-pre-gap in the space $\mathcal{P}(\omega \times \omega)$, and that for every $\alpha < \omega_1$ and all $k \in \omega$, the set*

$$\{\gamma < \omega_1 \mid a_\alpha \cap b_\gamma \subseteq k \times k\}$$

is finite. Then there is no $c \in [\omega \times \omega]^\omega$ such that for all $a \in A$ and for all $b \in B$, $a \setminus c$ and $c \cap b$ is finite. In other words, (A, B) is a gap.

Proof. Suppose not; that is, there is $c \in [\omega \times \omega]^\omega$ such that for every $\alpha < \omega_1$ there is $k_\alpha < \omega$ such that $a_\alpha \setminus c \subseteq k_\alpha \times k_\alpha$) and $c \cap b_\alpha \subseteq k_\alpha \times k_\alpha$. Then there is an uncountable $X \subseteq \omega_1$ and $k < \omega$ such that for every $\alpha \in X$, $k_\alpha = k$. As X is uncountable we can find $\alpha \in X$, $\alpha \geq \omega$, such that $X \cap \alpha$ is infinite. Then since $a_\alpha \cap b_\gamma \subseteq (a_\alpha \setminus c) \cup (b_\gamma \cap c) \subseteq k \times k$ for all $\gamma \in X \cap \alpha$, the set

$$\{\gamma \in X \cap \alpha \mid a_\alpha \cap b_\gamma \subseteq k \times k\}$$

is infinite, a contradiction. □

Theorem 9.2. *Suppose A, B are Σ^1_2-definable subsets such that (A, B) is a Hausdorff gap. Then there exists a Hausdorff gap (A', B') in $\mathcal{P}(\omega \times \omega)$ such that A', B' are Π^1_1.*

Proof. Let A, B be as above, and let $F, G \subseteq [\omega]^\omega \times 2^\omega$ be Π^1_1 graphs of functions such that

$$a \in A \Leftrightarrow \exists x \in 2^\omega((a, x) \in F) \quad \text{and} \quad b \in B \Leftrightarrow \exists y \in \omega^\omega((b, y) \in G).$$

Define a functions continuous functions $g_A, g_B\colon [\omega]^\omega \times 2^\omega \to [\omega\times\omega]^\omega$ by letting

$$g_A(a,x) = (\{0\}\times a) \cup \bigcup_{n\in\omega} (\{2n+2\} \times (a \setminus a(n+x(n)))),$$

and

$$g_B(b,y) = (\{0\}\times b) \cup \bigcup_{n<\omega} (\{2n+1\} \times (b \setminus b(n+y(n)))).$$

Let $A' := g_A[F]$, and $B' := g_B[G]$. First, we have that both A', B' are linearly ordered by the relation $\subseteq^*$ on $[\omega\times\omega]^\omega$, as the proof goes through exactly as for the case of towers (see Section 5, Theorem 5.1).

To see A' and B' is a pre-gap, for any $c\in A'$ and $d \in B'$, $c_0 \cap d_0$ is finite as (A,B) was a pre-gap, and for $n\geq 1$, $c_n \cap b_n = \emptyset$, as $A' \cap \bigcup_{n<\omega}\{2n+1\}\times\omega = \emptyset = B' \cap \bigcup_{n<\omega}\{2n\}\times\omega$.

Write $A' = \{c_\alpha \mid \alpha<\omega_1\}$ and $B' = \{d_\alpha \mid \alpha < \omega_1\}$. We claim that for every $\alpha<\omega_1$ and for all $k\in\omega$, the set

$$\{\gamma < \alpha \mid a'_\alpha \cap b'_\gamma \subseteq (k,k)\}$$

is finite. Suppose not, and let $\alpha<\omega_1$ and $k<\omega$ be such that there are infinitely many ordinals $\gamma<\alpha$ with $a'_\alpha \cap b'_\gamma \subseteq (k,k)$. But notice if $(n,m)\in a'_\alpha \cap b'_\gamma$, by the previous observation it must be $n=0$ and $m \in a_\alpha \cap b_\gamma$, where $a_\alpha\in A$ and $b_\gamma \in B$ are such that $g_A(a_\alpha, x_\alpha) = a'_\alpha$ and $g_B(b_\gamma, y_\gamma) = b'_\gamma$. Therefore

$$\{\gamma<\alpha : a_\alpha \cap b_\gamma \subseteq k\}$$

is infinite, contradicting the hypothesis on (A,B). The previous proposition then shows that (A', B') is indeed a gap.

To see that A', B' have Π^1_1 definitions, we show that

$$c\in A' \Leftrightarrow \exists (a,x) \in \Delta^1_1(c)[(a,x)\in F \wedge g_A(a,x) = z],$$

and similarly for membership in B'. Given $c\in\mathcal{P}(\omega\times\omega)$, we check if $c\cap\bigcup_{n<\omega} 2_n \times\omega \neq \emptyset$ and $c\cap \bigcup_{n\in\omega}\{2n+1\}\times\omega = \emptyset$; when this is the case, one uses the set c_0 to effectively define a real $x\in 2^\omega$, by letting

$$x(n) = \begin{cases} 0 & \text{if } \min c_{2n+2} = c_0(n+1); \\ 1 & \text{if } \min c_{2n+2} = c_0(n+2). \end{cases}$$

Therefore there are some $(a,x)\in\mathcal{P}(\omega)\times\omega^\omega$ constructible from c. To check if $g_A(a,x) = c$ is Δ^1_1, and $(a,x)\in F$ is Π^1_1, so the result follows. The case for B' is similar. □

10. Concluding remarks

We conclude with some open questions, including those from the authors discussed in this paper which have remained unanswered.

Above we showed positive instances of when there is a method of coding two reals into one in such a way that a desired combinatorial property was preserved. Of course, it seems that coming up with an appropriate coding for preserving other properties just amounts to more clever coding, however it would be interesting if there is a case which this can provably not be done.

Question 1. Do there exist families for which one can prove that there consistently exist Σ^1_2 but no Π^1_1 examples?

All of the proofs presented in this paper were specific to the classes Σ^1_2 and Π^1_1, with the exception of Theorem 8.1. The obstruction for lifting the proofs in the other cases is the use of Π^1_1 uniformization and the Spector-Gandy theorem. One axiom sufficient for obtaining the higher analogue of the latter is Projective Determinacy (see [MM22]), however this would be of little help because PD also imposes Baire measurability of sets in those classes.

Question 2. If there exist liftings of the above proofs to higher projective pointclasses?

Reviving a question posed in [FKK14] and [Kho14], the following seems to still be unanswered:

Question 3. Does AD imply there are no gaps ?

The following also seems to be open:

Question 4. Does AD imply there are no mad families?

The goal of this survey has been to overview methods which perhaps will give authors ideas for answering the following:

Question 5. Are there ZFC derivations of Π^1_1 examples from Σ^1_2 for the following:

- Hamel bases (see, for example, [Vid14, Corollary 4.11] for a Π^1_1 construction in L);
- Cofinitary groups (see, for example, [FST17] for Π^1_1 construction in L which is Cohen indestructible);
- Van Douwen mad families (see [Rag08], where it is shown that such families cannot be analytic).

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