A COMPLETE FILTER ASSOCIATED WITH A THIN LIST

KENTA TSUKUURA

Abstract. In this paper, we construct a κ-complete filter $F_{\overline{d}}$ over λ using a thin $\mathcal{P}_\kappa(2^\lambda)$-list $\overline{d}$. We show that if $\overline{d}$ has no branch, then $F_{\overline{d}}$ cannot be extended to a κ-complete ultrafilter. This result allows us to demonstrate that if every κ-complete filter over λ can be extended to a κ-complete ultrafilter, then $\square(2^\lambda)$ fails, without relying on the compactness theorem of $\mathcal{L}_{\kappa\kappa}$.

1. Introduction

The notion of strong compactness has been widely studied in the contexts of infinitary logic, large cardinals, and infinitary combinatorics. The original definition is due to Tarski. For a regular cardinal κ, the infinitary logic $\mathcal{L}_{\kappa\kappa}$ is a logic allows conjunctions of $<\kappa$-many formulas and the use of $<\kappa$-many quantifiers to define formulas. $\mathcal{L}_{\omega\omega}$, as classical first-order logic, satisfies the compactness theorem. For an uncountable cardinal κ, κ is strongly compact if and only if $\mathcal{L}_{\kappa\kappa}$ satisfies the compactness theorem. As is widely known, strong compactness has many characterizations. The following are equivalent:

(1) κ is strongly compact.
(2) $\mathcal{P}_\kappa\lambda$ carries a fine ultrafilter for all $\lambda \geq \kappa$.
(3) For every κ-complete filter F, F can be extended to a κ-complete ultrafilter, that is, there exists a κ-complete ultrafilter U such that $F \subseteq U$.
(4) A two-cardinal tree property $\mathrm{TP}(\kappa, \lambda)$ holds for all $\lambda \geq \kappa$, and κ is inaccessible.

In [1], Hayut analyzed these relations.

Theorem 1 (Hayut [1]). *For regular cardinals $\kappa \leq \lambda = \lambda^{<\kappa}$, the following are equivalent:*

(1) *Every κ-complete filter over λ can be extended to a κ-complete ultrafilter.*
(2) *$\mathcal{L}_{\kappa\kappa}$ satisfies the compactness theorem for a theory of size 2^λ. That is, for every $\mathcal{L}_{\kappa\kappa}$-theory T with 2^λ-many symbols, if T is $<\kappa$-consistent, then T is consistent.*

In particular, if every κ-complete filter over λ can be extended to a κ-complete ultrafilter, then $\square(\mu, <\kappa)$ fails for all regular $\lambda \leq \mu \leq 2^\lambda$.

Hayut also pointed out the following implications concering compactness of κ:

	2^λ-strongly compact	$\Rightarrow$	λ-compact
$\Leftrightarrow$	$\mathcal{L}_{\kappa\kappa}$ satisfies the comp. thm. for T of size 2^λ	$\Rightarrow$	λ-strongly compact

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Here, λ-strongly compactness asserts the existence of a fine ultrafilter over $\mathcal{P}_\kappa(\lambda)$. λ-compactness is (1) in Theorem 1.

The proof of Theorem 1 involves interpreting $\mathcal{L}_{\kappa 1}$-formulas and this make the proof a bit complex to understand. To better understand Theorem 1, we focus on the combinatorial aspects of strong compactness. We also provide a combinatorial proof of $\neg\square(\mu, {<}\kappa)$. Specifically:

Theorem 2. *For every regular cardinals $\aleph_2 \leq \kappa \leq \lambda = \lambda^{<\kappa} \leq \mu \leq 2^\lambda$, if $\neg\mathrm{TP}(\kappa,\mu)$, then there exists a κ-complete filter F over λ (generated by $\mu^{<\kappa}$-many sets) that cannot be extended to a κ-complete ultrafilter. In particular, a $\square(\mu, {<}\kappa)$-sequence defines such a κ-complete filter over λ.*

The organization of this paper is as follows: Section 2 provides preliminaries. Section 3 is devoted to the proof of Theorem 2. Section 4 explores Namba forcings.

2. Preliminaries

We use [3] as a reference for set theory in general. Our notation is standard. Throughout this paper, κ and λ denote regular cardinals greater than $\aleph_2$, unless otherwise stated. Typically, we assume $\aleph_2 \leq \kappa \leq \lambda = \lambda^{<\kappa}$. The symbol μ is used to denote an infinite cardinal.

An important concept in this paper is the two-cardinal tree property $\mathrm{TP}(\kappa,\lambda)$. A $\mathcal{P}_\kappa\lambda$*-list* is a sequence $\overline{d} = \langle d_a \mid a \in \mathcal{P}_\kappa\lambda\rangle$, where $d_a \subseteq a$ for all $a \in \mathcal{P}_\kappa\lambda$. We say that $\overline{d}$ is *thin* if there exists a club $C \subseteq \mathcal{P}_\kappa\lambda$ such that $|\{d_a \cap c \mid c \subsetneq a\}| < \kappa$ for all $c \in C$. We denote by $\mathrm{Lev}_c(\overline{d})$ the set $\{d_a \cap c \mid c \subseteq a\}$. A *branch* of $\overline{d}$ is a set $d \subseteq \lambda$ such that, for all $a \in \mathcal{P}_\kappa\lambda$, $d \cap a \in \mathrm{Lev}_a(\overline{d})$. The property $\mathrm{TP}(\kappa,\lambda)$ asserts the *nonexistence* of a thin $\mathcal{P}_\kappa\lambda$-list with *no* branches. Note that $\mathrm{TP}(\kappa,\kappa)$ is equivalent to the *nonexistence* of a κ-Aronszajn tree.

For Theorem 2, the following lemma is central:

Lemma 3. *For regular cardinals $\kappa \leq \lambda = \lambda^{<\kappa}$, there exists a family $\langle A_\alpha \mid \alpha < 2^\lambda\rangle$ of subsets of λ such that, for every $a, b \in \mathcal{P}_\kappa 2^\lambda$, if $a \cap b = \emptyset$, then*

$$|\bigcap_{\alpha\in a} A_\alpha \setminus \bigcup_{\alpha \in b} A_\alpha| = \lambda.$$

Proof. See [1]. □

3. Proof of Theorem 2

For regular cardinals $\kappa \leq \lambda = \lambda^{<\kappa} \leq \mu \leq 2^\lambda$ and a thin $\mathcal{P}_\kappa\mu$-list $\overline{d}$, let us define a κ-complete filter $F_{\overline{d}}$ over λ. Let C be a club such that, for all $a \in \mathcal{P}_\kappa\lambda$, there exists $c_a \in C$ with $a \subseteq c_a$ and $|\mathrm{Lev}_{c_a}(\overline{d})| < \kappa$.

For $a \in \mathcal{P}_\kappa\mu$, define B_a by

$$B_a = \bigcup_{d \in \mathrm{Lev}_{c_a}(\overline{d})} \left(\bigcap_{\alpha \in d\cap a} A_\alpha \setminus \bigcup_{\alpha \in a \setminus d} A_\alpha \right).$$

Here, $\langle A_\alpha \mid \alpha < 2^\lambda\rangle$ comes from Lemma 3. By Lemma 3, $|B_a| = \lambda$.

Lemma 4. *The family $\{B_a \mid a \in \mathcal{P}_\kappa\mu\}$ generates a κ-complete filter.*

Proof. For every $X \in [\mathcal{P}_\kappa\mu]^{<\kappa}$, fix $c \in C$ such that $\bigcup_{a\in X} c_a \subseteq c$. We can choose $d \in \mathrm{Lev}_c(\overline{d})$. Then the following hold:

- $d \cap c_a \in \mathrm{Lev}_{c_a}(\overline{d})$.
- $|\bigcap_{\alpha\in d} A_\alpha \setminus \bigcup_{\alpha\in c\setminus d} A_\alpha| = \lambda$.

Let $B' = \bigcap_{\alpha\in d} A_\alpha \setminus \bigcup_{\alpha\in c\setminus d} A_\alpha$. It suffices to prove that $B' \subseteq \bigcap_{a\in X} B_a$. For each $a \in X$, $B' \subseteq \bigcap_{\alpha\in(d\cap c_a)\cap a} A_\alpha \setminus \bigcup_{\alpha\in a\setminus(d\cap c_a)} A_\alpha$. By $d\cap c_a \in \mathrm{Lev}_{c_a}(\overline{d})$, $B' \subseteq B_a$. Thus, the proof is complete. □

Let $F_{\overline{d}}$ be the filter generated by $\{B_a \mid a \in \mathcal{P}_\kappa\mu\}$. The following key lemma is crucial for proving our main theorem:

Lemma 5. *If there exists a κ-complete ultrafilter U over λ such that $F_{\overline{d}} \subseteq U$, then $\overline{d}$ has a branch.*

Proof. Define $d \subseteq \mu$ by $\alpha \in d$ if and only if $A_\alpha \in U$. We claim that d is a branch of $\overline{d}$. For every $a \in \mathcal{P}_\kappa\mu$, since $B_a \in U$ and U is a κ-complete ultrafilter, there exists $e \in \mathrm{Lev}_{c_a}(\overline{d})$ such that

$$\bigcap_{\alpha\in e\cap a} A_\alpha \setminus \bigcup_{\alpha\in a\setminus e} A_\alpha \in U.$$

Thus, we have:

- For all $\alpha \in e\cap a$, $A_\alpha \in U$.
- For all $\alpha \in a\setminus e$, $A_\alpha \notin U$.

By the definition of d, $d\cap a = e\cap a$. Since $e \in \mathrm{Lev}_{c_a}(\overline{d})$, there exists $c' \in \mathcal{P}_\kappa\mu$ such that $a \subseteq c_a \subseteq c'$ and $e = d_{c'} \cap c_a$. Therefore, $d\cap a = e\cap a = d_{c'}\cap a \in \mathrm{Lev}_a(\overline{d})$, as desired. □

Proof of Theorem 2. Suppose there exists a thin $\mathcal{P}_\kappa\mu$-list $\overline{d}$ with *no* branch. By Lemma 5, $F_{\overline{d}}$ cannot be extended to a κ-complete ultrafilter.

It is also known that a $\square(\mu, {<}\kappa)$-sequence provides such a thin $\mathcal{P}_\kappa\mu$-list. For a proof, see [4]. Therefore, a $\square(\mu, {<}\kappa)$-sequence defines the required κ-complete filter over λ. □

Our proof can also help in understanding Theorem 1.

Theorem 6 (Jech [2])**.** *For regular cardinals $\kappa \leq \lambda = \lambda^{<\kappa}$, the following are equivalent:*

1. *$\mathcal{L}_{\kappa\kappa}$ satisfies the compactness theorem for a theory of size λ.*
2. *κ is inaccessible and* $\mathrm{TP}(\kappa, \lambda)$.

Corollary 7. *For regular cardinals $\kappa \leq \lambda$, the following are equivalent:*

1. *Every κ-complete filter over λ can be extended to a κ-complete ultrafilter.*
2. *$\mathcal{L}_{\kappa\kappa}$ satisfies the compactness theorem for a theory of size 2^λ.*
3. *κ is inaccessible and* $\mathrm{TP}(\kappa, 2^\lambda)$.

4. Semiproperness of Namba forcings

The notions of thin $\mathcal{P}_\kappa\lambda$-lists $\overline{d}$ and $F_{\overline{d}}$ can be defined even if κ is a successor cardinal. To study the compactness properties underlying $F_{\overline{d}}$, we introduce Namba forcing.

For a κ-complete fine ideal I over $Z \subseteq \mathcal{P}(X)$, an ($I$-)Namba tree p is a set $p \subseteq [Z]^{<\omega}$ satisfying the following conditions:

1. p is a tree. That is, each $s \in p$ is $\subseteq$-increasing, and p is closed under initial segments.

(2) There exists a maximal $s \in p$ such that $\forall t \in p(s \subseteq t \vee t \subseteq s)$. This s is called the trunk, denoted $\mathrm{tr}(p)$, of p.
(3) For each $s \in p$, if $s \sqsupseteq \mathrm{tr}(p)$, then $\mathrm{Suc}(s) = \{a \in Z \mid s^\frown\langle a\rangle \in p\} \in I^+$.

Let $\mathrm{Nm}(Z, I)$ denote the set of all I-Namba trees. For a filter F, $\mathrm{Nm}(Z, F)$ is defined as $\mathrm{Nm}(Z, F^*)$. We write $\mathrm{Nm}(\kappa)$ and $\mathrm{Nm}(\kappa, \lambda)$ to refer to $\mathrm{Nm}(\kappa, \text{bounded ideal})$ and $\mathrm{Nm}(\mathcal{P}_\kappa\lambda, \text{bounded ideal})$, respectively. $\mathrm{Nm}(\aleph_2)$ corresponds to the original Namba forcing. Since every Namba forcing $\mathrm{Nm}(Z, I)$ is ω_1-stationary preserving, every Namba forcing can be semiproper.

The semiproperness of Namba forcings is connected to reflection principles (see Lemma 9). A typical example is the following:

Theorem 8. *For a κ-complete ideal I over λ, if $\mathrm{Nm}(\lambda, I)$ is semiproper, then $\square(\lambda)$ fails.*

Note that $\mathrm{Nm}(\lambda, I)$ forces $\dot{\mathrm{cf}}(\lambda) = \omega$. Theorem 8 follows from Lemma 9 and Theorems 10 and 11.

Lemma 9 (Tsukuura [7])**.** *For regular cardinals $\aleph_2 \leq \kappa \leq \lambda$ and a semistationary subset $S \subseteq [\lambda]^\omega$, if $\mathrm{Nm}(\kappa, \lambda)$ forces that S is semistationary, then there exists $a \in \mathcal{P}_\kappa\lambda$ such that $S \cap [a]^\omega$ is semistationary. Conversely, if we assume $\forall \mu < \kappa(\mu^\omega < \kappa)$ or $\kappa = \aleph_2$, the reverse direction also holds.*

Theorem 10 (Sakai–Veličković [5])**.** *If every semistationary subset $S \subseteq [\lambda]^\omega$ reflects to some $a \in \mathcal{P}_\kappa\lambda$, then $\square(\lambda)$ fails.*

Theorem 11 (Shelah [6] for (1) $\leftrightarrow$ (3), Tsukuura [7] for (1) $\leftrightarrow$ (2))**.** *For a regular cardinal $\kappa \geq \aleph_2$, the following are equivalent:*

(1) *$\mathrm{Nm}(\kappa)$ is semiproper.*
(2) *$\mathrm{Nm}(\kappa, \kappa)$ is semiproper.*
(3) *There exists a semiproper forcing that forces $\dot{\mathrm{cf}}(\kappa) = \omega$.*

Strong compactness directly effects the semiproperness of Namba forcings.

Proposition 12. *For a κ-complete filter F over λ, if F can be extended to a κ-complete ultrafilter, then $\mathrm{Nm}(\lambda, F)$ is semiproper.*

The extendability of $F_{\overline{d}}$ connects both the semiproperness of $\mathrm{Nm}(\lambda, F_{\overline{d}})$ and the existence of a branch in $\overline{d}$. However, these two properties do not imply each other.

It is straightforward to construct a model where $\mathrm{Nm}(\lambda, F_{\overline{d}})$ is *not* semiproper, but $\overline{d}$ has a branch. Indeed, a thin $\mathcal{P}_\kappa\lambda$-list $\overline{d}$ with a branch always exists. On the other hand, obtaining the semiproperness of $\mathrm{Nm}(\lambda, F_{\overline{d}})$ requires the failure of $\square(\lambda)$.

Theorem 13. *It is consistent that there exists a thin $\mathcal{P}_\kappa\lambda$-list $\overline{d}$ such that $\overline{d}$ has* no *branch, but $\mathrm{Nm}(\lambda, F_{\overline{d}})$ is semiproper.*

Proof. Every $\mathrm{Nm}(\lambda, F_{\overline{d}})$ is ω_1-stationary preserving. It is known that if a strongly compact cardinal κ is collapsed to $\aleph_2$ by $\mathrm{Coll}(\aleph_1, {<}\kappa)$, then (†) and CH hold in the extension. Let us consider the extension. Note that CH implies the existence of an $\aleph_2$-Aronszajn tree, and thus $\mathrm{TP}(\aleph_2, \aleph_2)$ fails. Let $\overline{d}$ be a thin $\mathcal{P}_{\aleph_2}\aleph_2$-list with *no* branches. By (†), $\mathrm{Nm}(\aleph_2, F_{\overline{d}})$ must be semiproper. □

The next question concerns whether there exists a thin $\mathcal{P}_\kappa\lambda$-list $\overline{d}$ such that $\overline{d}$ has *no* branches, and $\mathrm{Nm}(\lambda, F_{\overline{d}})$ is *not* semiproper. It seems that stronger anti-compactness principles are needed.

Question 14. *Suppose that* $\square(2^\lambda)$ *holds. Is there an* $\aleph_2$*-complete filter* F *over* λ *such that* $\mathrm{Nm}(\lambda, F)$ *is* not *semiproper? What about* $F_{\vec{d}}$ *for a thin* $\mathcal{P}_\kappa 2^\lambda$*-list defined by* $\square(2^\lambda)$*?*

We conclude this paper with the following diagram, which motivates and illustrates our question. For details on the implications of Namba forcings, see [7]. Note that F denotes a κ-complete filter, and we assume $\aleph_2 \leq \kappa \leq \lambda = \lambda^{<\kappa}$.

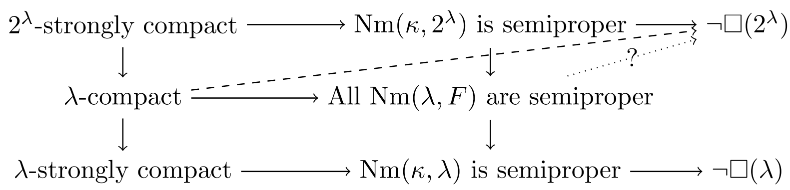

Figure 1. Our question concerns the "dotted" arrow. The "dashed" arrow comes from Theorem 1.

References

[1] Yair Hayut. Partial strong compactness and squares. *Fund. Math.*, 246(2):193–204, 2019.
[2] Thomas Jech. Some combinatorial problems concerning uncountable cardinals. *Ann. Math. Logic*, 5(3):165–198, 1973.
[3] Akihiro Kanamori. *The higher infinite*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, second edition, 2003. Large cardinals in set theory from their beginnings.
[4] Hiroshi Sakai. TP and weak square (accessed: 2025-01-24). `https://www2.kobe-u.ac.jp/~hsakai/Research/notes/tp_weak_square.pdf`.
[5] Hiroshi Sakai and Boban Veličković. Stationary reflection principles and two cardinal tree properties. *J. Inst. Math. Jussieu*, 14(1):69–85, 2015.
[6] Saharon Shelah. *Proper and improper forcing*. Perspectives in Mathematical Logic. Springer-Verlag, Berlin, second edition, 1998.
[7] Kenta Tsukuura. On semiproperness of namba forcings and ideals in prikry extensions. submitted.

Department of Fisheries Distribution and Management, National Fisheries University, Shimonoseki, 759-6533, Japan
Email address: `tsukuura@fish-u.ac.jp`