VARIANTS OF ŁOŚ'S THEOREM

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ABSTRACT. We study Łoś's theorem in a choiceless context. We introduce some variants of Łoś's theorem. These variants seem weaker than Łoś's theorem, but we prove that these are equivalent to Łoś's theorem.

1. INTRODUCTION

Łoś's theorem is a fundamental theorem of ultrapowers and ultraproducts. First we recall it. Throughout this paper, a *filter* over a set means a proper filter, and a *structure* does a first order structure with some language. Let U be an ultrafilter over a set I, and $\{M_i \mid i \in I\}$ an indexed family of structures with language $\mathcal{L}$. For $f \in \prod_{i \in I} M_i$, let $[f]$ denote the equivalence class of f modulo U. Let $\prod_{i \in} M_i/U$ denote the ultraproduct of the family $\{M_i \mid i \in I\}$ by U. If every M_i is the same to the structure M, then the ultraproduct $\prod_{i \in I} M_i/U$ is denoted by ${}^I M/U$, it is the ultrapower of M by U.

Theorem 1.1 (Łoś's fundamental theorem of ultraproducts [4]). *For every $\mathcal{L}$-formula $\varphi(v_0, \dots, v_n)$ and $f_0, \dots, f_n \in \prod_{i \in I} M_i$,*

$$\prod_{i \in I} M_i/U \models \varphi([f_0], \dots, [f_n])$$

$$\iff \{i \in I \mid M_i \models \varphi(f_0(i), \dots, f_n(i))\} \in U.$$

In [5], we proved that some variants of Łoś's theorem and weak choice principles are equivalent to Łoś's theorem in ZF:

Theorem 1.2 ([5]). *In ZF, the following are equivalent:*

    (1) Łoś's theoreom, that is, for every family $\{M_i \mid i \in I\}$ of structures with same language and ultrafilter U over I, if $\prod_{i \in I} M_i \neq \emptyset$ then Łoś's fundamental theorem holds for the ultraproduct $\prod_{i \in I} M_i/U$.

    (2) For every structure M and ultrafilter U over I, the ultrapower ${}^I M/U$ is elementarily equivalent to M.

    (3) For every structure M and ultrafilter U over I, Łoś's fundamental theorem holds for the ultrapower ${}^I M/U$.

    (4) U-AC_I holds (Karagila-Yuan [3]) for every ultrafilter U over I. Where, for an ultrafilter U over I, U-AC_I is the assertion that for every indexed family $\{A_i \mid i \in I\}$ of non-empty sets, there is a function f on I such that $\{i \in I \mid f(i) \in A_i\} \in U$.

The *ultrapower embedding* $j : M \to {}^I M/U$ is the map defined by $j(x) = [c_x]$ for $x \in M$, where $c_x : I \to M$ is the constant map with value x. Under Łoś's theorem, the ultrapower embedding is an elementary embedding.

Theorem 1.3 ([5], (1) $\iff$ (4) is Bilinsky ([1])). *In* ZF, *the following are equivalent:*

(1) *Łoś's theorem.*

(2) *For every structure M and ultrafilter U over I, the ultrapower embedding $j : M \to {}^I M/U$ is an elementary embedding.*

(3) *For every structure M and ultrafilter U over I, if the ultrapower embedding $j : M \to {}^I M/U$ is an elementary embedding, then Łoś's fundamental theorem holds for ${}^I M/U$.*

(4) *For every structure M and ultrafilter U over I, if ${}^I M/U$ is elementarily equivalent to M, then the ultrapower embedding $j : M \to {}^I M/U$ is an elementary embedding.*

In this paper, we study more variants of Łoś's theorem. Variants which we will consider seem weaker than Łoś's theoreom, but we prove that these are in fact equivalent to Łoś's theoreom in ZF.

First we prove that $U\text{-}\mathsf{AC}_I$ can be characterized by bounded forms of Łoś's fundamental theorem:

Theorem 1.4. *In* ZF, *let U be an ultrafilter over a set I. Then the following are equivalent:*

(1) $U\text{-}\mathsf{AC}_I$ *holds.*

(2) *For every structure M, the ultrapower ${}^I M/U$ is Σ_2-elementarily equivalent to M.*

(3) *For every family $\{M_i \mid i \in I\}$ of pairwise isomorphic structures with same language, if $\prod_{i \in I} M_i \neq \emptyset$ then $\prod_{i \in I} M_i/U$ is Σ_1-elementarily equivalent to M_i for every $i \in I$.*

(4) *For every two structures M and N with same language, if there are elementary embeddings $j : M \to N$ and $j' : N \to M$, then ${}^I M/U$ and ${}^I N/U$ are Σ_2-elementarily equivalent.*

(5) *For every two families $\{M_i \mid i \in I\}$, $\{N_i \mid i \in I\}$ of structures with same language, if each M_i is isomorphic to N_i and both $\prod_{i \in I} M_i$ and $\prod_{i \in I} N_i$ are non-empty, then $\prod_{i \in I} M_i/U$ is Σ_1-elementarily equivalent to $\prod_{i \in I} N_i/U$.*

(6) *For every two families $\{M_i \mid i \in I\}$, $\{N_i \mid i \in I\}$ of structures with same language, if each M_i is isomorphic to N_i and both $\prod_{i \in I} M_i$ and $\prod_{i \in I} N_i$ are non-empty, then $\prod_{i \in I} M_i/U$ is isomorphic to $\prod_{i \in I} N_i/U$.*

It is known that $U\text{-}\mathsf{AC}_I$ holds if, and only if, for every family $\{M_i \mid i \in I\}$ of structures with same language, Łoś's fundamental theorem holds for the ultraproduct $\prod_{i \in I} M_i/U$ (e.g., see [5]). As an immediate consequence, we have:

Theorem 1.5. *In* ZF, *the following are equivalent:*

(1) *Łoś's theorem.*

(2) *For every structure M and ultrafilter U over I, the ultrapower ${}^I M/U$ is Σ_2-elementarily equivalent to M.*

(3) *For every family $\{M_i \mid i \in I\}$ of pairwise isomorphic structures with same language and ultrafilter U over I, if $\prod_{i \in I} M_i \neq \emptyset$ then $\prod_{i \in I} M_i/U$ is Σ_1-elementarily equivalent to M_i for every $i \in I$.*

(4) *For every two structures M and N with same language and ultrafilter U over I, if there are elementary embeddings $j : M \to N$ and $j' : N \to M$, then $^I M/U$ and $^I N/U$ are Σ_2-elementarily equivalent.*

(5) *For every two families $\{M_i \mid i \in I\}$ and $\{N_i \mid i \in I\}$ of structures with same language and ultrafilter U over I, if each M_i is isomorphic to N_i and both $\prod_{i \in I} M_i$ and $\prod_{i \in I} N_i$ are non-empty, then $\prod_{i \in I} M_i/U$ is Σ_1-elementarily equivalent to $\prod_{i \in I} N_i/U$.*

(6) *For every two families $\{M_i \mid i \in I\}$ and $\{N_i \mid i \in I\}$ of structures with same language and ultrafilter U over I, if each M_i is isomorphic to N_i and both $\prod_{i \in I} M_i$ and $\prod_{i \in I} N_i$ are non-empty, then $\prod_{i \in I} M_i/U$ is isomorphic to $\prod_{i \in I} N_i/U$.*

We also study a generic version of Łoś's theorem. It is known that, in ZF, Łoś's theorem is not provable, and Łoś's theorem does not imply the Axiom of Choice AC. Unlike this, we show that a generic version of Łoś's theorem is equivalent to AC in ZF.

2. Proof of Theorem 1.4

We start the proof of Theorem 1.4. First we prove (1) $\iff$ (2) $\iff$ (3).

Proof of (1) $\iff$ (2). The direction (1) $\Rightarrow$ (2) is clear.

(2) $\Rightarrow$ (1). Take a family $\{A_i \mid i \in I\}$ of non-empty sets. We shall find a function f on I such that $\{i \in I \mid f(i) \in A_i\} \in U$. Now we may assume that the family $\{A_i \mid i \in I\}$ is pairwise disjoint and $I \cap \bigcup_{i \in I} A_i = \emptyset$. Consider the structure $M = \langle I \cup \bigcup_{i \in I} A_i; I, R\rangle$, where I is identified with a unary predicate, and R is a binary relation defined by $R(x,y) \iff x \in I$ and $y \in A_x$. It is clear that the Π_2-sentence $\forall x(I(x) \to \exists y R(x,y))$ holds in M. Let $M^* = \langle M^*; I^*, R^*\rangle$ be the ultrapower of M by U, where I^* and R^* are the relations corresponding to I and R. By the assumption (2), M^* is Σ_2-elementarily equivalent to M. Hence $\forall x(I^*(x) \to \exists y R^*(x,y))$ holds in M^*. Let $f : I \to I$ be the identity function. We have $I^*([f])$, hence there is $[g] \in M^*$ such that $R^*([f],[g])$. Then $\{i \in I \mid R(i,g(i))\} \in U$ by the construction of M^*, so $\{i \in I \mid g(i) \in A_i\} \in U$. $\square$

Proof of (1) $\iff$ (3). (1) $\Rightarrow$ (3) is clear.

(3) $\Rightarrow$ (1). Take a family $\{A_i \mid i \in I\}$ of non-empty sets. We may assume that $\{A_i \mid i \in I\}$ is a pairwise disjoint family. Take a large limit ordinal θ with $A_i \in V_\theta$ for every $i \in I$. Let $B_i = A_i \times V_\theta$. Clearly there is an injection from V_θ into B_i, and since θ is limit with $A_i \in V_\theta$, we have $B_i \subseteq V_\theta$. Thus for each $i, j \in I$ there is a bijection from B_i onto B_j. Let $S = \bigcup_{i \in I} B_i$, and for $i \in I$, let $R_i = B_i$. Let M_i be the structure $\langle S; R_i\rangle$. A Σ_1-sentence $\exists x R_i(x)$ holds in M_i.

Claim 2.1. *For $i, j \in I$, M_i is isomorphic to M_j.*

Proof. Fix a bijection $f : B_i \to B_j$. Define $\pi : S \to S$ as follows. For $x \in S$,

$$\pi(x) = \begin{cases} x & \text{if } x \notin B_i \cup B_j. \\ f(x) & \text{if } x \in B_i. \\ f^{-1}(x) & \text{if } x \in B_j. \end{cases}$$

π is a bijection from S onto S with $\pi``B_i = B_j$. Hence π is an isomorphism between M_i and M_j. □

Let $M^* = \langle M^* ; R^* \rangle$ be the ultraproduct $\prod_{i \in I} M_i / U$. By the assumption (3), $\exists x R^*(x)$ holds in M^*. Pick $[f] \in M^*$ with $R^*([f])$. Then $X = \{i \in I \mid R_i(f(i))\} \in U$. For $i \in X$, by the definition of R_i, $f(i)$ is an element of $A_i \times V_\theta$. Let $g(i)$ be the first coordinate of $f(i)$. We have $\{i \in I \mid g(i) \in A_i\} \in U$. □

Next we prove (1) $\iff$ (4).

Proof of (1) $\iff$ (4). It is clear that (1) $\Rightarrow$ (4). For (4) $\Rightarrow$ (1), take a family $\{A_i \mid i \in I\}$ of non-empty sets. We may assume that $\{A_i \mid i \in I\}$ is a pairwise disjoint family and $I \cap \bigcup_{i \in I} A_i = \emptyset$. We shall construct two structures M and N such that N is an elementary substructure of M and there is an elementary embedding $j : M \to N$.

First, let $A_i' = A_i \times \omega$. For $n < \omega$ and $i \in I$, let $A_{i,n}' = A_i' \times \{i\} \times \{n\}$. Fix a point $p \notin I \cup (I \times \omega) \cup \bigcup_{i \in I} A_i' \cup \bigcup_{n < \omega, i \in I_n} A_{i,n}'$, and let $B_{i,n} = A_{i,n}' \cup \{\langle p, i, n \rangle\}$. Let $M = I \cup (I \times \omega) \cup \bigcup_{i \in I} A_i' \cup \bigcup_{i \in I, n < \omega} B_{i,n}$ and $N = (I \times \omega) \cup \bigcup_{i \in I, n < \omega} B_{i,n}$. We identify M and N as structures $\langle M ; I_0, R_0 \rangle$ and $\langle N ; I_0, R_1 \rangle$, where $I_0 = I \cup (I \times \omega)$, $I_1 = I \times \omega$ are unary predicates, and

- $R_0(x, y) \iff x \in I$ and $y \in A_x'$, or $x = \langle i, n \rangle \in I \times \omega$ and $y \in B_{i,n}$.
- $R_1(x, y) \iff x = \langle i, n \rangle \in I \times \omega$ and $y \in B_{i,n}$.

Note that N is a substructure of M.

Claim 2.2. *N is an elementary substructure of M.*

Proof. Fix a formula φ, and take $\langle i_0, n_0 \rangle, \ldots, \langle i_k, n_k \rangle \in I \times \omega$ and $\langle x_0, j_0, m_0 \rangle, \ldots, \langle x_l, j_l, m_l \rangle \in \bigcup_{i \in I, n < \omega} B_{i,n}$. Fix a large natural number h with $h > n_0, \ldots, n_k, m_0, \ldots, m_l$. We work in some generic extension $V[G]$ in which M and N are countable. Since A_i' and $B_{i,n}$ are countably infinite, we can find a bijection π from N onto M such that:

(1) $\pi(\langle i, h \rangle) = i$ and $\pi(\langle i, n \rangle) = \langle i, n \rangle$ for $i \in I$ and $n < h$.
(2) $\pi(\langle i, n \rangle) = \langle i, n - 1 \rangle$ for $i \in I$ and $n > h$.
(3) $\pi \upharpoonright \{\langle i_0, n_0 \rangle, \ldots, \langle i_k, n_k \rangle\}$ is identity.
(4) $\pi``B_{i,h} = A_i'$ for $i \in I$.
(5) $\pi``B_{i,n} = B_{i,n}$ for $n < h$ and $i \in I$.
(6) $\pi``B_{i,n} = B_{i,n-1}$ for $n > h$ and $i \in I$.
(7) $\pi \upharpoonright \{\langle x_0, j_0, m_0 \rangle, \ldots, \langle x_l, j_l, m_l \rangle\}$ is identity.

Then it is routine to check that π is an isomorphism from N onto M, hence we have

$$M \models \varphi(\langle i_0, n_0 \rangle, \ldots, \langle i_k, n_k \rangle, \langle x_0, j_0, m_0 \rangle, \ldots, \langle x_l, j_l, m_l \rangle)$$
$$\iff N \models \varphi(\langle i_0, n_0 \rangle, \ldots, \langle i_k, n_k \rangle, \langle x_0, j_0, m_0 \rangle, \ldots, \langle x_l, j_l, m_l \rangle).$$

□

Next we define an embedding $j : M \to N$ as follows:

(1) $j(i) = \langle i, 0 \rangle$ and $j(x) = \langle x, i, 0 \rangle$ for $i \in I$ and $x \in A_i'$.

(2) $j(\langle i, n \rangle) = \langle i, n+1 \rangle$ and $j(\langle x, i, n \rangle) = \langle x, i, n+1 \rangle$ for $i \in I$ and $x \in A_{i,n}'$.

Claim 2.3. $j : M \to N$ *is an elementary embedding.*

Proof. Take a generic extension $V[G]$ in which M and N are countable. In $V[G]$, since each A_i' and $B_{i,0}$ are countably infinite, for every finite set $C \subseteq \bigcup_{i \in I} A_i'$, we can take a bijection π from $\bigcup_{i \in I} A_i'$ onto $\bigcup_{i \in I} B_{i,0}$ such that $\pi \upharpoonright C = j \upharpoonright C$. Then it is easy to see that $\pi \cup j \upharpoonright (I \cup (I \times \omega) \cup \bigcup_{i \in I, n < \omega} B_{i,n})$ is an isomorphism, so j is an elementary embedding as before. $\square$

Let $M^* = \langle M^*; I_0^*, R_0^* \rangle$ and $N^* = \langle N^*; I_1^*, R_1^* \rangle$ be ultrapowers of M and N by U respectively.

Claim 2.4. $\forall x(I_1^*(x) \to \exists y R_1^*(x, y))$ *holds in* N^*,

Proof. Take $[f] \in \prod_{i \in I} N_i / U$, and suppose $I_1^*([f])$. Then $X = \{i \in I \mid f(i) \in I_1\} \in U$. For $i \in X$, we know that $f(i)$ is of the form $\langle g(i), n \rangle$ for some $g(i) \in I$ and $n < \omega$. Since $\langle p, g(i), n \rangle \in B_{g(i),n}$, we have $h(i) = \langle p, g(i), n \rangle \in B_{g(i),n}$ and $R_1(f(i), h(i))$ holds. Then clearly $R_1^*([f], [h])$ holds in N^*. $\square$

By the assumption (4), M^* is Σ_2-elementarily equivalent to N^*. Hence $\forall x(I_0^*(x) \to \exists y R_0^*(x, y))$ holds in M^*. Let $f \in {}^I M$ be the map $f(i) = i$. We know $I_0^*([f])$ holds in M^*, so there is some $[g] \in M^*$ with $R_0^*([f], [g])$ in M^*. Then $X = \{i \in I \mid R_0(i, g(i))\} \in U$. By the definition of R_0, $g(i)$ is an element of $A_i \times \omega$ for $i \in X$. Let $h(i) \in A_i$ be the first coordinate of $g(i)$. Then $\{i \in I \mid h(i) \in A_i\} = X \in U$. $\square$

Finally we show (1) $\iff$ (5) $\iff$ (6).

Proof of (1) $\iff$ (5) $\iff$ (6). For (1) $\Rightarrow$ (6), take families $\{M_i \mid i \in I\}$ and $\{N_i \mid i \in I\}$ of structures with same language such that each M_i is isomorphic to N_i, and $\prod_{i \in I} M_i, \prod_{i \in I} N_i \neq \emptyset$. For $i \in I$, let A_i be the set of all isomorphisms from M_i onto N_i. By the assumption (1), we can find a function σ on I such that $X = \{i \in I \mid \sigma(i) \in A_i\} \in U$. Fix $h_0 \in \prod_{i \in I} N_i$. Define $\pi : \prod_{i \in I} M_i \to \prod_{i \in I} N_i$ by

$$\pi(f)(i) = \begin{cases} \sigma(i)(f(i)) & \text{if } i \in X, \\ h_0(i) & \text{if } i \notin X. \end{cases}$$

It is routine to check that the assignment $[f] \mapsto [\pi(f)]$ is an isomorphism from $\prod_{i \in I} M_i / U$ onto $\prod_{i \in I} N_i / U$.

(6) $\Rightarrow$ (5) is clear.

For (5) $\Rightarrow$ (1), take a family $\{A_i \mid i \in I\}$ of non-empty sets. For $i \in I$, let $A_i' = A_i \times \omega$. It is clear that A_i' contains a countably infinite subset.

Fix two points $p, q \notin \bigcup_{i \in I} A_i'$. For $i \in I$, let M_i and N_i be the structures that $M_i = \langle A_i' \cup \{p\}; p \rangle$ and $N_i = \langle A_i' \cup \{p, q\}; p \rangle$, where we identify p as a constant symbol. Since each A_i' contains a countably infinite subset, we can take a bijection from A_i' onto $A_i' \cup \{q\}$. This bijection induces an isomorphism from M_i onto N_i.

We have $\prod_{i\in} M_i, \prod_{i\in I} N_i \neq \emptyset$, hence $\prod_{i\in I} M_i/U = \langle M^*; p^* \rangle$ is Σ_1-elementarily equivalent to $\prod_{i\in I} N_i/U = \langle N^*; p^{**} \rangle$ by the assumption (5). We know $\prod_{i\in I} N_i/U \models [c_q] \neq p^{**}$ (where c_q is the constant function with value q), so $\prod_{i\in I} N_i/U \models \exists x(x \neq p^{**})$ and we have $\prod_{i\in I} M_i/U \models \exists x(x \neq p^*)$. Pick $[g] \in \prod_{i\in I} M_i/U$ with $[g] \neq p^*$. Then we have $X = \{i \in I \mid g(i) \neq p\} \in U$. For each $i \in X$, since $g(i) \neq p$ we have $g(i) \in A_i' = A_i \times \omega$. Let $f(i)$ be the first coordinate of $g(i)$. Then we have $\{i \in I \mid f(i) \in A_i\} = X \in U$. $\qquad \square$

Note 2.5. By the construction of ultrapowers, in ZF it is easy to show that a structure M is Σ_1-elementarily equivalent to an ultrapower ${}^I M/U$. In this sense, Σ_2-elementarity in (2) and (4) of Theorem 1.4 is optimal.

Variants of (3) and (6) in Theorem 1.4 are still equivalent to U-AC_I. For sets A, B, let us say that A and B are *equipotent*, denoted by $A \simeq B$, if there is a bijection from A onto B. For an ultrafilter U over I and a family $\{A_i \mid i \in I\}$ of non-empty sets, let $\prod_{i\in I} A_i/U$ be just the set $\{[f] \mid f \in \prod_{i\in I} A_i\}$, where $[f]$ denotes the equivalence class of f modulo U.

Proposition 2.6. *In* ZF, *let U be an ultrafilter over I. Then the following are equivalent:*

(1) U-AC_I holds.

(2) For every two families $\{A_i \mid i \in I\}$, $\{B_i \mid i \in I\}$ of non-empty sets, if $A_i \simeq B_i$ for every $i \in I$ and both $\prod_{i\in I} A_i, \prod_{i\in I} B_i$ are non-empty, then $\prod_{i\in I} A_i/U \simeq \prod_{i\in I} B_i/U$.

(3) For every family $\{A_i \mid i \in I\}$ of pairwise equipotent sets, if each A_i has at least two elements and $\prod_{i\in I} A_i \neq \emptyset$, then $\prod_{i\in I} A_i/U$ has two elements.

Proof. $(1) \Rightarrow (2)$ is immediate from (6) in Theorem 1.4. For $(2) \Rightarrow (1)$, we repeat the proof of $(5) \Rightarrow (1)$ in Theorem 1.4. Take a family $\{A_i \mid i \in I\}$ of non-empty sets, and let $A_i' = A_i \times \omega$. Fix $p, q \notin \bigcup_{i\in I} A_i'$. Then for each $i \in I$, we have $A_i' \cup \{p\} \simeq A_i' \cup \{p, q\}$. Moreover $\prod_{i\in I} (A_i' \cup \{p\})$ and $\prod_{i\in I} (A_i' \cup \{p, q\})$ are non-empty. By the assumption, there is a bijection $f : \prod_{i\in I} (A_i' \cup \{p\})/U \to \prod_{i\in I} (A_i' \cup \{p, q\})/U$. Clearly $\prod_{i\in I} (A_i' \cup \{p, q\})/U$ has at least two elements, hence so does $\prod_{i\in I} (A_i' \cup \{p\})/U$. Pick two elements $[f], [g] \in \prod_{i\in I} (A_i' \cup \{p\})/U$. Then $\{i \in I \mid f(i) \neq p\} \in U$ or $\{i \in I \mid g(i) \neq p\} \in U$. If $\{i \in I \mid f(i) \neq p\} \in U$, then $\{i \in I \mid f(i) \in A_i \times \omega\} \in U$, and we can take a function f' with $\{i \in I \mid f'(i) \in A_i\}$. The case $\{i \in I \mid g(i) \neq p\} \in U$ is similar.

$(1) \Rightarrow (3)$. For a given family $\{A_i \mid i \in I\}$, since $\prod_{i\in I} A_i \neq \emptyset$, we can fix $f \in \prod_{i\in I} A_i$. For each $i \in A_i$, since A_i has two elements, we have $A_i \setminus \{f(i)\} \neq \emptyset$. By (1), we can take a function g on I with $\{i \in I \mid g(i) \in A_i \setminus \{f(i)\}\} \in U$. Let h be the function on I defined by $h(i) = g(i)$ if $g(i) \in A_i \setminus \{f(i)\}$, and $h(i) = f(i)$ otherwise. We have $[f] \neq [h]$, so $\prod_{i\in I} A_i/U$ has two elements.

$(3) \Rightarrow (1)$. Let $\{A_i \mid i \in I\}$ be a family of non-empty sets. In this case we repeat the proof of $(3) \Rightarrow (1)$ in Theorem 1.4. Fix a large limit ordinal θ with $A_i \in V_\theta$ for every $i \in I$. We know $A_i \times V_\theta \simeq A_j \times V_\theta$ for every $i, j \in I$. Fix a point $p \notin \bigcup_{i\in I} (A_i \times V_\theta)$, and let $B_i = (A_i \times V_\theta) \cup \{p\}$. Each B_i has at least two elements, $B_i \simeq B_j$ for $i, j \in I$, and $\prod_{i\in I} B_i \neq \emptyset$, hence

$\prod_{i \in I} B_i/U$ has two elements by (3). In particular there is $[f] \in \prod_{i \in I} B_i/U$ with $[f] \neq [c_p]$. Then $\{i \in I \mid f(i) \neq p\} \in U$, hence if $f'(i)$ is the first coordinate of $f(i)$ then $\{i \in I \mid f'(i) \in A_i\} \in U$. $\square$

Corollary 2.7. *In* ZF, *the following are equivalent:*

(1) *Łoś's theorem.*

(2) *For every two families* $\{A_i \mid i \in I\}$, $\{B_i \mid i \in I\}$ *of non-empty sets and ultrafilter* U *over* I, *if* $A_i \simeq B_i$ *for every* $i \in I$ *and both* $\prod_{i \in I} A_i, \prod_{i \in I} B_i$ *are non-empty, then* $\prod_{i \in I} A_i/U \simeq \prod_{i \in I} B_i/U$.

(3) *For every family* $\{A_i \mid i \in I\}$ *of pairwise equipotent sets and ultrafilter* U *over* I, *if each* A_i *has at least two elements and* $\prod_{i \in I} A_i \neq \emptyset$ *then* $\prod_{i \in I} A_i/U$ *has two elements.*

Question 2.8. In the spirit of Theorem 1.3, the following statement would be considerable: For every M and U, if ${}^I M/U$ is Σ_2-elementarily equivalent to M, then ${}^I M/U$ is elementarily equivalent to M. In ZF, is this statement equivalent to Łoś's theorem?

3. Generic Łoś's theorem

First we recall generic ultrapower and generic ultraproduct.

Definition 3.1. For a filter F over I, let F^+ be the set $\{X \in \mathcal{P}(I) \mid X \cap Y \neq \emptyset$ for every $Y \in F\}$. An element of F^+ is called an F-positive set. Let $\mathbb{P}_F$ be the poset F^+ with the order defined by $X \leq Y \iff X \setminus Y \notin F^+$.

If G is a $(V, \mathbb{P}_F)$-generic filter, then G is a V-ultrafilter extending F, that is, the following hold:

(1) $F \subseteq G$ and $\emptyset \notin G$.

(2) For $X \in G$ and $Y \in \mathcal{P}(I)^V$, if $X \subseteq Y$ then $Y \in G$.

(3) For $X, Y \in G$ we have $X \cap Y \in G$.

(4) For every $X \in \mathcal{P}(I)^V$, either $X \in G$ or else $I \setminus X \in G$.

For a family $\{M_i \mid i \in I\} \in V$ of structures, we can define the equivalence relation on $(\prod_i M_i)^V$ modulo G and the equivalence class $[f]$ for $f \in (\prod_{i \in I} M_i)^V$ as expected. Hence we can construct the *generic ultraproduct* $\prod_{i \in I} M_i/G$ in $V[G]$, and, similarly, for a structure $M \in V$, we can take the *generic ultrapower* ${}^I M/G$ in $V[G]$. Under AC in V, we have the following generic version of Łoś's fundamental theorem: For every formula $\varphi(v_0, \ldots, v_n)$ and $f_0, \ldots, f_n \in (\prod_{i \in I} M_i)^V$,

$$\prod_{i \in I} M_i/G \models \varphi([f_0], \ldots, [f_n])$$

$$\iff \{i \in I \mid M_i \models \varphi(f_0(i), \ldots, f_n(i))\} \in G.$$

In particular, under AC in V, the generic ultrapower ${}^I M/G$ is elementarily equivalent to M.

Proposition 3.2. *In* ZF, *the following are equivalent:*

(1) *The Axiom of Choice* AC.

(2) *For every structure* M, *filter* F *over* I, *and* $(V, \mathbb{P}_F)$-*generic filter* G, *the generic ultrapower* ${}^I M/G$ *is* Σ_2-*elementarily equivalent to* M.

(3) For every structure M, filter F over I, and $(V, \mathbb{P}_F)$-generic filter G, the generic ultrapower $^I M/G$ is elementarily equivalent to M.

(4) For every family $\{M_i \mid i \in I\}$ of pairwise isomorphic structures with same language, filter F over I, and $(V, \mathbb{P}_F)$-generic filter G, if $\prod_{i \in I} M_i \neq \emptyset$ then the generic ultraproduct $\prod_{i \in I} M_i/G$ is Σ_1-elementarily equivalent to M_i for every $i \in I$.

Proof. (1) $\Rightarrow$ (3), (4) are well-known, and (3) $\Rightarrow$ (2) is trivial.

(2) $\Rightarrow$ (1). Our proof is based on Howard's one ([2]). Take an indexed family $\{A_i \mid i \in I\}$ of non-empty sets, and suppose to the contrary that this family has no choice function. Let F be the set of all $X \subseteq I$ such that $\{A_i \mid i \in I \setminus X\}$ has a choice function ($X = I$ is possible). One can check that F is a filter over I. We define the structure M as the following. Let $S = I \cup \bigcup_{i \in I} A_i$, and let $R \subseteq S^2$ be the binary relation defined by $R(x,y) \iff x \in I$ and $y \in A_x$. Let M be the structure $\langle S; I, R \rangle$. We know that the Π_2-sentence $\forall x(I(x) \to \exists y R(x,y))$ holds in M.

Take a $(V, \mathbb{P}_F)$-generic G, and we work in $V[G]$. Let $M^* = (M^*; I^*, R^*)$ be the generic ultrapower of M by G. By (2), we have that $M^* \models \forall x(I^*(x) \to \exists y R^*(x,y))$. If id is the identity map on I, we have $M^* \models I^*([\text{id}])$ by the construction of the generic ultrapower. Hence there is $f \in (^I M)^V$ such that $M^* \models R^*([\text{id}], [f])$. Then $X = \{i \in I \mid R(i, f(i))\} = \{i \in I \mid f(i) \in A_i\} \in G$. Because $f \in V$, the family $\{A_i \mid i \in X\}$ has a choice function f in V. This means that $I \setminus X \in F$. Since $F \subseteq G$, we have $I \setminus X \in G$, but this contradicts $X \in G$.

(4) $\Rightarrow$ (1). Take a family $\{A_i \mid i \in I\}$ of non-empty sets which has no choice function. Let F be the filter as the above, and let $\{M_i = (M_i; R_i) \mid i \in I\}$ the family of pairwise isomorphic structures in the proof of (3) $\Rightarrow$ (1) in Theorem 1.4. Take a $(V, \mathbb{P}_F)$-generic G, and work in $V[G]$. If $M^* = (M^*; R^*)$ is the generic ultraproduct $\prod_{i \in I} M_i/G$, we have $\exists x R^*(x)$ holds in M^*, hence there is $[f] \in M^*$ with $R^*([f])$ in M^*. Then $f \in V$ and $\{i \in I \mid R_i(f(i))\} \in G$, and we can choose $g \in V$ with $\{i \in I \mid g(i) \in A_i\} \in G$. Then we can derive a contradiction as the above. $\square$

Note 3.3. In ZF, a structure M is always Σ_1-elementarily equivalent to a generic ultrapower $^I M/G$. Hence Σ_2-elementarity of (2) in Proposition 3.2 is optimal.

We can prove that generic versions of (4)–(6) in Theorem 1.5 are equivalent to AC, and we omit the proof.

References

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