NOTES ON SLALOM PREDICTION

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Abstract. We study a concept of evasion and prediction associated with slaloms, called slalom prediction. This article collects ZFC-provable properties on the slalom prediction.

1. Introduction

Functions $\omega \to [\omega]^{<\omega}$ are called slaloms and it is known that many cardinal invariants can be characterized using relational systems regarding slaloms (e.g. Theorem 5.1,5.2). Let us introduce them:

Definition 1.1.
- $\mathbf{R} = \langle X, Y, \sqsubset \rangle$ is a relational system if X and Y are non-empty sets and $\sqsubset \subseteq X \times Y$.
- We call an element of X a challenge, an element of Y a response, and "$x \sqsubset y$" "x is met by y".
- $F \subseteq X$ is $\mathbf{R}$-unbounded if no response meets all challenges in F.
- $F \subseteq Y$ is $\mathbf{R}$-dominating if every challenge is met by some response in F.
- $\mathbf{R}$ is non-trivial if X is $\mathbf{R}$-unbounded and Y is $\mathbf{R}$-dominating. For non-trivial $\mathbf{R}$, define
  - $\mathfrak{b}(\mathbf{R}) := \min\{|F| : F \subseteq X \text{ is } \mathbf{R}\text{-unbounded}\}$, and
  - $\mathfrak{d}(\mathbf{R}) := \min\{|F| : F \subseteq Y \text{ is } \mathbf{R}\text{-dominating}\}$.

Definition 1.2. Let $b \in (\omega+1)^\omega$ and $h \in \omega^\omega$.
- Let $\prod b := \prod_{n<\omega} b(n)$, $\mathrm{seq}_{<\omega}(b) := \bigcup_{n<\omega} \prod_{i<n} b(i)$ and $\mathcal{S}(b,h) := \prod_{n<\omega} [b(n)]^{\leq h(n)}$, the set of all (b,h)-slaloms. We often use "localizer" instead of "slalom".
- For $x \in \prod b$ and $\varphi \in \mathcal{S}(b,h)$, we write:
  - $x \in^* \varphi$ if $x(n) \in \varphi(n)$ for all but finitely many $n < \omega$, and
  - $x \in^\infty \varphi$ if $x(n) \in \varphi(n)$ for infinitely many $n < \omega$.
- Denote the relational system $\mathbf{GLc}(b,h) := \langle \prod b, \mathcal{S}(b,h), \in^* \rangle$ and $\mathfrak{b}^{\mathrm{GLc}}_{b,h} := \mathfrak{b}(\mathbf{GLc}(b,h))$ and $\mathfrak{d}^{\mathrm{GLc}}_{b,h} := \mathfrak{d}(\mathbf{GLc}(b,h))$. (short for "global localization")
- Denote the relational system $\mathbf{ILc}(b,h) := \langle \prod b, \mathcal{S}(b,h), \in^\infty \rangle$ and $\mathfrak{b}^{\mathrm{ILc}}_{b,h} := \mathfrak{b}(\mathbf{ILc}(b,h))$ and $\mathfrak{d}^{\mathrm{ILc}}_{b,h} := \mathfrak{d}(\mathbf{ILc}(b,h))$. (short for "infinite localization")

To avoid triviality, we always assume $1 \leq h(n) < b(n)$ for all $n < \omega$.

There are several types of notation denoting the same cardinal invariants we deal with, such as $\mathfrak{c}^{\forall}_{b,h}$, $\mathfrak{v}^{\exists}_{b,h}$, $\mathfrak{d}_h(\in^*)$, $\mathfrak{b}^{\mathrm{Lc}}_{b,h}$, $\mathfrak{d}^{\mathrm{aLc}}_{b,h}$, $\mathfrak{sl}_{\mathrm{t}}(b,h,\mathrm{Fin})$, $\mathfrak{sl}^{\perp}_{\mathrm{e}}(b,h,\mathrm{Fin})$ (see the list in [CGMRS24, Remark 3.3]) and our notation is based on Cardona and Mejía's $\mathfrak{b}^{\mathrm{Lc}}_{b,h}$, $\mathfrak{d}^{\mathrm{aLc}}_{b,h}$ in [CM23] (they used the names "localizations" and "anti-localizations").

On the other hand, Blass studied the Specker phenomenon in group theory and consequently introduced the combinatorial notion of evasion and prediction in [Bla94] as follows:

Definition 1.3.
- A pair $\pi = (D, \{\pi_n : n \in D\})$ is a predictor if $D \in [\omega]^\omega$ and each π_n is a function $\pi_n \colon \omega^n \to \omega$. Pred denotes the set of all predictors.
- $\pi \in \mathrm{Pred}$ predicts $f \in \omega^\omega$ if $f(n) = \pi_n(f{\restriction}n)$ for all but finitely many $n \in D$. f evades π if π does not predict f.
- The prediction number $\mathfrak{pr}$ and the evasion number $\mathfrak{e}$ are defined as follows[1]:

$$\mathfrak{pr} := \min\{|\Pi| : \Pi \subseteq \mathrm{Pred}, \forall f \in \omega^\omega\ \exists \pi \in \Pi\ \pi \text{ predicts } f\},$$
$$\mathfrak{e} := \min\{|F| : F \subseteq \omega^\omega, \forall \pi \in \mathrm{Pred}\ \exists f \in F\ f \text{ evades } \pi\}.$$

Both localizations and predictions are a kind of a guess, but predictions have the following two differences with localizations:

(1) Predictors have their infinite set D on which they guess.
(2) Predictors can see the previous values of functions as a hint of the next value.

Now we consider possible patterns of the guess combining the two notions localization and prediction. Blass made a similar approach in [Bla09] (and our names "global", "local" and "infinite" come from his work), but he assumed $b = \omega$ (the constant function of the value ω). Our following framework contains the case when $b \in \omega^\omega$ as well.

Firstly, we introduce localizations with their infinite set on which localizers guess:

Definition 1.4. ([BS96, Definition 2.4], [Spi98, Definition 1.2], [Car24, Definition 2.3]). Let $b \in (\omega+1)^\omega$ and $h \in \omega^\omega$.

- A local (b,h)-slalom is a pair (D,φ) where $D \in [\omega]^\omega$ and $\varphi \in \mathcal{S}(b,h)$. $\mathcal{LS}(b,h)$ denotes the set of all local (b,h)-slaloms.
- For $x \in \prod b$ and $(D,\varphi) \in \mathcal{LS}(b,h)$, we write $x \in^* (D,\varphi)$ if $x(n) \in \varphi(n)$ for all but finitely many $n \in D$.
- Denote the relational system $\mathbf{LLc}(b,h) := \langle \prod b, \mathcal{LS}(b,h), \in^* \rangle$ and $\mathfrak{b}^{\mathrm{LLc}}_{b,h} := \mathfrak{b}(\mathbf{LLc}(b,h))$ and $\mathfrak{d}^{\mathrm{LLc}}_{b,h} := \mathfrak{d}(\mathbf{LLc}(b,h))$.

We often write $x \in^*_D \varphi$ instead of $x \in^* (D,\varphi)$.

Secondly, we introduce predictions with slaloms and split into three cases (global/local/infinite):

[1] While the name "prediction number" and the notation "$\mathfrak{pr}$" are not common, we use them in this article.

Definition 1.5. Let $b \in (\omega+1)^\omega$ and $h \in \omega^\omega$.

- A global (b,h)-predictor is a function π with its domain $\mathrm{seq}_{<\omega}(b)$ such that given $n < \omega$ and $\sigma \in \prod_{i<n} b(i)$, $\pi(\sigma)$ is a subset of $b(n)$ of size $\leq h(n)$. $\mathcal{GPR}(b,h)$ denotes the set of all global (b,h)-predictors.
- A local (b,h)-predictor is a pair (D,π) where $D \in [\omega]^\omega$ and $\pi \in \mathcal{GPR}(b,h)$. $\mathcal{LPR}(b,h)$ denotes the set of all local (b,h)-predictors.
- For $(D,\pi) \in \mathcal{LPR}(b,h)$ and $f \in \prod b$, (D,π) predicts f if $f(n) \in \pi(f{\restriction}n)$ for all but finitely many $n \in D$ and we write $f \lhd^* (D,\pi)$ or $f \lhd^*_D \pi$.
- For $\pi \in \mathcal{GPR}(b,h)$ and $f \in \prod b$;
  - π predicts f if $f \lhd^*_\omega \pi$ and we write $f \lhd^* \pi$.
  - π infinitely predicts f if $f \lhd^*_D \pi$ for some $D \in [\omega]^\omega$ and we write $f \lhd^\infty \pi$.
- Define the following relational systems and cardinal invariants:
  - $\mathbf{GPr}(b,h) := \langle \prod b, \mathcal{GPR}(b,h), \lhd^* \rangle$, $\mathfrak{e}^{\mathrm{G}}_{b,h} := \mathfrak{b}(\mathbf{GPr}(b,h))$ and $\mathfrak{pr}^{\mathrm{G}}_{b,h} := \mathfrak{d}(\mathbf{GPr}(b,h))$.
  - $\mathbf{LPr}(b,h) := \langle \prod b, \mathcal{LPR}(b,h), \lhd^* \rangle$, $\mathfrak{e}^{\mathrm{L}}_{b,h} := \mathfrak{b}(\mathbf{LPr}(b,h))$ and $\mathfrak{pr}^{\mathrm{L}}_{b,h} := \mathfrak{d}(\mathbf{LPr}(b,h))$.
  - $\mathbf{IPr}(b,h) := \langle \prod b, \mathcal{GPR}(b,h), \lhd^\infty \rangle$, $\mathfrak{e}^{\mathrm{I}}_{b,h} := \mathfrak{b}(\mathbf{IPr}(b,h))$ and $\mathfrak{pr}^{\mathrm{I}}_{b,h} := \mathfrak{d}(\mathbf{IPr}(b,h))$.

The local case when $b = \omega$ and $h = 1$ is equivalent to the standard prediction. In particular, $\mathfrak{e} = \mathfrak{e}^{\mathrm{L}}_{\omega,1}$ and $\mathfrak{pr} = \mathfrak{pr}^{\mathrm{L}}_{\omega,1}$.

In this article, we focus on ZFC-provable properties on local localizations and slalom predictions, particularly when $b \in \omega^\omega$.

In Section 2, we study the local localization comparing it with the global and infinite versions following [CM23]. In Section 3, we see several properties of slalom predictions. In Section 4, we investigate the connection between localizations and slalom predictions. In Section 5, we study the relationship between these numbers and some ideals on the reals and conclude the article with Figure 1 which illustrates the inequalities we know.

In the rest of this section, we review a basic property of relational systems.

Definition 1.6. For relational systems $\mathbf{R} = \langle X, Y, \sqsubset \rangle, \mathbf{R}' = \langle X', Y', \sqsubset' \rangle$, $(\Phi_-, \Phi_+) : \mathbf{R} \to \mathbf{R}'$ is a Tukey connection from $\mathbf{R}$ into $\mathbf{R}'$ if $\Phi_- : X \to X'$ and $\Phi_+ : Y' \to Y$ are functions such that:

$$\forall x \in X\ \forall y' \in Y'\ \Phi_-(x) \sqsubset' y' \Rightarrow x \sqsubset \Phi_+(y').$$

We write $\mathbf{R} \preceq_T \mathbf{R}'$ if there is a Tukey connection from $\mathbf{R}$ into $\mathbf{R}'$ and call $\preceq_T$ the Tukey order. Note that $\mathbf{R} \preceq_T \mathbf{R}'$ implies $\mathfrak{b}(\mathbf{R}') \leq \mathfrak{b}(\mathbf{R})$ and $\mathfrak{d}(\mathbf{R}) \leq \mathfrak{d}(\mathbf{R}')$.

2. Local localization

For any function $x \in \prod b$ and (b,h)-slalom φ, $x \in^* \varphi \Rightarrow x \in^*_D \varphi \Rightarrow x \in^\infty \varphi$ holds for any $D \in [\omega]^\omega$, so:

Lemma 2.1. $\mathbf{ILc}(b,h) \preceq_T \mathbf{LLc}(b,h) \preceq_T \mathbf{GLc}(b,h)$ and hence $\mathfrak{b}^{\mathrm{GLc}}_{b,h} \leq \mathfrak{b}^{\mathrm{LLc}}_{b,h} \leq \mathfrak{b}^{\mathrm{ILc}}_{b,h}$ and $\mathfrak{d}^{\mathrm{ILc}}_{b,h} \leq \mathfrak{d}^{\mathrm{LLc}}_{b,h} \leq \mathfrak{d}^{\mathrm{GLc}}_{b,h}$ for any b,h.

The monotonicity on b,h also holds:

Lemma 2.2. Let $b,b' \in (\omega+1)^\omega$ and $h,h' \in \omega^\omega$. If $b(n) \leq b'(n)$ and $h'(n) \leq h(n)$ for all but finitely many $n < \omega$, then $\mathbf{GLc}(b,h) \preceq_T \mathbf{GLc}(b',h')$, $\mathbf{ILc}(b,h) \preceq_T \mathbf{ILc}(b',h')$ and $\mathbf{LLc}(b,h) \preceq_T \mathbf{LLc}(b',h')$ and hence $\mathfrak{b}^{\mathrm{GLc}}_{b',h'} \leq \mathfrak{b}^{\mathrm{GLc}}_{b,h}$, $\mathfrak{b}^{\mathrm{LLc}}_{b',h'} \leq \mathfrak{b}^{\mathrm{LLc}}_{b,h}$, $\mathfrak{b}^{\mathrm{ILc}}_{b',h'} \leq \mathfrak{b}^{\mathrm{ILc}}_{b,h}$ and $\mathfrak{d}^{\mathrm{GLc}}_{b,h} \leq \mathfrak{d}^{\mathrm{GLc}}_{b',h'}$, $\mathfrak{d}^{\mathrm{LLc}}_{b,h} \leq \mathfrak{d}^{\mathrm{LLc}}_{b',h'}$ and $\mathfrak{d}^{\mathrm{ILc}}_{b,h} \leq \mathfrak{d}^{\mathrm{ILc}}_{b',h'}$.

Global and infinite localizations are studied well in [CM23], so let us look at the local version. The case when $b = \omega$ is studied in [Bla09]. For example:

Fact 2.3. ([Bla09], see also [BS96, Lemma 2.5]) $\mathfrak{b}^{\mathrm{LLc}}_{\omega,h} = \min\{\mathfrak{e}, \mathfrak{b}\}$ and $\mathfrak{d}^{\mathrm{LLc}}_{\omega,h} = \max\{\mathfrak{pr}, \mathfrak{d}\}$ if $h \in \omega^\omega$ goes to infinity[2].

Let us focus on the case $b \in \omega^\omega$. Following [CM23], we investigate what kind of properties that hold for global and infinite localizations will also hold for local cases.

First, they proved the following (non-trivial) Tukey relation of global localizations with different parameters:

Lemma 2.4. ([CM23, Lemma 3.14]) Let $b \in \omega^\omega$ and $h, h^+ \in \omega^\omega$. Assume that $\langle I_k \rangle_{k<\omega}$ is an interval partition of ω satisfying $h(i) \geq h^+(k)$ for all $k < \omega$ and $i \in I_k$ and define $b^+ \in \omega^\omega$ by $b^+(k) \coloneqq \prod_{i \in I_k} b(i)$. Then, $\mathbf{GLc}(b,h) \preceq_T \mathbf{GLc}(b^+,h^+)$.

This lemma holds in the local (and infinite) cases as well:

Lemma 2.5. Under the assumption of Lemma 2.4, $\mathbf{LLc}(b,h) \preceq_T \mathbf{LLc}(b^+,h^+)$ and $\mathbf{ILc}(b,h) \preceq_T \mathbf{ILc}(b^+,h^+)$.

Proof. Their proof of Lemma 2.4 gives functions $\Psi_- \colon \prod b \to \prod b^+$ and $\Psi_+ \colon \mathcal{S}(b^+,h^+) \to \mathcal{S}(b,h)$ such that for any $x \in \prod b$, $\varphi \in \mathcal{S}(b^+,h^+)$, $k < \omega$ and $i \in I_k$, $\Psi_-(x)(k) \in \varphi(k)$ implies $x(i) \in \Psi_+(\varphi)(i)$, which witness $\mathbf{GLc}(b,h) \preceq_T \mathbf{GLc}(b^+,h^+)$ and $\mathbf{ILc}(b,h) \preceq_T \mathbf{ILc}(b^+,h^+)$ as well. Moreover, for any $D \in [\omega]^\omega$, $\Psi(x) \in^* (D,\varphi)$ implies $x \in^* (\bigcup_{k \in D} I_k, \Psi_+(\varphi))$, which witnesses $\mathbf{LLc}(b,h) \preceq_T \mathbf{LLc}(b^+,h^+)$. □

We introduce the limits of localization numbers:

Definition 2.6. Let $h \in \omega^\omega$ go to infinity and $h' \in \omega^\omega$.

[2]While Blass proved only the former equation, it is easy to prove the latter from his proof.

- $\text{minGLc}_h \coloneqq \min\{\mathfrak{b}^{\text{GLc}}_{b,h} : b \in \omega^\omega\}$, $\text{supGLc}_h \coloneqq \sup\{\mathfrak{d}^{\text{GLc}}_{b,h} : b \in \omega^\omega\}$.
- $\text{minLLc}_h \coloneqq \min\{\mathfrak{b}^{\text{LLc}}_{b,h} : b \in \omega^\omega\}$, $\text{supLLc}_h \coloneqq \sup\{\mathfrak{d}^{\text{LLc}}_{b,h} : b \in \omega^\omega\}$.
- $\text{minILc}_{h'} \coloneqq \min\{\mathfrak{b}^{\text{ILc}}_{b,h'} : b \in \omega^\omega\}$, $\text{supILc}_{h'} \coloneqq \sup\{\mathfrak{d}^{\text{ILc}}_{b,h'} : b \in \omega^\omega\}$.

They proved that the parameter h is irrelevant for the global and infinite numbers:

Theorem 2.7. ([CM23, Theorem 3.19])

- For $h, h' \in \omega^\omega$ going to infinity, $\text{minGLc}_h = \text{minGLc}_{h'}$ and $\text{supGLc}_h = \text{supGLc}_{h'}$.
- For $h, h' \in \omega^\omega$, $\text{minILc}_h = \text{minILc}_{h'}$ and $\text{supILc}_h = \text{supILc}_{h'}$.

Thus, we omit the subscripts.

This holds in the local case as well:

Theorem 2.8. For $h, h' \in \omega^\omega$ going to infinity, $\text{minLLc}_h = \text{minLLc}_{h'}$ and $\text{supLLc}_h = \text{supLLc}_{h'}$. Thus, we omit the subscripts.

Proof. We only prove $\text{minLLc}_h = \text{minLLc}_{h'}$. Let $h^+ \in \omega^\omega$ such that $h^+(0) = 0$ and $h^+ \geq^* h, h'$. By Lemma 2.2, $\text{minLLc}_{h'}, \text{minLLc}_h \leq \text{minLLc}_{h^+}$, so it suffices to show that $\text{minLLc}_{h^+} \leq \text{minLLc}_h, \text{minLLc}_{h'}$. Let $b \in \omega^\omega$ be arbitrary. Since h and h' go to infinity, we can find some interval partitions $\langle I_k \rangle_{k<\omega}$ and $\langle I'_k \rangle_{k<\omega}$ as in Lemma 2.5 and corresponding $b^+, (b')^+$ such that $\mathbf{LLc}(b, h) \preceq_T \mathbf{LLc}(b^+, h^+)$ and $\mathbf{LLc}(b, h') \preceq_T \mathbf{LLc}((b')^+, h^+)$. Therefore we have $\text{minLLc}_{h^+} \leq \mathfrak{b}^{\text{GLc}}_{b^+,h^+} \leq \mathfrak{b}^{\text{GLc}}_{b,h}$ and $\text{minLLc}_{h^+} \leq \mathfrak{b}^{\text{GLc}}_{(b')^+,h^+} \leq \mathfrak{b}^{\text{GLc}}_{b,h'}$. Since b was arbitrary, we get $\text{minLLc}_{h^+} \leq \text{minLLc}_h, \text{minLLc}_{h'}$. □

They proved that these limits are used to characterize the global localization cardinals for $b = \omega$:

Theorem 2.9. ([CM23, Theorem 3.20]) $\mathfrak{b}^{\text{GLc}}_{\omega,h} = \min\{\mathfrak{b}, \text{minGLc}_h\}$ and $\mathfrak{d}^{\text{GLc}}_{\omega,h} = \max\{\mathfrak{d}, \text{supGLc}_h\}$ for $h \in \omega^\omega$ going to infinity.

This theorem holds in the local cases as well:

Theorem 2.10. $\mathfrak{b}^{\text{LLc}}_{\omega,h} = \min\{\mathfrak{b}, \text{minLLc}_h\}$ and $\mathfrak{d}^{\text{LLc}}_{\omega,h} = \max\{\mathfrak{d}, \text{supLLc}_h\}$ for $h \in \omega^\omega$ going to infinity. Equivalently, $\min\{\mathfrak{b}, \mathfrak{e}\} = \min\{\mathfrak{b}, \text{minLLc}\}$ and $\max\{\mathfrak{d}, \mathfrak{pr}\} = \max\{\mathfrak{d}, \text{supLLc}\}$ by Fact 2.3.

Proof. We only prove $\mathfrak{b}^{\text{LLc}}_{\omega,h} = \min\{\mathfrak{b}, \text{minLLc}_h\}$ and it is clear that $\mathfrak{b}^{\text{LLc}}_{\omega,h} = \min\{\mathfrak{b}, \mathfrak{e}\} \leq \mathfrak{b}$ and $\mathfrak{b}^{\text{LLc}}_{\omega,h} \leq \text{minLLc}_h$ hold, so we show $\mathfrak{b}^{\text{LLc}}_{\omega,h} \geq \min\{\mathfrak{b}, \text{minLLc}_h\}$. Let $F \subseteq \omega^\omega$ of size $< \min\{\mathfrak{b}, \text{minLLc}_h\}$. Since $|F| < \mathfrak{b}$, some $b \in \omega^\omega$ dominates all $x \in F$ (in the sense of $<^*$) and we may assume that $h \in \prod b$. For $x \in F$, define $x' \in \prod b$ by $x'(n) \coloneqq \min\{x(n), b(n)-1\}$ and note that $x(n) = x'(n)$ for all but finitely many $n < \omega$. Since $|F| < \text{minLLc}_h \leq \mathfrak{b}^{\text{LLc}}_{b,h}$, some local (b, h)-slalom (D, φ) locally localizes all functions in $\{x' : x \in F\}$ and hence all $x \in F$. Therefore, $|F| < \mathfrak{b}^{\text{LLc}}_{\omega,h}$. □

For any function $f \in \prod b$ and global (b,h)-predictor π, $f \triangleleft^* \pi \Rightarrow f \triangleleft^*_D \pi \Rightarrow f \triangleleft^\infty \pi$ holds for any $D \in [\omega]^\omega$, so:

Lemma 3.1. $\mathbf{IPr}(b,h) \preceq_T \mathbf{LPr}(b,h) \preceq_T \mathbf{GPr}(b,h)$ and hence $\mathfrak{e}^{\mathrm{G}}_{b,h} \leq \mathfrak{e}^{\mathrm{L}}_{b,h} \leq \mathfrak{e}^{\mathrm{I}}_{b,h}$ and $\mathfrak{pr}^{\mathrm{I}}_{b,h} \leq \mathfrak{pr}^{\mathrm{L}}_{b,h} \leq \mathfrak{pr}^{\mathrm{G}}_{b,h}$.

The monotonicity on b, h holds as well:

Lemma 3.2. Let $b, b' \in (\omega+1)^\omega$ and $h, h' \in \omega^\omega$. If $b(n) \leq b'(n)$ and $h'(n) \leq h(n)$ for all but finitely many $n < \omega$, then $\mathbf{GPr}(b,h) \preceq_T \mathbf{GPr}(b',h')$, $\mathbf{LPr}(b,h) \preceq_T \mathbf{LPr}(b',h')$ and $\mathbf{IPr}(b,h) \preceq_T \mathbf{IPr}(b',h')$ and hence $\mathfrak{e}^{\mathrm{G}}_{b',h'} \leq \mathfrak{e}^{\mathrm{G}}_{b,h}$, $\mathfrak{e}^{\mathrm{L}}_{b',h'} \leq \mathfrak{e}^{\mathrm{L}}_{b,h}$, $\mathfrak{e}^{\mathrm{I}}_{b',h'} \leq \mathfrak{e}^{\mathrm{I}}_{b,h}$ and $\mathfrak{pr}^{\mathrm{G}}_{b,h} \leq \mathfrak{pr}^{\mathrm{G}}_{b',h'}$, $\mathfrak{pr}^{\mathrm{L}}_{b,h} \leq \mathfrak{pr}^{\mathrm{L}}_{b',h'}$ and $\mathfrak{pr}^{\mathrm{I}}_{b,h} \leq \mathfrak{pr}^{\mathrm{I}}_{b',h'}$.

The case when $b = \omega$ is studied in [Bla09]:

Fact 3.3 ([Bla09])**.** Let $2 \leq k < \omega$ and $h \in \omega^\omega$ go to infinity.

(1) $\aleph_1 = \mathfrak{e}^{\mathrm{G}}_{\omega,1} \leq \mathfrak{m}_k \leq \mathfrak{e}^{\mathrm{G}}_{\omega,k} \leq \mathrm{add}(\mathcal{N}) = \mathfrak{e}^{\mathrm{G}}_{\omega,h}$, where $\mathfrak{m}_k$ denotes Martin's number for σ-k-linked forcings.
(2) $\mathfrak{e}^{\mathrm{L}}_{\omega,1} = \mathfrak{e} \leq \mathfrak{e}^{\mathrm{L}}_{\omega,k} \leq \mathfrak{e}^{\mathrm{L}}_{\omega,h} \leq \mathrm{cov}(\mathcal{M}), \mathrm{non}(\mathcal{M})$.
(3) $\mathfrak{e}^{\mathrm{I}}_{\omega,1} = \mathfrak{e}^{\mathrm{I}}_{\omega,k} = \mathfrak{e}^{\mathrm{I}}_{\omega,h} = \mathrm{cov}(\mathcal{M})$.

It is not hard to see that the dual inequalities hold as follows:

Lemma 3.4. Let $2 \leq k < \omega$ and $h \in \omega^\omega$ go to infinity.

(1) $2^{\aleph_0} = \mathfrak{pr}^{\mathrm{G}}_{\omega,1} \geq \mathfrak{pr}^{\mathrm{G}}_{\omega,k} \geq \mathrm{cof}(\mathcal{N}) = \mathfrak{pr}^{\mathrm{G}}_{\omega,h}$.
(2) $\mathfrak{pr}^{\mathrm{L}}_{\omega,1} = \mathfrak{pr} \geq \mathfrak{pr}^{\mathrm{L}}_{\omega,k} \geq \mathfrak{pr}^{\mathrm{L}}_{\omega,h} \geq \mathrm{cov}(\mathcal{M}), \mathrm{non}(\mathcal{M})$.
(3) $\mathfrak{pr}^{\mathrm{I}}_{\omega,1} = \mathfrak{pr}^{\mathrm{I}}_{\omega,k} = \mathfrak{pr}^{\mathrm{I}}_{\omega,h} = \mathrm{non}(\mathcal{M})$.

The local cases when $b \in \omega^\omega$ and $h = 1$ are studied in [Bre95]:

Fact 3.5 ([Bre95])**.**

(1) $\mathfrak{e} \geq \min\{\mathfrak{b}, \mathfrak{e}_{ubd}\}$, where $\mathfrak{e}_{ubd} \coloneqq \min\{\mathfrak{e}^{\mathrm{L}}_{b,1} : b \in \omega^\omega\}$.
(2) All $\mathfrak{e}^{\mathrm{L}}_{n,1}$ for $2 \leq n < \omega$ are the same value $\mathfrak{e}_{fin}$, where the subscript n denotes the constant function of the value $n < \omega$.
(3) $\mathfrak{e}_{fin} \geq \mathfrak{s}$, the splitting number.

It is not hard to see that the dual inequalities hold as follows:

Lemma 3.6.

(1) $\mathfrak{pr} \leq \max\{\mathfrak{d}, \mathfrak{pr}_{ubd}\}$, where $\mathfrak{pr}_{ubd} \coloneqq \sup\{\mathfrak{pr}^{\mathrm{L}}_{b,1} : b \in \omega^\omega\}$.
(2) All $\mathfrak{pr}^{\mathrm{L}}_{n,1}$ for $2 \leq n < \omega$ are the same value $\mathfrak{pr}_{fin}$.
(3) $\mathfrak{pr}_{fin} \leq \mathfrak{r}$, the reaping number.

The same Tukey connections as in Lemma 2.4 and 2.5 also hold by a similar proof:

Lemma 3.7. Let $b \in \omega^\omega$ and $h, h^+ \in \omega^\omega$. Assume that $\langle I_k \rangle_{k<\omega}$ is an interval partition of ω satisfying $h(i) \geq h^+(k)$ for all $k < \omega$ and $i \in I_k$ and define $b^+ \in \omega^\omega$ by $b^+(k) \coloneqq \prod_{i \in I_k} b(i)$. Then, $\mathbf{GPr}(b,h) \preceq_T \mathbf{GPr}(b^+,h^+)$, $\mathbf{LPr}(b,h) \preceq_T \mathbf{LPr}(b^+,h^+)$ and $\mathbf{IPr}(b,h) \preceq_T \mathbf{IPr}(b^+,h^+)$.

Moreover, in the local and infinite cases, h and h' do not require any assumptions:

Lemma 3.8. Let $b \in \omega^\omega$ and $h, h^+ \in \omega^\omega$. Then, there exists $b^+ \in \omega^\omega$ such that $\mathbf{LPr}(b,h) \preceq_T \mathbf{LPr}(b^+,h^+)$ and $\mathbf{IPr}(b,h) \preceq_T \mathbf{IPr}(b^+,h^+)$.

Proof. We may assume $h = 1$. Let $\langle I_k : k < \omega \rangle$ be an interval partition of ω satisfying $|I_k| > h^+(k)$ for all $k < \omega$ and define b^+ by $b^+(k) \coloneqq \prod_{i \in I_k} b(i)$. Let π be any global (b^+,h^+)-predictor and fix $k < \omega$ and $\sigma \in \prod_{k'<k} b^+(k')$. Let $S \coloneqq \pi(\sigma) \subseteq b^+(k) = \prod_{i \in I_k} b(i)$, which we may assume as a set of functions on I_k. Since $|S| = |\pi(\sigma)| \leq h^+(k) < |I_k|$, there is $j_k \coloneqq j \in I_k$ which is not a branching point of any two $t, t' \in S$, namely, $t{\restriction} I_k \cap j = t'{\restriction} I_k \cap j$ implies $t(j) = t'(j)$ for any $t, t' \in S$. Thus, the function $g_{k,\sigma} : \{t{\restriction} I_k \cap j : t \in S\} \to b(j)$, $t{\restriction} I_k \cap j \mapsto t(j)$ is well-defined. Unfixing k and σ, let π' be a global $(b,1)$-predictor satisfying for all $k < \omega$ and $\tau \in \prod_{i<j_k} b(i) = \prod_{k'<k} b^+(k') \prod_{i \in I_k \cap j_k} b(i)$, $\pi'(\tau) = \{g_{k,\sigma}(\tau{\restriction} I_k \cap j_k)\}$ where $\sigma \coloneqq \langle \tau{\restriction} I_{k'} : k' < k \rangle \in \prod_{k'<k} b^+(k')$. By construction, for any $f \in \prod b$ and $k < \omega$,

$$f{\restriction} I_k \in \pi(\langle f{\restriction} I_{k'} : k' < k \rangle) \text{ implies } f(j_k) \in \pi'(f{\restriction} j_k),$$

which induces $\mathbf{LPr}(b,1) \preceq_T \mathbf{LLc}(b^+,h^+)$ and $\mathbf{IPr}(b,1) \preceq_T \mathbf{ILc}(b^+,h^+)$. □

We introduce the limits of slalom prediction numbers:

Definition 3.9. Let $h \in \omega^\omega$.

- $\mathrm{minGPr}_h \coloneqq \min\{\mathfrak{e}^{\mathrm{G}}_{b,h} : b \in \omega^\omega\}, \mathrm{supGPr}_h \coloneqq \sup\{\mathfrak{pr}^{\mathrm{G}}_{b,h} : b \in \omega^\omega\}$.
- $\mathrm{minLPr}_h \coloneqq \min\{\mathfrak{e}^{\mathrm{L}}_{b,h} : b \in \omega^\omega\}, \mathrm{supLPr}_h \coloneqq \sup\{\mathfrak{pr}^{\mathrm{L}}_{b,h} : b \in \omega^\omega\}$.
- $\mathrm{minIPr}_h \coloneqq \min\{\mathfrak{e}^{\mathrm{I}}_{b,h} : b \in \omega^\omega\}, \mathrm{supIPr}_h \coloneqq \sup\{\mathfrak{pr}^{\mathrm{I}}_{b,h} : b \in \omega^\omega\}$.

By a similar proof to Theorem 2.10, we obtain the following lemma:

Lemma 3.10. Let $h \in \omega^\omega$.

- $\mathfrak{e}^{\mathrm{G}}_{\omega,h} = \min\{\mathfrak{b}, \mathrm{minGPr}_h\}$ and $\mathfrak{pr}^{\mathrm{G}}_{\omega,h} = \max\{\mathfrak{d}, \mathrm{supGPr}_h\}$.
- $\mathfrak{e}^{\mathrm{L}}_{\omega,h} \geq \min\{\mathfrak{b}, \mathrm{minLPr}_h\}$ and $\mathfrak{pr}^{\mathrm{L}}_{\omega,h} \leq \min\{\mathfrak{d}, \mathrm{supLPr}_h\}$.
- $\mathrm{cov}(\mathcal{M}) = \mathfrak{e}^{\mathrm{I}}_{\omega,h} \geq \min\{\mathfrak{b}, \mathrm{minIPr}_h\}$ and $\mathrm{non}(\mathcal{M}) = \mathfrak{pr}^{\mathrm{I}}_{\omega,h} \leq \max\{\mathfrak{d}, \mathrm{supIPr}_h\}$.

4. Connections between localizations and slalom predictions

In this section we see the connection between localizations and slalom predictions. First, predictions are easier than localizations because of the hints of initial segments of functions, so we immediately have:

Lemma 4.1. $\mathbf{GPr}(b,h) \preceq_T \mathbf{GLc}(b,h)$, $\mathbf{LPr}(b,h) \preceq_T \mathbf{LLc}(b,h)$ and $\mathbf{IPr}(b,h) \preceq_T \mathbf{ILc}(b,h)$ for any b and h.

However, regardless of such hints, predictions can be harder than localizations when the width of slaloms of localizations is sufficiently wider:

Definition 4.2. For $h \in \omega^\omega$, define $h'_h \in \omega^\omega$ by $h'_h(n) := n \cdot \prod_{i \le n} h(i)$.

Lemma 4.3. ([Bla09]) Let $b \in (\omega+1)^\omega$ and $h \in \omega^\omega$ such that $b > h' := h'_h$, where $<$ denotes the total strict domination on $(\omega+1)^\omega$. Then, $\mathbf{GLc}(b, h') \preceq_T \mathbf{GPr}(b, h)$, $\mathbf{LLc}(b, h') \preceq_T \mathbf{LPr}(b, h)$ and $\mathbf{ILc}(b, h') \preceq_T \mathbf{IPr}(b, h)$.

Proof. Let π be a global (b, h)-predictor. For $s \in \mathrm{seq}_{<\omega}(b)$, take a tree $T^s \subseteq \mathrm{seq}_{<\omega}(b)$ of stem s which contains all $t \in \mathrm{seq}_{<\omega}(b)$ such that $s \subseteq t$ and $t(n) \in \pi(t{\restriction}n)$ for all $n \in |t| \setminus |s|$ and we may assume that every $t \in T^s \cap \omega^n$ is $h(n)$-branching for all $n \ge |s|$. Put $\pi_s(n) := \bigcup\{t(n) : t \in T^s \cap \omega^{n+1}\}$ for $n < \omega$ and note that $|\pi_s(n)| \le \prod_{i \le n} h(i)$. Enumerate $\{s_i : i < \omega\} = \mathrm{seq}_{<\omega}(b)$ and let $\varphi_\pi(n) := \bigcup\{\pi_{s_i}(n) : i < n\}$ for $n < \omega$ and note that $|\varphi_\pi(n)| \le \prod_{i \le n} h(i) \cdot n = h'_h(n)$, so φ_π is a (b, h'_h)-slalom. Take $x \in \prod b$ and $n_0 < \omega$ arbitrarily and put $s_i := x{\restriction}n_0$. Then, for any $n > i$, $x(n) \in \pi(x{\restriction}n)$ implies $x(n) \in \varphi_\pi(n)$, which induces the three Tukey connections. □

Consequently, localizations and predictions have the same limit value in the following sense:

Theorem 4.4. Let $h \in \omega^\omega$.

(1) $\mathrm{minGPr}_h \le \mathrm{minGLc}$ and $\mathrm{supGPr}_h \ge \mathrm{supGLc}$. In particular, $\mathrm{minGPr}_h = \mathrm{minGLc}$ and $\mathrm{supGPr}_h = \mathrm{supGLc}$ if h goes to infinity by Lemma 4.1 and Theorem 2.7.

(2) (e.g. [Laf97, Proposition 2.2]) $(\mathfrak{e}_{ubd} =)\mathrm{minLPr}_h = \mathrm{minLLc}$ and $(\mathfrak{pr}_{ubd} =)\mathrm{supLPr}_h = \mathrm{supLLc}$.

(3) $\mathrm{minIPr}_h = \mathrm{minILc}$ and $\mathrm{supIPr}_h = \mathrm{supILc}$.

Proof. We only prove the former (in)equalities.

(1) Since $h' := h'_h$ goes to infinity, $\mathrm{minGLc} = \mathrm{minGLc}_{h'} = \min\{\mathfrak{b}^{\mathrm{GLc}}_{b,h'} : b \in \omega^\omega\} = \min\{\mathfrak{b}^{\mathrm{GLc}}_{b,h'} : b \in \omega^\omega, b > h'\}$ holds. Let $b \in \omega^\omega$ be arbitrary with $b > h'$. By Lemma 4.3, $\mathrm{minGPr}_h \le \mathfrak{e}^{\mathrm{G}}_{b,h} \le \mathfrak{b}^{\mathrm{GLc}}_{b,h'}$. Since $b > h'$ was arbitrary, we have $\mathrm{minGPr}_h \le \mathrm{minGLc}$.

(2) If h goes to infinity, $\mathrm{minLPr}_h = \mathrm{minLLc}$ is obtained similarly, so we may assume h is bounded. Since $\mathrm{minLPr}_h \le \mathrm{minLLc}$ is proved similarly as well, we show the converse. Let $b \in \omega^\omega$ be arbitrary, $h^+ \ge h$ be any function going to infinity and b^+ witness $\mathbf{LPr}(b, h) \preceq_T \mathbf{LPr}(b^+, h^+)$ by Lemma 3.8. Since h^+ goes to infinity, $\mathrm{minLLc} = \mathrm{minLPr}_{h^+} \le \mathfrak{e}^{\mathrm{L}}_{b^+,h^+} \le \mathfrak{e}^{\mathrm{L}}_{b,h}$. Since b was arbitrary, we obtain $\mathrm{minLLc} \le \mathrm{minLPr}_h$.

(3) $\mathrm{minIPr}_h \le \mathrm{minILc}$ is proved similarly and $\mathrm{minIPr}_h \ge \mathrm{minILc}_h = \mathrm{minILc}$ by Lemma 4.1 and Theorem 2.7.

□

Remark 4.5. If h is bounded, $\mathrm{minGPr}_h < \mathrm{minGLc}$ might happen: It is not hard to see that σ-$(k+1)$-linked forcings are $\mathbf{GPr}(k+1,k)$-good and hence $\mathrm{minGPr}_k < \mathrm{add}(\mathcal{N}) = \mathfrak{b}^{\mathrm{GLc}}_{\omega,h} \leq \mathrm{minGLc}$ holds in the Amoeba model (see also Theorem 5.1, [BS01, Lemma 3.1, Theorem 3.8(a)]).

In the local and infinite versions, the same phenomenon that predictions can be harder than localizations happens in another case, namely, when the whole space of predictions is sufficiently larger:

Theorem 4.6. Let $b, h, h^- \in \omega^\omega$ and assume h goes to infinity. Then, there is $b^+ \in \omega^\omega$ such that $\mathbf{LLc}(b,h) \preceq_T \mathbf{LPr}(b^+, h^-)$ and $\mathbf{ILc}(b,h) \preceq_T \mathbf{IPr}(b^+, h^-)$.

Proof. Inductively take natural numbers $0 = i_{-1} < i_0 < \cdots$ which satisfy for all $k < \omega$,

$$\prod_{i<i_{k-1}} b(i) \cdot h^-(k) \leq h(i_k - 1),$$

which is possible since h goes to infinity. Define b^+ by $b^+(k) := \prod_{i \in I_k} b(i)$ where $I_k := [i_{k-1}, i_k)$ for each $k < \omega$. Assume we are given a global (b^+, h^-)-predictor π. For $k < \omega$, define:

$$A_k := \left\{ t(i_k - 1) : t \in \pi(\langle \sigma \restriction I_{k'} : k' < k \rangle),\ \sigma \in \prod_{i<i_{k-1}} b(i) \right\},$$

which is valid since if $\sigma \in \prod_{i<i_{k-1}} b(i)$, by identifying functions and natural numbers, $\langle \sigma \restriction I_{k'} : k' < k \rangle \in \prod_{k'<k} b^+(k')$ and if $t \in \pi(\langle \sigma \restriction I_{k'} : k' < k \rangle)$, t belongs to $b^+(k) = \prod_{i \in I_k} b(i)$ and again by identification t is a function and $t(i_k - 1) \in b(i_k - 1)$. Also note that $|A_k| \leq \prod_{i<i_{k-1}} b(i) \cdot h^-(k) \leq h(i_k - 1)$. Therefore, $A_k \in [b(i_k - 1)]^{\leq h(i_k - 1)}$, so there is a (b,h)-slalom φ_π satisfying $\varphi_\pi(i_k - 1) = A_k$ for all $k < \omega$. By construction, for any $k < \omega$ and $f \in \prod b$,

$$f \restriction I_k \in \pi(\langle f \restriction I_{k'} : k' < k \rangle) \text{ implies } f(i_k - 1) \in \varphi_\pi(i_k - 1),$$

which induces $\mathbf{LLc}(b,h) \preceq_T \mathbf{LPr}(b^+, h^-)$ and $\mathbf{ILc}(b,h) \preceq_T \mathbf{IPr}(b^+, h^-)$. □

5. Connection with ideals

There are many known results on the connections between ideals on the reals and global/infinite localization cardinals. Here are such examples:

Theorem 5.1. ([Bar84], [CM23, Theorem 4.2]) $\mathfrak{b}^{\mathrm{GLc}}_{\omega,h} = \mathrm{add}(\mathcal{N})$ and $\mathfrak{d}^{\mathrm{GLc}}_{\omega,h} = \mathrm{cof}(\mathcal{N})$ for $h \in \omega^\omega$ going to infinity.

Theorem 5.2. ([Mil82], [Bar87], [CM23, Theorem 5.1]) $\mathfrak{d}^{\mathrm{ILc}}_{\omega,h} = \mathrm{non}(\mathcal{M})$ and $\mathfrak{b}^{\mathrm{ILc}}_{\omega,h} = \mathrm{cov}(\mathcal{M})$ for $h \in \omega^\omega$.

Theorem 5.3. ([KM22, Lemma 2.3],[CM23, Lemma 6.2])

(1) If $\sum_{n<\omega} \frac{h(n)}{b(n)} < \infty$, then $\mathrm{cov}(\mathcal{N}) \leq \mathfrak{d}^{\mathrm{ILc}}_{b,h}$ and $\mathfrak{b}^{\mathrm{ILc}}_{b,h} \leq \mathrm{non}(\mathcal{N})$.

(2) If $\sum_{n<\omega} \frac{h(n)}{b(n)} = \infty$, then $\mathrm{cov}(\mathcal{E}) \leq \mathfrak{b}^{\mathrm{ILc}}_{b,h}$ and $\mathfrak{d}^{\mathrm{ILc}}_{b,h} \leq \mathrm{non}(\mathcal{E})$, where $\mathcal{E}$ denotes the σ-ideal generated by closed null sets.

Now let us look at local cases. First, we easily have the following by considering the set of functions predicted by a single local predictor:

Lemma 5.4.

- Let $b \in (\omega+1)^\omega$ and $h \in \omega^\omega$. Then, $\mathfrak{b}^{\mathrm{LLc}}_{b,h} \leq \mathfrak{e}^{\mathrm{L}}_{b,h} \leq \mathrm{non}(\mathcal{M})$ and $\mathrm{cov}(\mathcal{M}) \leq \mathfrak{pr}^{\mathrm{L}}_{b,h} \leq \mathfrak{d}^{\mathrm{LLc}}_{b,h}$.
- Let $b, h \in \omega^\omega$. If $\limsup_n \frac{h(n)}{b(n)} < 1$, then $\mathfrak{b}^{\mathrm{LLc}}_{b,h} \leq \mathfrak{e}^{\mathrm{L}}_{b,h} \leq \mathrm{non}(\mathcal{E})$ and $\mathrm{cov}(\mathcal{E}) \leq \mathfrak{pr}^{\mathrm{L}}_{b,h} \leq \mathfrak{d}^{\mathrm{LLc}}_{b,h}$.

Remark 5.5. If $\limsup_n \frac{h(n)}{b(n)} = 1$, $\mathrm{non}(\mathcal{E}) < \mathfrak{b}^{\mathrm{LLc}}_{b,h}$ and $\mathfrak{d}^{\mathrm{LLc}}_{b,h} < \mathrm{cov}(\mathcal{E})$ might happen: The author introduced in [Yam24] the forcing notion $\mathbb{LE}_b$ which adds an $\mathbf{LLc}(b, b-1)$-unbounded real and proved that $\mathbb{LE}_b$ keeps $\mathrm{non}(\mathcal{E})$ small if $b(n) \geq 2^n$ for $n < \omega$. Consequently, in his model constructed in [Yam24, Theorem 5.9], $\mathrm{non}(\mathcal{E}) < \mathfrak{b}^{\mathrm{LLc}}_{b,b-1} = \mathrm{non}(\mathcal{M}) < \mathrm{cov}(\mathcal{M}) = \mathfrak{d}^{\mathrm{LLc}}_{b,b-1} < \mathrm{cov}(\mathcal{E})$ holds.

Let us focus on the uniformity of the null and meager additive ideals $\mathcal{NA}$ and $\mathcal{MA}$.

Definition 5.6. Let $\mathcal{I}$ be an ideal on 2^ω. $\mathcal{IA}$ denotes the set of all $\mathcal{I}$-additive sets, i.e., the collection of $X \subseteq 2^\omega$ such that $A + X \in \mathcal{I}$ for all $A \in \mathcal{I}$. Here, the addition $+$ on 2^ω is defined by identifying $2^\omega \cong (\mathbb{Z}/2\mathbb{Z})^\omega$.

The additivity and uniformity of $\mathcal{NA}$ can be characterized using the limit global number:

Fact 5.7. ([Paw85, Lemma 2.2], [CMR24, Corollary 1.7, Theorem A]) $\mathrm{add}(\mathcal{NA}) = \mathrm{non}(\mathcal{NA}) = \mathrm{minGLc}$.

In particular, $\mathrm{non}(\mathcal{NA}) = \mathrm{minGLc} \leq \mathrm{minLLc} = \mathrm{minLPr} = \mathfrak{e}_{ubd}$ holds, which is already shown by Cardona in [Car24, Theorem 2.1]. However, in the case of $\mathcal{MA}$, we show that the opposite direction holds:

Theorem 5.8. $\mathfrak{e}_{ubd} \leq \mathrm{non}(\mathcal{MA})$.

To prove this theorem, let us introduce the following relational system to characterize the uniformity of $\mathcal{MA}$:

Definition 5.9. ([CMR24, Definition 2.7]) Let $\mathbb{IP}$ denote the set of all interval partitions of ω.

(1) For $f, g \in \omega^\omega$ and $I = \langle I_k : k < \omega \rangle \in \mathbb{IP}$, $f \sqsubset^\bullet (I, g)$ if for all but finitely many $k < \omega$, there exists $i \in I_k$ such that $f(k) = g(k)$.
(2) For $b \in \omega^\omega$, denote the relational system $\mathbf{R}_b := \langle \prod b, \mathbb{IP} \times \prod b, \sqsubset^\bullet \rangle$.

Fact 5.10. ([BJ94, Theorem 2.2], [CMR24, Lemma 2.10]) $\mathrm{non}(\mathcal{MA}) = \min\{\mathfrak{b}(\mathbf{R}_b) : b \in \omega^\omega\}$.

$\mathbf{R}_b$ is Tukey below some $\mathbf{LLc}(b^+, h)$:

Lemma 5.11. Let $b, h \in \omega^\omega$. Then, there exists $b^+ \in \omega^\omega$ such that $\mathbf{R}_b \preceq_T \mathbf{LLc}(b^+, h)$.

Proof. Take some $I = \langle I_k : k < \omega \rangle \in \mathbb{IP}$ such that $|I_k| \geq h(k)$. Define $b^+(k) := \prod_{i \in I_k} b(i)$. Let φ be any (b^+, h)-slalom and fix $k < \omega$. Let $S := \varphi(k) \in b^+(k) := \prod_{i \in I_k} b(i)$, which is assumed to be a set of functions on I_k. Enumerate $\{t_i : i < |S|\} = S$ and let $I^- := \{\min I_k + i : i < |S|\} \subseteq I_k$, since $|S| \leq h(k) \leq |I_k|$. define a function g_k on I^- by:

$$g_k(\min I_k + i) := t_i(i) \in b(i).$$

Unfix k and by collecting all g_k together, let $g = g_\varphi \in \prod b$ be some function which satisfies $g(\min I_k + i) = g_k(\min I_k + i)$ for all $k < \omega$ and $i < |\varphi(k)|$. By construction, note that for any $x \in \prod b$ and $k < \omega$,

$$x \restriction I_k \in \varphi(k) \text{ implies } x(i) = g_\varphi(i) \text{ for some } i \in I_k.$$

Let $D \in [\omega]^\omega$ and $I_D = \langle I'_k : k < \omega \rangle \in \mathbb{IP}$ be such that if k is the j-th element of D, then $I_k \subseteq I'_j$. Again by construction, for $x \in \prod b$, $\langle x \restriction I_k : k < \omega \rangle \in^* (D, \varphi)$ implies $x \sqsubset^\bullet (I_D, g_\varphi)$, which shows $\mathbf{R}_b \preceq_T \mathbf{LLc}(b^+, h)$. $\square$

Proof of Theorem 5.8. Let h be any function going to infinity. By Theorem 4.4, we have $\mathfrak{e}_{ubd} = \mathrm{minLPr}_h = \mathrm{minLLc}_h$, so we show $\mathrm{minLLc}_h \leq \mathrm{non}(\mathcal{MA})$ instead. Let $b \in \omega^\omega$ be arbitrary. By Lemma 5.11, some $b^+ \in \omega^\omega$ satisfies $\mathbf{R}_b \preceq_T \mathbf{LLc}(b^+, h)$, so we have $\mathrm{minLLc}_h \leq \mathfrak{b}^{\mathrm{LLc}}_{b^+,h} \leq \mathfrak{b}(\mathbf{R}_b)$. Since b was arbitrary, by Fact 5.10 we have $\mathrm{minLLc}_h \leq \min\{\mathfrak{b}(\mathbf{R}_b) : b \in \omega^\omega\} = \mathrm{non}(\mathcal{MA})$. $\square$

Remark 5.12. $\mathfrak{e}_{ubd} < \mathrm{non}(\mathcal{MA})$ holds in the Hechler model: Brendle (essentially) showed in [Bre95] that $\mathfrak{e}_{ubd} \leq \mathfrak{e}_{\mathrm{id}} = \aleph_1$ in the Hechler model (id denotes the identity function on ω), and in that model $\mathrm{non}(\mathcal{MA}) \geq \mathrm{add}(\mathcal{M}) = \min\{\mathfrak{b}, \mathrm{cov}(\mathcal{M})\} = 2^{\aleph_0} > \aleph_1$ holds.

Figure 1 illustrates the relationship of the cardinal invariants below $\mathrm{non}(\mathcal{M})$ we have seen and here are additional explanations:

- $\mathrm{cov}(\mathcal{N}) \leq \mathrm{supI}$ by Theorem 5.3(1).
- $\mathrm{non}(\mathcal{MA}) \leq \mathrm{non}(\mathcal{E})$ by e.g. [CMR24, Corollary 1.12].
- $\mathrm{add}(\mathcal{M}) \leq \mathrm{non}(\mathcal{MA})$ by $\mathrm{add}(\mathcal{M}) \leq \mathrm{add}(\mathcal{MA}) \leq \mathrm{non}(\mathcal{MA})$.
- $\min\{\mathfrak{e}, \mathfrak{b}\} \leq \mathrm{add}(\mathcal{M})$ by $\mathrm{add}(\mathcal{M}) = \min\{\mathfrak{b}, \mathrm{cov}(\mathcal{M})\}$ and $\mathfrak{e} \leq \mathrm{cov}(\mathcal{M})$.

- $\mathfrak{d}^{\mathrm{ILc}}_{b,h}$ can be finite by [CM23, Theorem 3.13].

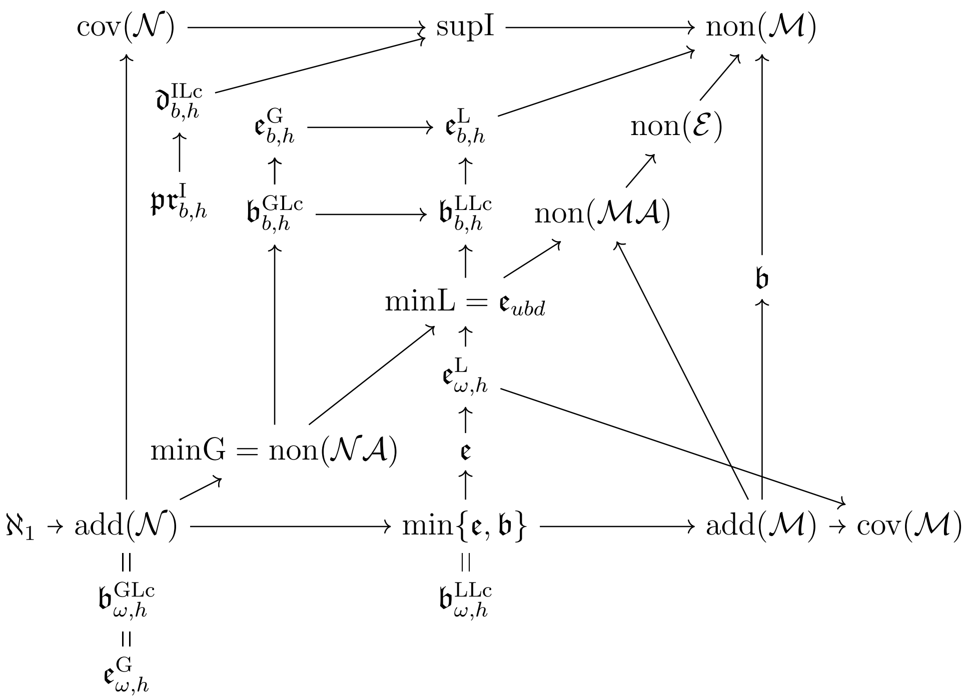

FIGURE 1. Diagram of cardinal invariants below $\mathrm{non}(\mathcal{M})$. Here, $b \in (\omega \setminus 2)^\omega$ and $h \in \prod b$ go to infinity and $\mathrm{minG} := \mathrm{minGLc} = \mathrm{minGPr}_h$, $\mathrm{minL} := \mathrm{minLLc} = \mathrm{minLPr}_{h'}$, $\mathrm{supI} := \mathrm{supILc} = \mathrm{supIPr}_{h'}$ for $h' \geq 1$.

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