FORCING THE FAILURE OF $\mho$ BY FINITE APPROXIMATIONS

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Abstract. We demonstrate that the Y-Proper Forcing Axiom implies the failure of $\mho$. This is indirectly proved in [6], but the proof in this paper is a direct way and easier than the argument in [6].

1. Introduction

Moore formulated the axiom $\mho$ in [8] to show that his solution of the five element basis problem for the uncountable linear orders in [7] needs his Mapping Reflection Principle in some sense. In fact, $\mho$ implies the existence of an Aronszajn line containing no Countryman suborders ([8]). The Proper Forcing Axiom implies the Mapping Reflection Principle, and the Mapping Reflection Principle implies the failure of $\mho$.

Cohen forcing adds a witness of $\mho$, and so a finite support iteration of ccc forcing notions of limit length forces $\mho$. Thus it seems to be difficult to show that it is consistent that $\mho$ fails and the size of the continuum greater than $\aleph_2$. Asperó and Mota introduced the forcing axiom $\mathsf{PFA}^{\mathrm{fin}}(\omega_1)$ for finitely proper forcing notions and proved that $\mathsf{PFA}^{\mathrm{fin}}(\omega_1)$ implies the negations of $\mho$, and $\mathsf{PFA}^{\mathrm{fin}}(\omega_1)$ is consistent with the size of the continuum greater than $\aleph_2$ ([1]).

Chodounský and Zapletal introduced the properties of forcing notions called Y-cc and Y-properness, and they proved that, if there exists a supercompact cardinal, then there exists a Y-proper forcing notion which forces the Forcing Axiom for Y-proper forcing notions, call the Y-Proper Forcing Axiom ([3]). Miyamoto and the author proved that the Y-Proper Forcing Axiom implies the Mapping Reflection Principle ([6]), hence the Y-Proper Forcing Axiom implies the failure of $\mho$.

We demonstrate that the Y-Proper Forcing Axiom implies the failure of $\mho$ by a direct way. This is easier than the argument in [6].

2. The Y-properness

For a cardinal κ, $H(\kappa)$ denotes the set of all sets of hereditary cardinality less than κ. $H(\kappa)$ is always considered as the structure equipped with a fixed well-order.

Shelah introduced the notion of the properness of foricng notions ([9, 10]). Let $\mathbb{P}$ be a forcing notion, λ a regular cardinal with $\mathcal{P}(\mathbb{P}) \in H(\lambda)$, N a countable elementary submodel of $H(\lambda)$, and p a condition of the forcing notion $\mathbb{P}$. p is called a $(N, \mathbb{P})$-*generic* provided that, for any dense subset D of $\mathbb{P}$, if $D \in N$, then $D \cap N$ is predense below p in $\mathbb{P}$. A forcing notion $\mathbb{P}$ is *proper* if and only if, for any regular cardinal λ with $\mathcal{P}(\mathbb{P}) \in H(\lambda)$, there is a closed unbounded set of countable elementary submodels N of $H(\lambda)$ with $\mathbb{P} \in N$ such that every condition of $\mathbb{P}$ in N has an extension which is $(N, \mathbb{P})$-generic. Typical proper forcing notions are ccc forcing notions and σ-closed forcing notions. The Proper Forcing Axiom is the assertion that, for any proper forcing notion $\mathbb{P}$ and $\aleph_1$ many dense subsets $\{D_\alpha : \alpha \in \omega_1\}$ of $\mathbb{P}$, there exists a filter G of $\mathbb{P}$ which meets all the D_α's. Baumgartner proved that, if there exists a supercompact cardinal, then there exists a proper forcing notion which forces the Proper Forcing Axiom ([2, §3]).

Definition 2.1 (Chodounský and Zapletal [3, §1])**.** For a forcing notion $\mathbb{P}$, $\mathrm{RO}(\mathbb{P})$ is denoted by the regular open algebra of $\mathbb{P}$ (see e.g. [4, Ch. II 3.3. Lemma], [5, Lemma III.4.8]).

(1) Let $\mathbb{P}$ be a forcing notion, λ a regular cardinal with $\mathcal{P}(\mathbb{P}) \in H(\lambda)$, N a countable elementary submodel of $H(\lambda)$ with $\mathbb{P} \in N$, and p a condition of $\mathbb{P}$. p is called $(N, \mathbb{P})$-*Y-generic* if and only if, for any $r \leq_{\mathbb{P}} p$, there exists a filter $F \in N$ on $\mathrm{RO}(\mathbb{P})$ such that the set $\{s \in \mathrm{RO}(\mathbb{P}) \cap N : r \leq_{\mathrm{RO}(\mathbb{P})} s\}$ is included in the set F as a subset.

(2) A forcing notion $\mathbb{P}$ satisfies *Y-proper* provided that, for any regular cardinal λ with $\mathcal{P}(\mathbb{P}) \in H(\lambda)$, there is a closed unbounded set of countable elementary submodels N of $H(\lambda)$ with $\mathbb{P} \in N$ such that every condition of $\mathbb{P}$ in N has an extension which is $(N, \mathbb{P})$-generic and $(N, \mathbb{P})$-Y-generic.

A forcing notion $\mathbb{P}$ is called *Y-cc* provided that, for any regular cardinal λ with $\mathcal{P}(\mathbb{P}) \in H(\lambda)$, and any countable elementary submodel N of $H(\lambda)$ with $\mathbb{P} \in N$, every condition of $\mathbb{P}$ is $(N, \mathbb{P})$-Y-generic. Chodounský and Zapletal proved that a Y-cc forcing notion is ccc. It has not been known yet whether a Y-proper ccc forcing notion is Y-cc ([3, Question 4.13]).

Chodounský and Zapletal proved that it is consistent relative to the the existence of a supercompact cardinal that the forcing axiom for Y-proper forcing notions is consistent, by applying Neeman's forcing iteration with two types of models as side conditions ([3, §6]). Their forcing iteration is Y-proper. Chodounský and Zapletal presented many preservation theorems of Y-proper forcing notions in [3, §2].

3. The failure of ℧

Definition 3.1. ℧ (mho) is the assertion that there is a sequence $\langle f_\alpha : \alpha \in \omega_1 \cap \mathsf{Lim}\rangle$ such that each f_α is a continuous map from α (equipped with the order topology) into ω, and, for any club subset A of ω_1, there exists $\delta \in A$ such that $\mathrm{ran}(f_\alpha \restriction (A \cap \delta)) = \omega$.

Let $\alpha \in (\omega_1 \cap \mathsf{Lim}) \setminus \omega$ and f a continuous function from α into ω. Then for each $\xi \in \alpha \cap \mathsf{Lim}$, the value of $f(\xi)$ is eventually equal to the values $f(\zeta)$ for $\zeta < \xi$, and so the set $\{\xi \in \alpha; f(\xi+1) \neq f(\xi)\}$ is of order type $\leq \omega$. Thus there exists $B \subseteq \alpha$ such that B is of order type $\leq \omega$ and, for each $\xi \in \alpha$, the value $f(\xi)$ is decided by the cardinality of the set $B \cap \xi$.

Definition 3.2 (Asperó and Mota [1])**.**

- A forcing notion $\mathbb{P}$ is called *finitely proper* if and only if, for any large enough regular cardinal λ, any finite set $\{N_i : i < m\}$ of countable elementary submodels of $H(\lambda)$ which contain $\mathbb{P}$ as a member, and any condition p of $\mathbb{P}$ in all N_i, there exists an extension of p which is $(N_i, \mathbb{P})$-generic for every $i < m$.
- $\mathsf{PFA}^{\mathrm{fin}}(\omega_1)$ denote the forcing axiom for the class of finitely proper forcing notions of size $\aleph_1$ and for families of $\aleph_1$ many dense sets.

Asperó and Mota proved that $\mathsf{PFA}^{\mathrm{fin}}(\omega_1)$ implies the nagation of ℧, and, for any regular cardinal κ greater than $\aleph_1$ (with some additional assumptions on κ), there exists a forcing iteration, called Asperó–Mota iteration, which forces $\mathsf{PFA}^{\mathrm{fin}}(\omega_1)$ and $2^{\aleph_0} = \kappa$ ([1]).

In the rest of this section, let $\vec{f} = \langle f_\alpha : \alpha \in \omega_1 \cap \mathsf{Lim}\rangle$ be a sequence of continuous functions f_α from α into ω.

Definition 3.3. $\mathbb{P}$ is defined by the set of finite functions p such that

- $\mathrm{dom}(p)$ is a finite set of countable limit ordinals, and, for each $\alpha \in \mathrm{dom}(p)$, denote $p(\alpha) = \langle p_0(\alpha), p_1(\alpha)\rangle$ which is in $\omega \times \omega_1$,

- (working part) for each $\alpha \in \mathrm{dom}(p)$, $p_0(\alpha) \notin \mathrm{ran}(f_\alpha \restriction \mathrm{dom}(p))$ (which is equal to $\mathrm{ran}(f_\alpha \restriction (\mathrm{dom}(p) \cap \alpha))$),
- (side-condition part) for any α and β in $\mathrm{dom}(p)$, if $\alpha < \beta$, then $\alpha < p_1(\alpha) < \beta$.

The order is defined by $q \leq_{\mathbb{P}} p$ if and only if $q \supseteq p$.

This forcing notion seems to be different from the one in [1, Proposition 5.8], but these are essentially same. By use of the side condition method, the proof of the properness may be simpler than the one in [1].

We will prove that $\mathbb{P}$ is proper. If PFA holds, then there exists a filter G on $\mathbb{P}$ such that $\omega_1 \cap \mathsf{Lim}$ is included in the union of the intervals $[\alpha, p_1(\alpha))$ of ordinals for all α in the set $\bigcup_{p \in G} \mathrm{dom}(p)$. Then $\bigcup_{p \in G} \mathrm{dom}(p)$ is club in ω_1, and, by the definition of $\mathbb{P}$ (that is, by the role of the first coordinates of conditions of $\mathbb{P}$), $\bigcup_{p \in G} \mathrm{dom}(p)$ witnesses that $\vec{f}$ does not satisfy ℧.

In the rest of the paper, let λ be a regular cardinal such that $\mathcal{P}(\mathbb{P}) \in H(\lambda)$, and λ^* is a regular cardinal such that $H(\lambda) \in H(\lambda^*)$.

Proposition 3.4. *Suppose that $m \in \omega$, for each $i \in m$, N_i^* is a countable elementary submodel N_i^* of $H(\lambda^*)$ with $H(\lambda) \in N_i^*$, and F is a finite subset of ω. Then there exists $k \in \omega \setminus F$ such that, for any $i \in m$ and any $b \in N_i^* \cap H(\lambda)$, there exists a countable elementary submodel N of $H(\lambda)$ in N_i^* such that $b \in N$ and $f_{\omega_1 \cap N_i^*}(\omega_1 \cap N) \neq k$.*

Proof. Since each N_i^* is countable, we have an enumeration $\{b_i^n : n \in \omega\} = N_i^* \cap H(\lambda)$ of $N_i^* \cap H(\lambda)$. By elementarity of N_i^*, for each $n \in \omega$, there exists a countable elementary submodel N_i^n of $H(\lambda)$ such that $\left\{b_i^j : j \leq n\right\} \in N_i^n \in N_i^*$. Then we can find $k \in \omega \setminus F$ such that, for each $i \in m$, $f_{\omega_1 \cap N_i^*}(\omega_1 \cap N_i^n) \neq k$ holds for infinitely many $n \in \omega$. □

So it follows from the previous proposition that the following lemma implies that $\mathbb{P}$ is finitely proper.

Lemma 3.5. *Suppose that N^* is a countable elementary submodel of $H(\lambda^*)$ such that N^* contains $\vec{f}$ and $H(\lambda)$ as members. Then a condition p of $\mathbb{P}$ is $(N^*, \mathbb{P})$-generic (then p is also $(N^* \cap H(\lambda), \mathbb{P})$-generic), if $\mathrm{dom}(p)$ contains $\omega_1 \cap N^*$ as a member and p satisfies that*

(∗) *for any $b \in N^* \cap H(\lambda)$, there exists a countable elementary submodel N of $H(\lambda)$ in N^* such that $b \in N$ and $f_{\omega_1 \cap N^*}(\omega_1 \cap N) \neq p_0(\omega_1 \cap N^*)$.*

Proof. Let N^* and p be as in the assumption of the lemma, and let D be a dense subset of $\mathbb{P}$ in N^*. We will show that $D \cap N^*$ is predense below p.

To do this, let q be an extension of p in $\mathbb{P}$. Since $\omega_1 \cap N^* \in \mathrm{dom}(q)$, $q \cap N^* = q \restriction N^*$. By extending q if necessary, we may assume that q belongs to D. The point of the proof is that, for each $\alpha \in \mathrm{dom}(q)$ which is greater than $\omega_1 \cap N^*$, $f_\alpha(\omega_1 \cap N^*) \neq q_0(\alpha)$ holds, because $f_\alpha(\omega_1 \cap N^*)$ is in $\mathrm{ran}(f_\alpha \restriction \mathrm{dom}(q)))$ and the restricted function $f_\alpha \restriction (\omega_1 \cap N^*)$ is eventually constant (because the function f_α is continuous and $\omega_1 \cap N^*$ is a closure point of the domain of f_α, and then $f_\alpha \restriction (\omega_1 \cap N^*)$ belongs to N^*). However $f_{\omega_1 \cap N^*}$ (whose domain is $\omega_1 \cap N^*$) may not be eventually constant. There exists $\varepsilon \in \omega_1 \cap N^*$ such that, for each $\alpha \in \mathrm{dom}(q)$ which is greater than $\omega_1 \cap N^*$, the restricted function $f_\alpha \restriction [\varepsilon, \omega_1 \cap N^*)$ is constant. By (∗), there exists a countable elementary submodel N of $H(\lambda)$ in N^* such that N contains the set $\left\{\vec{f}, \mathbb{P}, D, q \cap N^*, \varepsilon\right\}$ as a member and $f_{\omega_1 \cap N^*}(\omega_1 \cap N) \neq q_0(\omega_1 \cap N^*)$. Then $f_{\omega_1 \cap N^*} \restriction (\omega_1 \cap N)$ is eventually constant (and $f_{\omega_1 \cap N^*} \restriction (\omega_1 \cap N)$ belongs to N). Let $\varepsilon' \in \omega_1 \cap N$ be such that $f_{\omega_1 \cap N^*} \restriction [\varepsilon', \omega_1 \cap N)$ is constant. Then $f_{\omega_1 \cap N^*}(\varepsilon') = f_{\omega_1 \cap N^*}(\omega_1 \cap N) \neq q_0(\omega_1 \cap N^*)$. Let $\delta := \max\{\varepsilon, \varepsilon'\}$, which is in N. Since q belongs to D and D is in N, by elementarity of N, there exists $r \in D \cap N$ such that

- $r \restriction \delta = q \cap N^*$ (which is equal to $q \restriction N$) and $r \leq_{\mathbb{P}} q \cap N^*$,
- r and q are same size,
- for each $\nu < |q \setminus N^*|$, if α is the ν-th member of $\mathrm{dom}(q) \setminus N^*$ and β is the ν-th member of $\mathrm{dom}(r) \setminus \delta$, then $q_0(\alpha) = r_0(\beta)$.

We note that, $\mathrm{dom}(r) \setminus \delta \subseteq N$, and, for each $\alpha \in \mathrm{dom}(q) \setminus ((\omega_1 \cap N^*) + 1)$,

$$\mathrm{ran}(f_\alpha \restriction (\mathrm{dom}(r) \setminus N)) = \{f_\alpha(\omega_1 \cap N)\} = \{f_\alpha(\omega_1 \cap N^*)\} \not\ni q_0(\alpha),$$

and

$$\mathrm{ran}(f_{\omega_1 \cap N^*} \restriction (\mathrm{dom}(r) \setminus N)) = \{f_{\omega_1 \cap N^*}(\omega_1 \cap N)\} \not\ni q_0(\omega_1 \cap N^*).$$

Therefore, q and r are compatible in $\mathbb{P}$. □

Theorem 3.6. *$\mathbb{P}$ is Y-proper.*

Proof. For a condition $p \in \mathbb{P}$, a finite sequence $\vec{k} = \langle k_\nu : \nu < l \rangle$ of members of ω of length l, and a subset $\mathcal{A}$ of $\mathbb{P}$, $E(p, \vec{k}, \mathcal{A})$ denotes the set of all countable ordinals δ such that there exists $q \in \mathcal{A}$ such that

- $q \restriction (\delta + 1) = q \restriction \delta = p$ (hence $\delta \notin \mathrm{dom}(q)$),
- $\mathrm{dom}(q) \setminus \delta$ is of size l,
- for each $\nu < l$, if α is the ν-th member of $\mathrm{dom}(q) \setminus \delta$, then $q_0(\alpha) = k_\nu$ and $f_\alpha(\delta) \neq k_\nu$,

and define that a subset $\mathcal{A}$ of $\mathbb{P}$ is $(p, \vec{k})$-large if and only if $E(p, \vec{k}, \mathcal{A})$ is stationary in ω_1.

Let $p \in \mathbb{P}$ and $\vec{k} = \langle k_\nu : \nu < l \rangle$ a finite sequence of members of ω of length l. We will show that $\{\bigvee \mathcal{A} : \mathcal{A} \subseteq \mathbb{P} \text{ is } (p, \vec{k})\text{-large}\}$ is a centered subset of $\mathrm{RO}(\mathbb{P})$. Let $n \in \omega$ and $\mathcal{A}_i$, $i \in n$, $(p, \vec{k})$-large subsets of $\mathbb{P}$. It suffices to find $q^i \in \mathcal{A}_i$, $i \in n$, such that $\{q^i : i \in n\}$ has a common extension in $\mathbb{P}$. To do this, take a sequence $\langle \lambda_i : i \in n+1 \rangle$ of regular cardinals such that $\lambda_0 = \lambda$ and, for each $i \in n$ with $i \geq 1$, $\lambda_{i+1} = (2^{\lambda_i})^+$. Denote $M_n = H(\lambda_n)$. By reverse induction on $i \in n$, we will find a countable elementary submodel M_i of $H(\lambda_i)$, $q^i \in \mathcal{A}_i \cap M_{i+1}$ and $\varepsilon_i \in \omega_1 \cap M_i$ such that

1. $\left\{\vec{f}, \mathbb{P}, p, \left\{[H(\lambda_j)]^{\aleph_0}, \mathcal{A}_j : j \in i\right\}, \{\varepsilon_j : j \in n \setminus (i+1)\}\right\} \in M_i \in M_{i+1}$,
2. $q^i \restriction ((\omega_1 \cap M_i) + 1) = p$ and $q^i \leq_{\mathbb{P}} p$,
3. $\mathrm{dom}(q^i) \setminus M_i$ is of size l,
4. for each $\nu < l$, if α is the ν-th member of $\mathrm{dom}(q^i) \setminus M_i$, then $q^i_0(\alpha) = k_\nu$ and $f_\alpha(\omega_1 \cap M_i) \neq k_\nu$, and
5. for each $\alpha \in \mathrm{dom}(q^i) \setminus M_i$, $f_\alpha \restriction [\varepsilon_i, \omega_1 \cap M_i)$ is constant.

Then, as seen in the proof of the properness, we can conclude that $\bigcup_{i \in n} q^i$ is a condition of $\mathbb{P}$, and so $\{q^i : i \in n\}$ has a common extension in $\mathbb{P}$. To find M_i, q^i and ε_i as above, we assume that we have $\{M_j, q^j, \varepsilon_j : j \in n \setminus (i+1)\}$. Since $[H(\lambda_i)]^{\aleph_0}$ and $\mathcal{A}_i$ are in M_{i+1} and $\mathcal{A}_i$ is $(p, \vec{k})$-large, by elementarity of M_{i+1}, we can take a countable elementary submodel M_i of $H(\lambda_i)$ in M_{i+1} such that $\omega_1 \cap M_i \in E(p, \vec{k}, \mathcal{A}_i)$ and M_i satisfies (1) above. Then, by elementarity of M_{i+1} again, there exists $q^i \in \mathcal{A}_i \cap M_{i+1}$ which satisfies (2) – (4) above. Then we take $\varepsilon_i \in \omega_1 \cap M_i$ which satisfies (5) above, which finishes the constructions of M_i, q^i and ε_i.

To show that $\mathbb{P}$ is Y-proper, suppose that N^* is a countable elementary submodel of $H(\lambda^*)$ such that N^* contains $\vec{f}$ and $[H(\lambda)]^{\aleph_0}$ as members, $p \in \mathbb{P}$, $\omega_1 \cap N^* \in$

$\mathrm{dom}(p)$, and p satisfies $(*)$ in Lemma 3.5. By Lemma 3.5, p is $(N^*, \mathbb{P})$-generic. Let us show that p is $(N^*, \mathbb{P})$-Y-generic.

Let r be an extension of p in $\mathbb{P}$. Denote $\vec{k} = \langle r_0(\alpha) : \alpha \in \mathrm{dom}(r) \setminus N^* \rangle$ (which is a non-empty sequence of length $|r \setminus N^*|$). Define F by the filter on $\mathrm{RO}(\mathbb{P})$ that is generated by the set $\{\bigvee \mathcal{A} : \mathcal{A} \subseteq \mathbb{P}$ is $(r \restriction N^*, \vec{k})$-large$\}$. Then F belongs to N^*. We will show that, for any $s \in \mathrm{RO}(\mathbb{P}) \cap N^*$, if $r \leq_{\mathrm{RO}(\mathbb{P})} s$, then $s \in F$. Let $s \in \mathrm{RO}(\mathbb{P}) \cap N^*$ be such that $r \leq_{\mathrm{RO}(\mathbb{P})} s$, and define $\mathcal{A}$ by the set of all $q \in \mathbb{P}$ such that $q \leq_{\mathrm{RO}(\mathbb{P})} s$. Then $\mathcal{A}$ is in N^*, and $\bigvee \mathcal{A} = s$. So it suffices to show that $\mathcal{A}$ is $(r \restriction N^*, \vec{k})$-large, because then $s = \bigvee \mathcal{A} \in F$.

We will show that $\mathcal{A}$ is $(r \restriction N^*, \vec{k})$-large, that is, $E(r \restriction N^*, \vec{k}, \mathcal{A})$ is stationary in ω_1. Since N^* contains $E(r \restriction N^*, \vec{k}, \mathcal{A})$ as a member, it suffices to show that N^* satisfies that $E(r \restriction N^*, \vec{k}, \mathcal{A})$ is stationary in ω_1. To do this, let I be a club subset of ω_1 in N^*. Let $\varepsilon \in \omega_1 \cap N^*$ be such that, for any $\alpha \in \mathrm{dom}(r) \setminus ((\omega_1 \cap N^*)+1)$, the restricted function $f_\alpha \restriction [\varepsilon, \omega_1 \cap N^*)$ is constant. By $(*)$, there exists a countable elementary submodel N of $H(\lambda)$ in N^* such that the set $\left\{\vec{f}, r \restriction N^*, \varepsilon, H(\lambda), \mathcal{A}, I\right\}$ is in N and $f_{\omega_1 \cap N^*}(\omega_1 \cap N) \neq p_0(\omega_1 \cap N^*)$. Then $\omega_1 \cap N$ belongs to I. Moreover, since

- $r \leq_{\mathrm{RO}(\mathbb{P})} s$ (hence $r \in \mathcal{A}$),
- $r \restriction ((\omega_1 \cap N) + 1) = r \restriction (\omega_1 \cap N) = p$ and $r \leq_{\mathbb{P}} p$,
- $\mathrm{dom}(r) \setminus N$ is of size l, and
- for each $\nu < l$, if α is the ν-th member of $\mathrm{dom}(r) \setminus N$, then $r_0(\alpha) = k_\nu$ and $f_\alpha(\omega_1 \cap N) \neq k_\nu$,

$\omega_1 \cap N$ belongs to $E(r \restriction N^*, \vec{k}, \mathcal{A})$. Therefore, $I \cap E(r \restriction N^*, \vec{k}, \mathcal{A})$ is not empty in N^*. □

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