MURMURATIONS OF MAASS FORMS

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1. Introduction

This article is a rough overview of the talk I gave at the Analytic Number Theory and Related Topics Symposia at RIMS in 2024. The main emphasis of that talk was to introduce the notion of murmurations and describe a new result in this area, joint with Andrew R. Booker, Min Lee, David Lowry-Duda and Nina Zubrilina. I would like to thanks RIMS for their hospitality and the organisers, Maki Nakasuji and Takashi Taniguchi, for the invitation and kindness I received during my stay. I would also like to thank Masao Tsuzuki for the invitation to Japan and for the hospitality received during my time there.

1.1. **What are murmurations?** Murmurations are a surprising correlation between the Dirichlet coefficients and the root number of an L-function. This phenomenon was discovered by chance by Yang-Hui He, Kyu-Hwan Lee, Thomas Oliver and Alexey Pozdynakov [HLOP24] in 2022, whilst trying to understand why recent work by the first three authors on using machine learning on elliptic curves was working so well. To show what they found, we first let p_n denote the nth prime, i.e. $p_1 = 2, p_2 = 3, \ldots$ and define

$$f_r(n) = \frac{1}{\#\mathcal{E}_r[N_1, N_2]} \sum_{E \in \mathcal{E}_r[N_1, N_2]} a_E(p_n)$$

for any $n > 1$, where $\mathcal{E}_r[N_1, N_2]$ is the family of elliptic curves with rank r and conductor between N_1 and N_2, and

$$a_E(p) = p + 1 - \#E(\mathbb{F}_p),$$

is the trace of Frobenius for the elliptic curve E modulo p. Essentially this sum is the average of $a_E(p)$ for a given prime p, over this family of elliptic curves. Plotting this function $f_r(n)$, for $r = 0$ or 1, as n increases gives the following rather surprising picture.

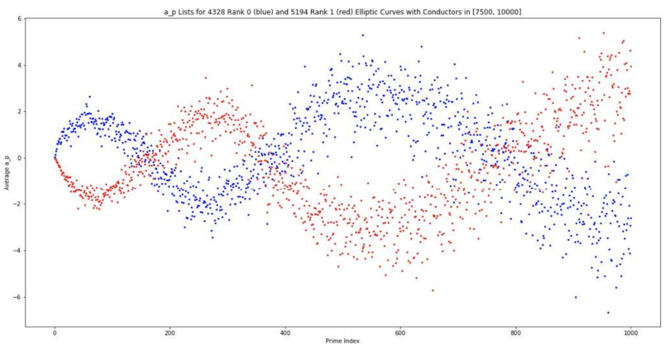

Figure 1. Plot of $f_r(n)$ for $N \in [7500, 10000]$ with $r = 0$ and $r = 1$. Original plot due to [HLOP24].

For those unfamiliar with the word, murmurations in the real world are a phenomenon in which certain birds, notably starlings, fly around in rather chaotic wave-like patterns in the sky, miraculously not hitting into each other! Figure 2 gives an example of such a phenomenon to compare to the elliptic curve plot.

FIGURE 2. Image of murmurations of birds. Alex Ramsay/Alamy Stock Photo

The plan for this article is as follows: I begin by giving some background on how this pattern was discovered. Then I will give some observations of what is actually happening and the relation of this phenomenon with other number theoretic objects. Following this, I will survey some known results that have been proven, including the recent result for Maass forms, and give a rough outline of the proof for this.

2. Background

To begin, we define an *elliptic curve* E to be the cubic equation

$$y^2 = x^3 + Ax + B,$$

where $A, B \in \mathbb{Z}$ satisfying

$$\Delta(E) = -16(4A^3 + 27B^2) \neq 0.$$

We call $\Delta(E)$ the *discriminant* of E. Further, let $E(\mathbb{Q})$ denote the set of rational points on E. Then the famous Mordell–Weil theorem tells us that $E(\mathbb{Q})$ is a finitely generated abelian group.

Theorem 2.1 (Mordell 1922, Weil 1929). *The set $E(\mathbb{Q})$ is a finitely generated abelian group. Namely*

$$E(\mathbb{Q}) \cong E(\mathbb{Q})_{\text{tors}} \oplus \mathbb{Z}^{r_E},$$

where $E(\mathbb{Q})_{\text{tors}}$ is the torsion subgroup and r_E is the rank of E.

The following theorem due to Mazur allows us to completely classify the torsion subgroup $E(\mathbb{Q})_{\text{tors}}$.

Theorem 2.2 (Mazur 1977-78). *The torsion subgroups $E(\mathbb{Q})_{\text{tors}}$ can be described as one of the following groups:*

- *C_n for $1 \leq n \leq 10$,*

- *C_{12},*
- *$C_{2n} \times C_2$ for $1 \leq n \leq 4$,*

where C_n denotes the cyclic group of order n.

To actually compute which group we have for a given elliptic curve, we use the following theorem due to Lutz–Nagell.

Theorem 2.3 (Lutz–Nagell 1937)**.** *Let $E/\mathbb{Q}$ be an elliptic curve. Then if (x, y) is a point of finite order on E, then:*

- *$x, y \in \mathbb{Z}$, and*
- *either $y = 0$ or $y^2 \mid \Delta_E$.*

The mysterious object in the Mordell–Weil theorem is the rank r_E. We expect most elliptic curves over $\mathbb{Q}$ to have ranks 0 or 1 [Gol79, KS99], although the highest known rank of an elliptic curve is 29 due to Elkies [Elk24]. Currently, it is still not known whether the maximum size of the rank is bounded or not.

Not all is lost however. Let $E(\mathbb{F}_p)$ be the set of integer points of E modulo a prime p and define the *conductor* N_E of E to be the product of all primes where $\Delta(E) = 0 \pmod{p}$. Further, define

$$a_E(p) := p + 1 - \#E(\mathbb{F}_p),$$

for some prime p. Due to Hasse, we have the inequality

$$|a_E(p)| \leq 2\sqrt{p}.$$

Now, we define the associated (incomplete) L-function to E by

$$L(E, s) := \prod_{p \nmid N_E} (1 - a_E(p)p^{-s} + p^{1-2s})^{-1}.$$

This definition is only convergent for $\mathrm{Re}(s) > 3/2$, however we can extend this to all $\mathbb{C}$ by analytic continuation, due to the modularity theorem for elliptic curves over $\mathbb{Q}$. This function has many interests to number theorists in giving information about the elliptic curve. One of the main ones is the following conjecture.

Conjecture 2.4. *(Birch–Swinnerton-Dyer (BSD) conjecture) We have that*

$$r_E = ord_{s=1} L(E, s).$$

This remarkable conjecture is telling us that this L-function defined using the traces of Frobenius, can tell us the rank.

2.1. **Computing the rank.** With BSD, computing the rank is as hard as computing the order of vanishing of the L-function, however this is still a conjecture, although some cases have been proven. Unconditionally, computing the rank is still hard, although large databases of elliptic curves and their ranks have been computed, for example on the "The L-functions and modular forms database" (LMFDB) [LMF25].

But it is now 2025, so maybe machine learning can help us? Yang-Hui He, Kyu-Hwan Lee and Thomas Oliver did exactly this in 2020 [HLO23]. They tested whether you can train a neural network on a list of $a_E(p)$ values and ranks. Then given a finite list of $a_E(p)$ for a elliptic curve E, could it predict the rank? Interestingly, it worked quite well, even for small lists of $a_E(p)$.

In a follow-up paper in 2022, He, Lee, Oliver and Alexey Pozdnyakov [HLOP24] further looked into why the technique was so accurate? To do this they did a principal component analysis on the data, which essentially reduces the large dimensional problem of the neural network, i.e. how long the list of $a_E(p)$ which are needed, and embeds this into $\mathbb{R}^2$. This is essentially done by computing some weights w_p and computing the sum $\sum_p w_p a_E(p)$. In doing so they uncovered a very interesting phenomenon appearing, murmurations!

2.2. **What is going on here?** The bias in the $a_E(p)$ is not too surprising. Actually, BSD tells us that

$$\lim_{X\to\infty} \underbrace{\frac{1}{\log X}\sum_{p\le X}\frac{a_E(p)\log p}{p}}_{\text{Mestre–Nagao sum}} = \frac{1}{2} - r_E,$$

showing that the average value of the $a_E(p)$ values get more negative as the rank increases. What is surprising though is that this murmuration correlation swaps sign, and then swaps signs again, and again!

2.2.1. *Root not rank.* Originally we just considered the incomplete L-function, but if we now consider the completed L-function (that also includes the bad prime factors), defined by

$$\Lambda(E,s) := (2\pi)^{-s}\Gamma(s)\prod_p \frac{1}{1-a_E(p)p^{-s}+\chi_0(p)p^{1-2s}} = \varepsilon N_E^{1-s}\Lambda(E,2-s),$$

where χ_0 is the trivial character modulo N_E and ε we called the *root number* of $\Lambda(E,s)$. BSD tells us that $\varepsilon = (-1)^{r_E}$.

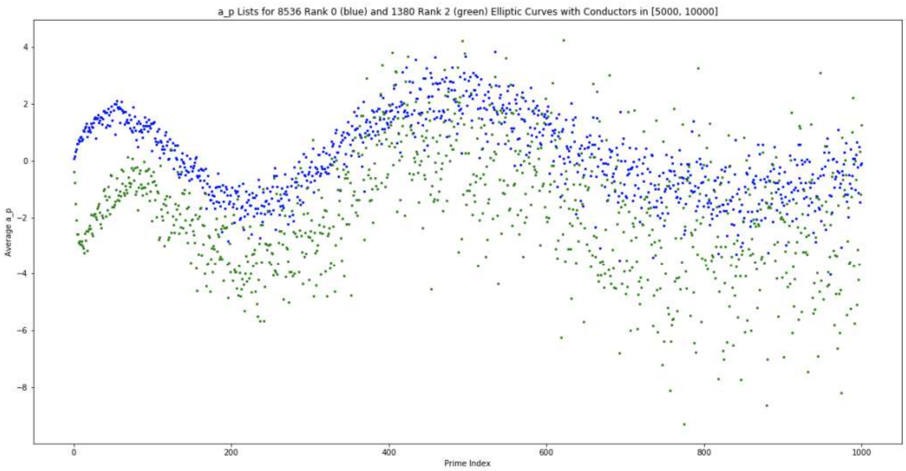

FIGURE 3. Plot of $f_r(n)$ for $N \in [5000, 10000]$ with $r = 0$ and $r = 2$. Original plot due to [HLOP24].

From Figure 3, we see the plots of rank 0 and 2 seem similar, albeit shifted by the negative bias that we see from BSD. Hence, we should view this phenomenon as a correlation between the $a_E(p)$ values and the root number of the elliptic curve!

2.2.2. *Scaling of prime p is important!* An observation, due to Jonathon Bober, is that the scale of p should be chosen relative to N_E. This means that if we choose families of elliptic curves with rather different conductors, but then choose p scaled appropriately, we should still see the phenomenon. This is illustrated in the pictures below in Figure 4 due to Andrew Sutherland. The range of conductors is different, but with the appropriate scaling of p relative to N_E, we still see the same pattern.

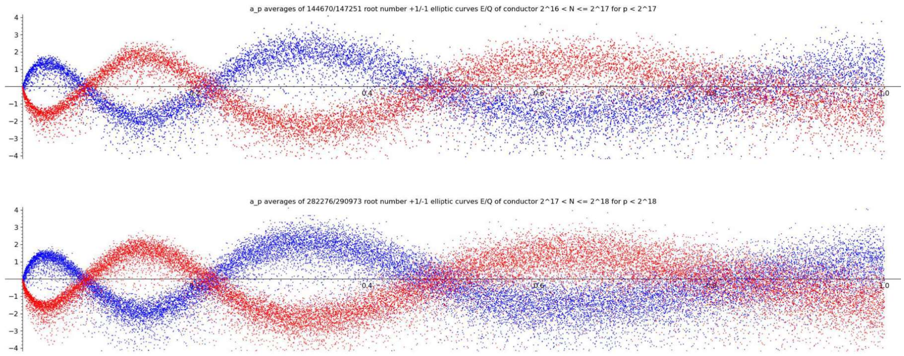

FIGURE 4. Murmuration plots for elliptic curves with different ranges of conductor, but with the primes p scaled relative to the conductor.

2.3. **Other families of objects.** The observations above seem to point to us that the elliptic curve itself is not the main object, but instead, it is the associated L-function. Hence we can ask the natural question, does this phenomenon occur for other Dirichlet coefficients a_p that are arithmetic and the associated root number of the L-function? The answer is yes! Below are some plots, due to Andrew Sutherland, that show the phenomenon for other various families of number theoretic objects.

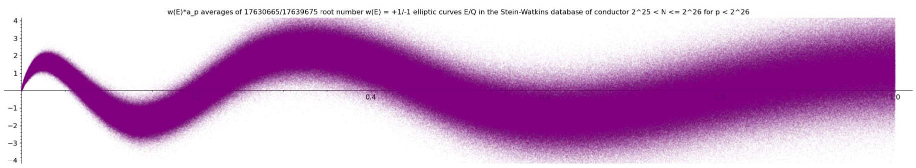

FIGURE 5. Elliptic curves from the Stein-Watkins database.

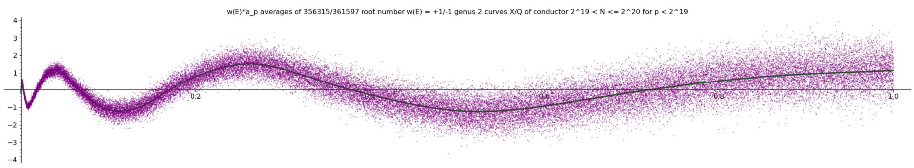

FIGURE 6. Genus 2 curves over $\mathbb{Q}$ with Sato-Tate group USp(4).

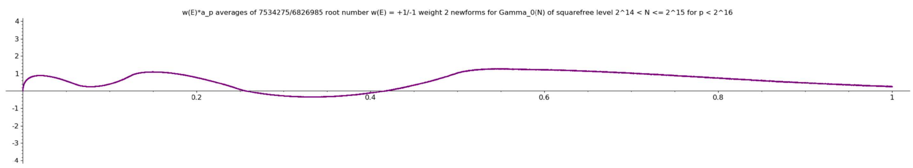

FIGURE 7. Weight 2 holomorphic newforms with squarefree level.

3. Proven results

The first breakthrough in proving this phenomenon came from Nina Zubrilina [Zub23], who proved the following result for holomorphic modular forms.

Theorem 3.1. *Fix $k \in 2\mathbb{Z}_{>0}$. Let X, Y and p be parameters $\to \infty$ with p prime; assume $Y = (1+o(1))X^{1-\delta_2}$ and $p \ll X^{1+\delta_1}$, $0 < 2\delta_1 < \delta_2 < 1$. Let $y = p/X$. Then*

$$\frac{\sum_{\substack{N\in[X,X+Y]\\ \text{square-free}}} \sum_{f\in H_k^{\text{new}}(N)} \varepsilon_f a_f(p)\sqrt{p}}{\sum_{\substack{N\in[X,X+Y]\\ \text{square-free}}} \sum_{f\in H_k^{\text{new}}(N)} 1} = M_k(y) + O_\varepsilon\left(X^{-\min\{\frac{\delta_2}{2},\frac{1+\delta_2}{9}\}+\delta_1+\varepsilon} + p^{-1}\right)$$

where

$$M_k(y) = \frac{12}{(k-1)\pi\prod_p(1-(p^2+p)^{-1})}\left\{\prod_p\left(1+\frac{p}{(p+1)^2(p-1)}\right)\sqrt{y}\right.$$
$$\left. +(-1)^{\frac{k}{2}-1}\prod_p\left(1-\frac{p}{(p^2-1)^2}\right)\sum_{1\le r\le 2\sqrt{y}} c(r)\sqrt{4y-r^2}U_{k-2}(r/(2\sqrt{y})) - \delta_{k=2}\pi y\right\}.$$

Here U_{k-2} is the Chebyshev polynomial $U_{k-2}(\cos(\theta)) = \frac{\sin((k-1)\theta)}{\sin(\theta)}$ and $c(r) = \prod_{p|r}\left(1+\frac{p^2}{(p^2-1)^2-p}\right)$.

Below is a plot of $M_k(y)$ compared to numerical data plotting the left side of Theorem 3.1.

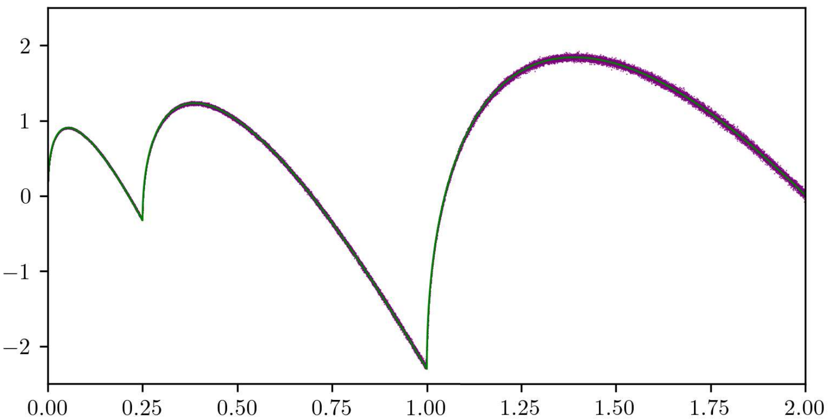

Figure 8. Comparison of Zubrilina's density function $M_2(y)$ function and numerical data. Computation and graph due to Andrew Sutherland.

This plot looks rather different to the one shown before, but this is due to the fact that the original plot averaged p/N in a dyadic interval. This means that it should tend to a convolution of Zubrilina's density function, which is indeed the case.

3.1. **Archimedean aspect.** Zubrilina's result is for a fixed weight and the level tending to infinity (non-archimedean aspect). A natural question to ask is, what happens when we fix the level and let the weight tend to infinity? This can be seeing as varying over the archimedean part of a family of modular forms.

In fact, at the workshop "Murmurations in Arithmetic" at ICERM in July 2023, Peter Sarnak asked the similar question, that is, is there an analogous murmuration phenomenon for Maass cusp forms of level 1, where we let the Laplace eigenvalues λ tend to infinity?

For the case of holomorphic modular forms, the archimedean aspect was proven by Andrew R. Booker, Jonathon Bober, Min Lee and David Lowry-Duda [BBLLD23]. We can see this by looking at the analytic conductor, defined by

$$\mathcal{N}(k) := \left(\frac{\exp\psi(k/2)}{2\pi}\right)^2 = \left(\frac{k-1}{4\pi}\right)^2 + O(1),$$

where ψ is the Digamma function, and letting it tend to infinity.

Theorem 3.2 ([BBLLD23])**.** *Assume GRH for the L-functions of Dirichlet characters and L-functions for modular forms. Fix $\varepsilon \in (0, \frac{1}{12})$, $\delta \in \{0,1\}$, and compact interval $E \subset \mathbb{R}_{>0}$ with $|E| > 0$. Let $K, H \in \mathbb{R}_{>0}$ with $K^{\frac{5}{6}+\varepsilon} < H < K^{1-\varepsilon}$ and set $N = \mathcal{N}(K)$. Then as $K \to \infty$, we have*

$$\frac{\sum_{\substack{p \text{ prime} \\ \frac{p}{N} \in E}} \log p \sum_{\substack{k \equiv 2\delta \bmod 4 \\ |k-K| \le H}} \sum_{f \in H_k(1)} \lambda_f(p)}{\sum_{\substack{p \text{ prime} \\ \frac{p}{N} \in E}} \log p \sum_{\substack{k \equiv 2\delta \bmod 4 \\ |k-K| \le H}} \sum_{f \in H_k(1)} 1} = \frac{(-1)^\delta}{\sqrt{N}} \left(\frac{\nu(E)}{|E|} + o_{E,\varepsilon}(1)\right),$$

where

$$\nu(E) = \frac{1}{\zeta(2)} \sideset{}{^*}\sum_{\substack{a,q \in \mathbb{Z}, \gcd(a,q)=1 \\ \left(\frac{a}{q}\right)^{-2} \in E}} \frac{\mu(q)^2}{\varphi(q)^2 \sigma(q)} \left(\frac{q}{a}\right)^3 = \frac{1}{2} \sum_{t=-\infty}^{\infty} \prod_{p \nmid t} \frac{p^2-p-1}{p^2-p} \cdot \int_E \cos\left(\frac{2\pi t}{\sqrt{y}}\right) dy,$$

and the $*$ means terms occurring at the end points of E are halved.

Figure 9 shows the plot of this function $\nu(E)$.

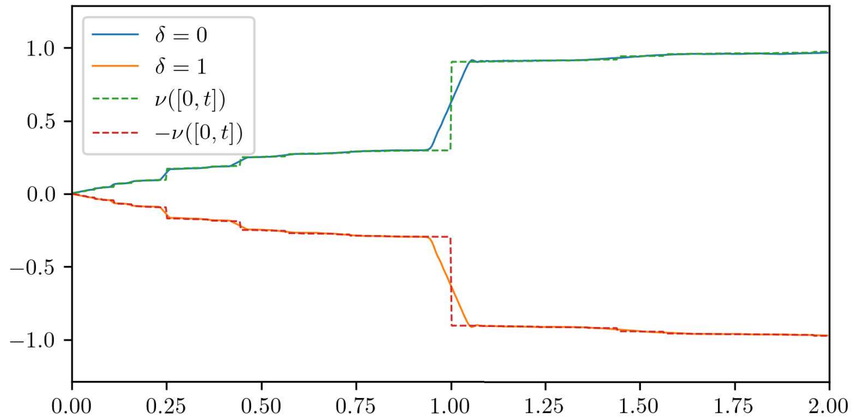

FIGURE 9. Cumulative plot of $(-1)^\delta \nu([0,t])$ and numerical data for $K = 3850, H = 100$ and $t \in [0,2]$ from Theorem 3.2.

3.2. **Maass murmurations.** Since the above archimedean result is written in terms of letting the analytic conductor tend to infinity, we can formulate Sarnak's question in this way as well for Maass forms. In a similar way to the holomorphic modular forms, we define the analytic conductor for Maass cusp forms of level 1 by

$$\mathcal{N}(R) := \frac{\exp\left(\psi\left(\frac{1/2+a+iR}{2}\right)+\psi\left(\frac{1/2+a-iR}{2}\right)\right)}{\pi^2} = \frac{R^2}{4\pi^2} + O(1).$$

In joint work with Andrew R. Booker, Min Lee, David Lowry-Duda and Nina Zubrilina, we were able to prove following result for Maass cusp forms, showing that we get the same result as for holomorphic modular forms.

Theorem 3.3 ([BLLD+24])**.** *Assume GRH for L-functions of Dirichlet characters and Maass forms. Let* $E \subset \mathbb{R}_{>0}$ *be a fixed compact interval with* $|E| > 0$*. Let* $R, H \in \mathbb{R}_{>0}$ *with* $R^{\frac{5}{6}+\varepsilon} < H < R^{1-\varepsilon}$ *for some* $\varepsilon > 0$ *and let* $N = \mathcal{N}(R)$*. As* $R \to \infty$*, we have*

$$\frac{\sum_{\substack{p \text{ prime} \\ \frac{p}{N}\in E}} \log p \sum_{|r_j - R| \le H} \varepsilon_j \lambda_j(p)}{\sum_{\substack{p \text{ prime} \\ \frac{p}{N}\in E}} \log p \sum_{|r_j - R| \le H} 1} = \frac{1}{\sqrt{N}}\left(\frac{\nu(E)}{|E|} + o_{E,\varepsilon}(1)\right).$$

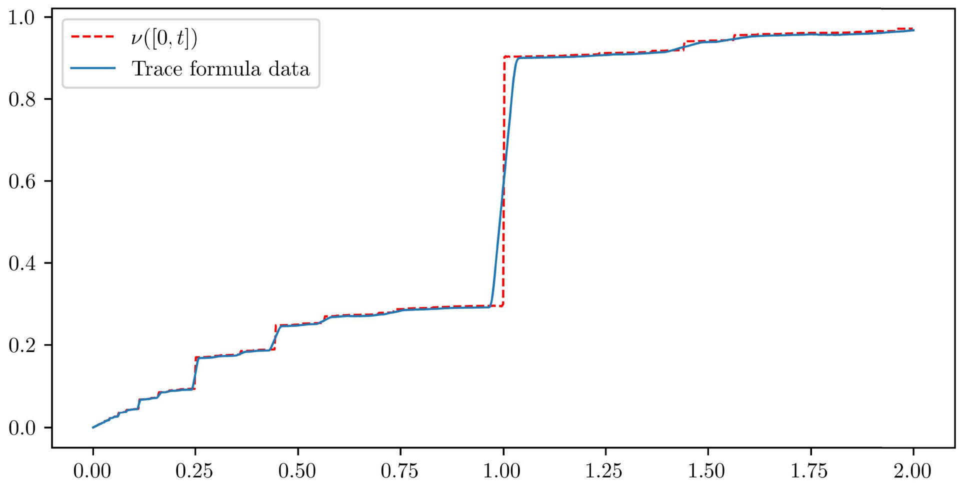

FIGURE 10. Plot of $\nu([0,t])$ scaled by $t\sqrt{N}$ and numerical data with $R = 6900, H = 100$ and $t \in [0,2]$ from Theorem 3.3. Note that this graph is identical to the plot for $\delta = 0$ from Figure 9.

3.3. **Rough steps of the proof.** The steps of the proof follow the ideas of [BBLLD23][Sec. 2.1] with the main differences coming from the fact that we use the Selberg trace formula with Hecke operators which will include class numbers with discriminants of the form $t^2 + 4n$. This is remedied in Step 4 below by replacing these class numbers with the special value of Dirichlet L-function and averaging over the L-functions. The rough steps of the proof of Theorem 3.3 are as follows:

Step 1: Choose a smooth test function that approximates the interval function and whose Fourier transform is compactly supported. Here $W_h(x) = W(x/h)/h$ and W is a function due to Ingham

[Ing34]:

$$\mathbf{1}_{[-R-H,-R+H]}(r) + \mathbf{1}_{[R-H,R+H]}(r) = (\mathbf{1}_{[-R-H,-R+H]} * W_h)(r) + (\mathbf{1}_{[R-H,R+H]} * W_h)(r) + A(R).$$

Plugging this into the spectral side of the trace formula will give an error due to $A(R)$. To get a good bound on this we use GRH for Maass cusp forms.

Step 2: Plug this test function into an explicit version of the Selberg trace formula, due to Strömbergsson [Str16], and bound terms on the geometric side to get that the numerator of the ratio we want to compute is:

$$\Sigma := \sum_{\substack{p \text{ prime} \\ p/N \in E}} \log p \sum_{\substack{t \in \mathbb{Z} \\ D = t^2+4p \\ t \neq \pm(p-1)}} L(1, \psi_D) \widehat{W}\left(\log\left(\frac{(|t|+\sqrt{D})^2}{4p}\right)\right) + O\left(\frac{R^{3+\varepsilon}}{h}\right).$$

Step 3: Approximate the term inside $\widehat{W}$ by its first order approximation

$$\Sigma = \sum_{t \in \mathbb{Z}} \sum_{\substack{p\,\text{prime} \\ 4\pi^2 p/R^2 \in E}} 2 \log p \sqrt{p} L(1, \psi_{t^2+4p}) \frac{\cos\left(R\frac{t}{\sqrt{p}}\right) \sin\left(H\frac{t}{\sqrt{p}}\right)}{\pi t} \widehat{W}\left(h \frac{t}{\sqrt{p}}\right) + O\left(\frac{R^{4+\varepsilon}}{h^3} + \frac{R^{3+\varepsilon}}{h}\right).$$

Step 4: Replace $L(1, \psi_{t^2+4p})$ by an averaged version. Here we use GRH for Dirichlet L-functions to get good bounds:

$$\Sigma = 4 \int_{\frac{R-H}{2\pi}}^{\frac{R+H}{2\pi}} u^2 \sum_{t \in \mathbb{Z}} L(1, \overline{\psi}_t) \int_{\lambda_u \cdot E^{-\frac{1}{2}}} \cos(2\pi \alpha t) \, \widehat{W}(t/\alpha_u) \, \frac{d\alpha}{\alpha^3} \, du + O\left(\frac{R^{4+\varepsilon}}{h^3} + \frac{R^{3+\varepsilon}}{h} + \frac{R^{3+\varepsilon}}{h^{1/5}}\right)$$

Step 5: Apply the circle method to the integral over α. The denominator is dealt with by using a refined Weyl law for the count of Maass cusp forms.

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