# TELHCIRID'S THEOREM ON ARITHMETIC PROGRESSIONS

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## 1. Introduction

We first recall the next famous theorem of Dirichlet, which is known as Dirichlet's theorem on arithmetic progressions (in this note, the letter $p$, with or without subscript, always denote prime numbers):

**Theorem A** (Dirichlet's theorem on A.P.). *For any $a, q \in \mathbb{Z}$ with $q \geq 1$ and $(a, q) = 1$,*

$$\#\{p \mid p \equiv a \pmod{q}\} = \infty.$$

In this note, we shall consider the *reverse* of Dirichlet's theorem on arithmetic progressions, which we may call *Telhcirid's theorem on arithmetic progressions*. What we do is not literally reading the whole statement of Theorem A backwards but read the digital representation of primes $p$ backwards.

We take and fix a base $g \in \mathbb{Z}_{\geq 2}$ and express $n \in \mathbb{Z}_{\geq 0}$ by the *base $g$ representation*

$$n = \sum_{i \geq 0} \varepsilon_i(n) g^i \quad \text{with} \quad \varepsilon_i(n) \in \{0, \ldots, g-1\}. \tag{1}$$

We then define the *length* $\mathrm{len}(n)$ of $n \in \mathbb{Z}_{\geq 0}$ by

$$\mathrm{len}(n) \coloneqq \min\{\ell \in \mathbb{Z}_{\geq 0} \mid \varepsilon_i(n) = 0 \text{ for all } i \geq \ell\}.$$

We also use a handy way to write the base $g$ representation (1):

$$(\varepsilon_{\mathrm{len}(n)-1}(n) \cdots \varepsilon_1(n)\varepsilon_0(n))_{(g)} \coloneqq \sum_{0 \leq i < \mathrm{len}(n)} \varepsilon_i(n) g^i. \tag{2}$$

To read a digital representation backwards, we introduce the *digital reverse* $\mathrm{rev}(n)$ of $n \in \mathbb{Z}_{\geq 0}$ defined by the reverse of the base $g$ representation (2), i.e.

$$\mathrm{rev}(n) \coloneqq (\varepsilon_0(n)\varepsilon_1(n) \cdots \varepsilon_{\mathrm{len}(n)-1}(n))_{(g)} = \sum_{0 \leq i < \mathrm{len}(n)} \varepsilon_i(n) g^{\mathrm{len}(n)-i-1}. \tag{3}$$

Now, the reverse of Dirichlet's theorem on arithmetic progressions is the following:

**Theorem 1** (Telhcirid's theorem on A.P.). *There is $G \in \mathbb{Z}_{\geq 2}$ such that for any*

$$g, a, q \in \mathbb{Z} \quad \text{with} \quad g \geq G, \quad q \geq 1$$

*with*

$$(a, q, g^2 - 1) = 1 \quad \text{and} \quad g \nmid (a, q), \tag{4}$$

*we have*

$$\#\{p \mid \mathrm{rev}(p) \equiv a \pmod{q}\} = \infty.$$

The conditions (4) are indeed necessary conditions as we can see easily:

- **The condition** $(a, q, g^2 - 1) = 1$**.** Since

$$g^2 \equiv 1 \pmod{(q, g^2 - 1)},$$

when $\mathrm{rev}(p) \equiv a \pmod{q}$, we have

$$p \equiv g^{\mathrm{len}(p)-1} \mathrm{rev}(p) \equiv g^{\mathrm{len}(p)-1} a \pmod{(q, g^2 - 1)}.$$

Therefore, in order to have the infinitude, we should have

$$1 = (g^{\mathrm{len}(p)-1} a, q, g^2 - 1) = (a, q, g^2 - 1),$$

where we used $(g, g^2 - 1) = 1$.

- **The condition** $g \nmid (a, q)$**.** By the definition of $\mathrm{rev}(p)$ and $p \geq 2$, we have

$$\varepsilon_0(\mathrm{rev}(p)) = \varepsilon_{\mathrm{len}(p)-1}(p),$$

which is non-zero by the definition of $\mathrm{len}(p)$. This implies

$$\mathrm{rev}(p) \not\equiv 0 \pmod{g}.$$

Thus, if we also have $\mathrm{rev}(p) \equiv a \pmod{q}$, then we should have $g \nmid (a, q)$.

In our preprint [1], we proved Theorem 1 with $G = 31699$. Recently, we succeeded in proving Theorem 1 for all bases, i.e. with $G = 2$. In this note, we shall sketch the proof of Theorem 1 with $G = 2$, the details of which will be made public in another forthcoming preprint.

A motivation for the digital reverse may be given by the following conjectures:

**Conjecture 1.** For any base $g \in \mathbb{Z}_{\geq 2}$, we have

$$\#\{p \mid \mathrm{rev}(p) = p\} = \infty.$$

(A prime $p$ satisfying $\mathrm{rev}(p) = p$ is called a *palindromic prime*.)

**Conjecture 2.** For any base $g \in \mathbb{Z}_{\geq 2}$, we have

$$\#\{p \mid \mathrm{rev}(p) : \text{prime}\} = \infty.$$

(A prime $p$ for which $\mathrm{rev}(p)$ is also a prime is called a *reversible prime*.)

Even though the above conjectures seem too difficult as Conjecture 1 is comparable to the infinitude of primes of the form $n^2 + 1$ and Conjecture 2 is comparable to the infinitude of twin primes, there are some partial results for these conjectures. Among those, we state the following two results (let $\Omega(n)$ be the number of prime factors of a positive integer $n$ counted with multiplicity):

**Theorem B** (Tuxanidy–Panario [9])**.** *For any base* $g \in \mathbb{Z}_{\geq 2}$*, we have*

$$\#\{n \in \mathbb{N} \mid \mathrm{rev}(n) = n \text{ and } \Omega(n) \leq 6\} = \infty,$$

*i.e. there are infinitely many palindromic* 6*-almost primes.*

**Theorem C** (Dartyge–Martin–Rivat–Shparlinski–Swaenepoel [2])**.** *For* $g = 2$*, we have*

$$\#\{n \in \mathbb{N} \mid \max(\Omega(n), \Omega(\mathrm{rev}(n))) \leq 8\} = \infty$$

*so there are infinitely many integers* $n$ *for which both of* $n, \mathrm{rev}(n)$ *are* 8*-almost prime.*

Note that Theorem C itself is weaker than Theorem B. However, in [2], Dartyge–Martin–Rivat–Shparlinski proved the "expected" lower bound

$$\#\{n \le x \mid \max(\Omega(n), \Omega(\mathrm{rev}(n))) \le 8\} \gg x(\log x)^{-2},$$

which cannot be deduced from the lower bound of Tuxanidy–Panario

$$\#\{n \le x \mid \mathrm{rev}(n) = n,\ \Omega(n) \le 6\} \gg x^{\frac{1}{2}}(\log x)^{-1}.$$

Also, it seems not so difficult to prove Theorem C for all $g \ge 2$ by the method of [2].

As the forms of the statements are telling, the proofs of Theorem B and Theorem C use sieve methods. From this point of view, by recalling Rényi's result [8, p. 58, Theorem 2] on the twin prime conjecture: there exists $R \in \mathbb{N}$ such that

$$\#\{p \mid \Omega(p+2) \le R\} = \infty,$$

it may be natural to try to prove that there is $R \in \mathbb{N}$ such that

$$\#\{p \mid \Omega(\mathrm{rev}(p)) \le R\} = \infty.$$

To obtain such a result, we need to get an asymptotic formula for the distribution of primes whose digital reverse satisfies a given congruence condition. Our result is still insufficient for such a purpose since the size of the modulus is too restricted, we have the next quantitative result, which we may call the *Zsiflaw–Legeis theorem*, the reversed version of the Siegel–Walfisz theorem. To avoid the irregular behavior of the counting function, for $N \in \mathbb{Z}_{\ge 1}$, we introduce the set $\mathscr{G}_N$ of integers of length $N$, i.e.

$$\mathscr{G}_N := [g^{N-1}, g^N) \cap \mathbb{Z}$$

and count primes in $\mathscr{G}_N$. We thus also let

$$\overleftarrow{\pi}_N(a,q) := \#\{p \in \mathscr{G}_N \mid \mathrm{rev}(p) \equiv a \pmod q\}.$$

**Theorem 2** (The Zsiflaw–Legeis theorem). *There is $G \in \mathbb{Z}_{\ge 2}$ such that for any*

$$g, a, q \in \mathbb{Z} \quad \text{with} \quad g \ge G, \quad q \ge 1, \quad (a, q, g^2-1) = 1, \quad g \nmid (a,q),$$

*and any $N \in \mathbb{Z}_{\ge 2}$, we have*

$$\overleftarrow{\pi}_N(a,q) = \frac{\rho_g(a,q)}{q}\frac{g^N}{\log g^N}\left(1 + O\left(\frac{1}{N}\right)\right) + O(g^N \exp(-c\sqrt{N}))$$

*provided*

$$q \le \exp(c\sqrt{N}), \tag{5}$$

*where $c \in (0,1)$ is some constant, the function $\rho_g(a,q)$ is given by*

$$\rho_g(a,q) := \begin{cases} \left(1 - \mathbb{1}_{(q,g)\mid a}\dfrac{(q,g)}{g}\right)\dfrac{(q,g^2-1)}{\varphi((q,g^2-1))} & \text{if } (a,q,g^2-1) = 1 \text{ and } g \nmid (a,q), \\[2ex] 0 & \text{otherwise} \end{cases}$$

*and $c$ and the implicit constant depend only on $g$ and are effectively computable.*

Indeed, Telhcirid's theorem on arithmetic progressions (Theorem 1) is a corollary of the Zsiflaw–Legeis theorem (Theorem 2). Note that the value of $G$ is the same as in Theorem 1, so $G = 31699$ is obtained in [1], and we sketch the proof of its improvement to $G = 2$ in this note.

Let us now compare the Zsiflaw–Legeis theorem with the classical Siegel–Walfisz theorem. For $a, q \in \mathbb{Z}$ with $q \geq 1$ and $x \geq 1$, let us write

$$\pi(x, a, q) \coloneqq \#\{p \leq x \mid p \equiv a \pmod{q}\}.$$

**Theorem D.** *For $a, q \in \mathbb{Z}$ and $x, A \geq 2$ with $q \geq 1$ and $(a, q) = 1$, we have*

$$\pi(x, a, q) = \frac{1}{\varphi(q)} \int_2^x \frac{du}{\log u} + O(x \exp(-c\sqrt{\log x}))$$

*with some constant $c \in (0, 1)$ provided*

$$q \leq (\log x)^A,$$

*where $c$ and the implicit constant depend only on $A$ and not effectively computable.*

One may find that the admissible level (5) of the Zsiflaw–Legeis theorem is larger than that of the usual Siegel–Walfisz theorem Theorem D (note that $N$ of (5) corresponds to $\log x$ of Theorem D), which is of the size comparable to $N^A$ with a fixed $A \geq 0$. Also, the implicit constants are effectively computable (once the base $g$ is given) in the Zsiflaw–Legeis theorem while not in the Siegel–Walfisz theorem. These phenomena probably come from the principle that the digital property of integers and the multiplicative structure of integers are orthogonal.

In our first preprint [1], we used the method used by Maynard [7]. Our new proof with arbitrary base $g \geq 2$ instead follows the method of Mauduit and Rivat [5], used for solving Gelfond's problem as in the next theorem (we state an explicit variant due to Drmota, Mauduit and Rivat [3, Proposition 2.1]).

**Theorem E.** *For $g, a, q \in \mathbb{Z}$ and $x \geq 1$ with $g \geq 2$ and $q \geq 1$, we have*

$$\#\{p \leq x \mid s_g(p) \equiv a \pmod{q}\} = \frac{(q, g-1)}{q} \pi(x, a, (q, g-1)) + O(x^{1-\frac{c}{q}})$$

*with a certain constant $c \in (0, 1)$, where $s_g(n)$ is the sum-of-digit function defined by*

$$s_g(n) \coloneqq \sum_{i \geq 0} \varepsilon_i(n)$$

*and $c$ and the implicit constant depend only on $g$.*

In [5], the primality and the digital property are considered simultaneously while they are independently treated in the method used in [7], which causes the superiority of the method of [5]. The method of [5] was later generalized by Martin, Mauduit and Rivat in [4], and we follow this generalization. A similar line of approach is, together with several other novelties, taken by Maynard for primes with missing digits in [6], which improved the admissible size of base for asymptotic formulas from $g \geq 2000000$ given in [7] to $g \geq 12$. (See the second last paragraph of Section 1 of [6].)

## 2. Setup of the proof

We do not consider the digital reverse $\operatorname{rev}(n)$ defined by (3) directly, but we consider the *relative digital reverse* $\operatorname{rev}_L(n)$ of order $L \in \mathbb{N}$ defined by

$$\operatorname{rev}_L(n) \coloneqq \sum_{0 \leq i < L} \varepsilon_i(n) g^{L-i-1}.$$

Note that we then have $\operatorname{rev}(n) = \operatorname{rev}_L(n)$ if $\operatorname{len}(n) = L$, and so there is no essential difference to consider $\operatorname{rev}_L(n)$ instead of $\operatorname{rev}(n)$.

It suffices to study the following counting function:

$$\overleftarrow{\psi}_L(x,a,q) \coloneqq \sum_{\substack{n\le x\\ \operatorname{rev}_L(n)\equiv a \pmod q}} \Lambda(n).$$

By the orthogonality, with writing $e(x) \coloneqq \exp(2\pi i x)$, we can expand $\overleftarrow{\psi}_L(x,a,q)$ as

$$\overleftarrow{\psi}_L(x,a,q) = \frac{1}{q}\sum_{0\le h<q} e\left(-\frac{ha}{q}\right)\sum_{n\le x}\Lambda(n)e\left(\frac{h\operatorname{rev}_L(n)}{q}\right).$$

By recalling the arguments for the necessary condition of Telhcirid's theorem, we find that the main contributions are coming from those $h$ with $q \mid g^L(g^2-1)h$. For these $h$ on the "major arc", we use the orthogonality backwards to obtain

$$\overleftarrow{\psi}_L(x,a,q) = \frac{(q, g^L(g^2-1))}{q}\overleftarrow{\psi}_L(x,a,(q,g^L(g^2-1))) + O\Bigg(\max_{\substack{0\le h<q\\ q\nmid g^L(g^2-1)h}}\left|\overleftarrow{S}\left(\frac{h}{q}\right)\right|\Bigg)$$

where

$$\overleftarrow{S}(\alpha) \coloneqq \sum_{n\le x}\Lambda(n)e(\alpha\operatorname{rev}_L(n)). \tag{6}$$

Our task is then to bound the exponential sum $\overleftarrow{S}(\alpha)$ non-trivially. The main term $\overleftarrow{\psi}_L(x,a,(q,g^L(g^2-1)))$ can be easily handled by some explicit form of the prime number theorem in arithmetic progressions.

Martin, Mauduit and Rivat estimated exponential sums of the type (6) not only for the sum-of-digit function $s_g(n)$ but for a general class of functions:

**Definition 1** (Digital function). A function $f\colon \mathbb{Z}_{\ge 0}\to\mathbb{R}$ is *digital* if there is a map

$$\alpha\colon \{0,\dots,g-1\}\to\mathbb{R}$$

such that

$$f(n) = \sum_{0\le i<\operatorname{len}(n)} \alpha(\varepsilon_i(n)).$$

Our relative digital reverse is unfortunately not a digital function since any digital function $f$ obeys a bound $f(n) \ll \log n$ while $\operatorname{rev}_L(n)$ does not. Therefore, we generalize this notion to the next "weakly digital function":

**Definition 2** (Weakly digital function). Let

$$\mathscr{A} \coloneqq \{(\alpha_i)_{i=0}^{\infty} \mid \alpha_i\colon \{0,\dots,g-1\}\to\mathbb{R} \text{ for all } i\in\mathbb{Z}_{\ge 0}\}.$$

For $L\in\mathbb{N}$ and $j\in\mathbb{Z}_{\ge 0}$, we define $f_L^{[j]}\colon \mathbb{Z}_{\ge 0}\to\mathbb{R}$ by

$$f_L^{[j]}(n) \coloneqq \sum_{0\le i<L}\alpha_i^{[j]}(\varepsilon_i(n)) \quad \text{with} \quad \alpha_i^{[j]} \coloneqq \alpha_{i+j}.$$

We call such a family of functions $f_L^{[j]}$ a *weakly digital function*.

Note that $\boldsymbol{\alpha}_L = (\alpha_{L,i})_{i=0}^{\infty}\in\mathscr{A}$ given by $\alpha_{L,i}(n) \coloneqq \alpha n g^{L-i-1}$ produces $f_L^{[0]}(n) = \alpha\operatorname{rev}_L(n)$ and so the notion of weakly digital function can capture $\operatorname{rev}_L$.

Since we are considering such general functions, we can apply our argument to a variety of the digital properties of primes. For example, we can consider the primes for which the base $g$ representation read in another base satisfies a given congruence condition, i.e. primes $p$ satisfying

$$\sum_{i\geq 0}\varepsilon_i(p)h^i \equiv a \pmod q$$

with some base $h \neq g$, where the digits $\varepsilon_i(p)$ is defined in terms of the base $g$. (If $h < g$, then the left-hand side is not a genuine base $h$ representation since some digit may be $\geq h$. However, we do not consider this subtlety here.)

Our task is thus obtaining a non-trivial bound for the exponential sum

$$S_L^{[j]} \coloneqq \sum_{n\leq x}\Lambda(n)e(f_L^{[j]}(n)). \tag{7}$$

By the standard application of combinatorial decompositions of $\Lambda(n)$, where Vaughan's identity is enough for our purpose here, the estimate of (7) is essentially reduced to the estimates of the bilinear sums

$$S_{\mathrm{I}} \coloneqq \sum_{\substack{mn\leq x\\ m\leq M}} a(m)e(f_L^{[j]}(mn)) \quad\text{and}\quad S_{\mathrm{II}} \coloneqq \sum_{\substack{mn\leq x\\ M<m\leq 2M\\ N<n\leq 2N}} a(m)b(n)e(f_L^{[j]}(mn)) \tag{8}$$

usually called the Type I and Type II sums, respectively. The coefficients $a(m), b(n)$ are arbitrary complex-valued coefficients with the normalization $|a(m)|, |b(n)| \leq 1$.

## 3. Product formula

We bound the Type I and Type II sums by using the orthogonality or the discrete Fourier analysis again. To this end, for $\lambda \in \mathbb{Z}_{\geq 0}$, we introduce the discrete Fourier transform of $e(f_\lambda^{[j]}(n))$ given by

$$F_\lambda^{[j]}(\beta) \coloneqq \frac{1}{g^\lambda}\sum_{0\leq n<g^\lambda} e(f_L^{[j]}(n) - \beta n). \tag{9}$$

By considering the base $g$ representation of $n$ in (9), we obtain

$$|F_\lambda^{[j]}(\beta)| = \frac{1}{g^\lambda}\prod_{0\leq i<\lambda}\varphi_i^{[j]}(\beta g^i), \tag{10}$$

where

$$\varphi_i^{[j]}(\beta) \coloneqq \left|\sum_{0\leq n<g} e(\alpha_i^{[j]}(n) - \beta n)\right|. \tag{11}$$

The *product formula* (10) is the key tool for studying the digital properties of integers in most existing works. The product formula enables us to accumulate small cancellations caused by the exponential sum over digits $\varphi_i^{[j]}(\beta)$ to obtain a substantial cancellation for the original $F_\lambda^{[j]}(\beta)$. Also, the product formula enables us to decompose the exponential sum $F_\lambda^{[j]}(\beta)$ "smoothly" to mix various bound efficiently. In the next sections, we shall use the product formula to obtain moment bounds for $F_\lambda^{[j]}(\beta)$.

## 4. The $L^\infty$ bound

We first prove the $L^\infty$ bound for $F_\lambda^{[j]}(\beta)$. We extract the cancellation for $L^\infty$ bound from the $f_L^{[j]}(n)$ side, and so it is highly correlated with the arithmetic information, e.g. $q \nmid g^L(g^2-1)h$ in our setting in Section 2.

To obtain a cancellation in $\varphi_i^{[j]}(\beta)$, we just consider the contribution of two terms, say the $m$-th and $n$-th term of (11) with distinct $m, n$. We then use the bound

$$|e(\alpha)+e(\beta)| = |1+e(\alpha-\beta)| = 2|\cos\pi(\alpha-\beta)| \le 2(1-4\|\alpha-\beta\|^2),$$

where $\|x\| \coloneqq \min_{n\in\mathbb{Z}} |x-n|$. By using the bound $1-x \le e^{-x}$, this gives

$$\varphi_i^{[j]}(\beta) \le g\exp(-c\|\alpha_i^{[j]}(m) - \alpha_i^{[j]}(n) - \beta(m-n)\|^2) \tag{12}$$

where $c = c(g)$ is a constant which may take different values at each occurrence. Since we now want to concentrate on the effect of $\boldsymbol{\alpha}$, we would like to remove the effect of $\beta$ from (12). We thus focus on the consecutive factors of (10) and use the fact

$$\begin{aligned}&g(\alpha_i^{[j]}(m) - \alpha_i^{[j]}(n) - \beta g^i(m-n)) - (\alpha_{i+1}^{[j]}(m) - \alpha_{i+1}^{[j]}(n) - \beta g^{i+1}(m-n))\\ &= (g\alpha_i^{[j]}(m) - \alpha_{i+1}^{[j]}(m)) - (g\alpha_i^{[j]}(n) - \alpha_{i+1}^{[j]}(n))\end{aligned}$$

and the triangle inequality of $\|x\|$ to get

$$(\varphi_i^{[j]}(\beta g^i)\varphi_{i+1}^{[j]}(\beta g^{i+1}))^{\frac{1}{2}} \le g^{1-\gamma_i^{[j]}(\boldsymbol{\alpha})} \tag{13}$$

with

$$\gamma_i^{[j]}(\boldsymbol{\alpha}) \coloneqq c \sum_{0\le m<n<g} \|(g\alpha_i^{[j]}(m) - \alpha_{i+1}^{[j]}(m)) - (g\alpha_i^{[j]}(n) - \alpha_{i+1}^{[j]}(n))\|^2.$$

On multiplying (13) over $i$ and using the product formula, we can get:

**Lemma 1** ($L^\infty$-bound)**.** *For $g \in \mathbb{Z}_{\ge 2}$, $\boldsymbol{\alpha} \in \mathscr{A}$, $\lambda, j \in \mathbb{Z}_{\ge 0}$ and $\beta \in \mathbb{R}$, we have*

$$|F_\lambda^{[j]}(\beta)| \ll g^{-\sigma_\lambda^{[j]}(\boldsymbol{\alpha})} \quad \text{with} \quad \sigma_\lambda^{[j]}(\boldsymbol{\alpha}) \coloneqq \sum_{0\le i<\lambda} \gamma_i^{[j]}(\boldsymbol{\alpha}),$$

*where the implicit constant depends only on $g$.*

For concrete problems, we need some lower bound for $\sigma_\lambda^{[j]}(\boldsymbol{\alpha})$. In the case of digital reverse, the argument used in, e.g. [1, 2, 9] gives the following bound:

**Lemma 2.** *Consider $g, a, q \in \mathbb{Z}$, $L \in \mathbb{Z}_{\ge 0}$ and $h, q \in \mathbb{Z}$ with $g \ge 2$, $q \ge 1$ and $q \nmid g^L(g^2-1)h$. For $\lambda \in \{0, \ldots, L\}$, we have*

$$\sigma_\lambda^{[0]}(\boldsymbol{\alpha}_L) \gg \frac{\lambda}{\log q} + O(1),$$

*where the implicit constants depend only on $g$.*

## 5. The $L^1$ bound

We also need the $L^1$ bound for $F_\lambda^{[j]}(\beta)$. We take the average over the $\beta$ side. In order to use the $L^1$ bound for the Type II estimate, we need to consider the discrete $L^1$ moment with a congruence condition given by

$$\sum_{\substack{0\le h<g^\lambda\\ h\equiv a \pmod q}} \left|F_\lambda^{[j]}\left(\frac{h+\beta}{g^\lambda}\right)\right| \quad \text{with} \quad \beta \in \mathbb{R}. \tag{14}$$

The basic idea goes as follows. We ignore the congruence condition $h \equiv a \pmod{q}$ for simplicity. We then extract the first factor of the product formula (10) to get

$$\left| F_\lambda^{[j]}\left(\frac{h+\beta}{g^\lambda}\right) \right| = \left| F_{\lambda-1}^{[j+1]}\left(\frac{h+\beta}{g^{\lambda-1}}\right) \right| \times \frac{1}{g}\varphi_j^{[0]}\left(\frac{h+\beta}{g^\lambda}\right). \tag{15}$$

Since the first factor on the right-hand side is $g^{\lambda-1}$ periodic with respect to $h$, we can rewrite the original summation variable $h$ as $h \rightsquigarrow h + rg^{\lambda-1}$ to get

$$\sum_{0\le h<g^\lambda} \left| F_\lambda^{[j]}\left(\frac{h+\beta}{g^\lambda}\right) \right| = \sum_{0\le h<g^{\lambda-1}} \left| F_{\lambda-1}^{[j+1]}\left(\frac{h+\beta}{g^{\lambda-1}}\right) \right| \times \frac{1}{g}\sum_{0\le r<g} \varphi_j^{[0]}\left(\frac{r+\beta_0}{g}\right), \tag{16}$$

where we write $\beta_0 \coloneqq (u+\beta)g^{-(\lambda-1)}$. We then prepare a non-trivial estimate for

$$\frac{1}{g}\sum_{0\le r<g} \varphi_j^{[0]}\left(\frac{r+t}{g}\right) \tag{17}$$

with $t \in \mathbb{R}$ and using the recursion formula (16) to accumulate the cancellation to obtain a non-trivial bound for the original $L^1$ moment (14).

When we apply the $L^1$ moment to the Type II sum estimate, we indeed use the orthogonality as if we use the circle method for a binary problem for both of the summation variables of the Type II sum. Thus, we need an $L^1$ bound that is better than the square root cancellation. However, the above idea does not provide a bound better than the square root cancellation. Thus, as Martin–Mauduit–Rivat did in [4], we need to extract the first two factors instead of one factor as in (15). This requires us to bound, instead of the single sum (17), the double sum

$$\Psi_i(t,R,S) \coloneqq \frac{1}{g^2}\sum_{0\le r<R} \varphi_{i+1}^{[0]}\left(\frac{g(r+t)}{RS}\right) \sum_{0\le s<S} \varphi_i^{[0]}\left(\frac{r+t}{RS} + \frac{s}{S}\right),$$

where $R, S \in \mathbb{N}$ satisfies $R, S \mid g$ and $(R, g/S) = 1$. This double sum $\Psi_i(t,R,S)$ differs from $\Psi(t,R,S)$ used in [4] for some technical reason, even if $\alpha_i$ is independent of $i$.

After all, by the above line of argument, we can prove the following:

**Lemma 3.** *For $\lambda, \delta, j \in \mathbb{Z}_{\ge 0}$, $g, k \in \mathbb{N}$, $a \in \mathbb{Z}$ and $\beta \in \mathbb{R}$ with*

$$g \ge 2, \quad \lambda \ge 0, \quad 0 \le \delta \le \lambda, \quad j \ge 0, \quad k \ge 1, \quad kg^\delta \mid g^\lambda, \quad g \nmid k,$$

*we have*

$$\sum_{\substack{0\le h<g^\lambda \\ h\equiv a \pmod{kg^\delta}}} \left| F_\lambda^{[j]}\left(\frac{h+\beta}{g^\lambda}\right) \right| \le g\left(\frac{g^\lambda}{kg^\delta}\right)^{\eta_g} \left| F_\delta^{[j+\lambda-\delta]}\left(\frac{a+\beta}{g^\delta}\right) \right|.$$

*for an exponent $\eta_g$ depending on $g$ and satisfying $\eta_g \in (0, \frac{1}{2})$.*

## 6. Estimating the exponential sum

By Lemma 1 and Lemma 3, we can bound the sums (8). For the Type I sum $S_{\mathrm{I}}$, we rewrite $mn$ to a single variable $m$ with a congruence condition $m \equiv 0 \pmod{n}$ and use the orthogonality. Then, by mixing the $L^\infty$ bound and the $L^1$ bound together with the Gallagher–Sobolev inequality, we obtain a sufficient bound. For the Type II sum $S_{\mathrm{II}}$, after the usual chain of the Cauchy–Schwarz inequality and the van der Corput differencing, we use the truncation trick of Mauduit–Rivat [5, Lemma 5]. Finally, applying the discrete circle method to both summation variables, we can use Lemma 1 and Lemma 3 to get a satisfactory bound.

Even though we cannot give the details because of the limitation of pages, the above argument leads to the following bound, which is sufficient for proving Theorem 2:

**Theorem 3.** *For $g \in \mathbb{Z}_{\geq 2}$, $\boldsymbol{\alpha} \in \mathscr{A}$, $L \in \mathbb{N}$ and $1 \leq x \leq g^L$, we have*

$$S := \sum_{n \leq x} \Lambda(n) e(f_L^{[0]}(n)) \ll x g^{-\kappa} (\log x)^4 \quad \textit{with} \quad \kappa := \frac{1}{10} \sigma_\xi^{[0]}(\boldsymbol{\alpha}) \textit{ and } \xi := \left[\frac{1}{4} \frac{\log x}{\log g}\right],$$

*where the implicit constant depends only on $g$.*

## References


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