Eichler–Selberg Relations for Singular Moduli

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Abstract

The Eichler–Selberg trace formula expresses the trace of Hecke operators on spaces of cusp forms as weighted sums of Hurwitz–Kronecker class numbers. We extend this formula to a natural class of relations for traces of singular moduli. This work is a joint project with Prof. Ken Ono and Prof. Toshiki Matsusaka [3].

1. Singular Moduli

Let $\mathbb{H}$ be the upper-half plane and $q := e^{2\pi i\tau}$ for $\tau = u+iv \in \mathbb{H}$. For $z \in \mathbb{C}$, we put $\mathbf{e}(z) := e^{2\pi i z}$. The elliptic modular j-function is defined by

$$j(\tau) := \frac{E_4(\tau)^3}{\Delta(\tau)} = \frac{1}{q} + 744 + 196884q + 21493760q^2 + 864299970q^3 + \cdots,$$

where

$$E_k(\tau) := 1 - \frac{2k}{B_k}\sum_{n=1}^{\infty}\sigma_{k-1}(n)q^n$$

is the Eisenstein series of weight k and

$$\Delta(\tau) := q\prod_{n=1}^{\infty}(1-q^n)^{24} = \frac{E_4(\tau)^3 - E_6(\tau)^2}{1728}$$

is a holomorphic cusp form of weight 12.

Let d be a positive integer such that $-d$ is congruent to 0 or 1 modulo 4, and $\mathcal{Q}_d$ the set of all positive definite binary quadratic forms $Q(X,Y) = [A,B,C] := AX^2 + BXY + CY^2$ $(A,B,C \in \mathbb{Z})$ of discriminant $-d$. The group $\Gamma := \mathrm{PSL}_2(\mathbb{Z})$ acts on $\mathcal{Q}_d$ by

$$\left(Q \circ \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)(X,Y) := Q(aX+bY, cX+dY).$$

For each $Q \in \mathcal{Q}_d$, we define $\alpha_Q \in \mathbb{H}$ as the unique root in $\mathbb{H}$ of $Q(\tau,1) = 0$. We write Γ_Q for the stabilizer of Q in Γ. It is well-known that

$$\#\Gamma_Q = \begin{cases} 3 & \text{if } Q \sim X^2+XY+Y^2, \\ 2 & \text{if } Q \sim X^2+Y^2, \\ 1 & \text{if otherwise.} \end{cases}$$

For any non-negative integer $m \geq 0$, let $j_m(\tau)$ be the unique polynomial in $j(\tau)$ satisfying $j_m(\tau) = q^{-m} + O(q)$. The set $\{j_m(\tau) : m \geq 0\}$ forms a basis of $M_0^!(\Gamma)$, the space of weakly holomorphic modular forms of weight 0 on Γ.

Example 1.1. The first first few examples are listed below.

$$\begin{aligned}
&j_0(\tau) = 1,\\
&j_1(\tau) = j(\tau) - 744 = q^{-1} + 196884q + \cdots,\\
&j_2(\tau) = j(\tau)^2 - 1488j(\tau) + 159768 = q^{-2} + 42987520q + \cdots,\\
&j_3(\tau) = j(\tau)^3 - 2232j(\tau)^2 + 1069956j(\tau) - 36866976 = q^{-3} + 2592899910q + \cdots.
\end{aligned}$$

Definition 1.2. For each $m \geq 0$ and d as above, we define the trace functions

$$\mathbf{t}_m(d) \coloneqq \sum_{Q \in \mathcal{Q}_d/\Gamma} \frac{1}{\#\Gamma_Q} j_m(\alpha_Q).$$

For $m = 0$, it gives the Kronecker–Hurwitz class number:

$$H(d) \coloneqq \mathbf{t}_0(d) = \sum_{Q \in \mathcal{Q}_d/\Gamma} \frac{1}{\#\Gamma_Q}.$$

2. Zagier's results on the generating series

In 1975, Zagier showed a modular aspect of $H(d)$.

Theorem 2.1. [12] *The generating function*

$$\mathcal{H}(\tau) \coloneqq -\frac{1}{12} + \sum_{\substack{d>0\\ d\equiv 0,3 \pmod 4}} H(d)q^d + \frac{1}{8\pi\sqrt{v}} + \frac{1}{4\sqrt{\pi}} \sum_{n=1}^{\infty} n\Gamma\left(-\frac{1}{2}, 4\pi n^2 v\right) q^{-n^2}$$

is a harmonic Maass form of weight $3/2$ *on* $\Gamma_0(4)$.

Its holomorphic part

$$\mathcal{H}^+(\tau) \coloneqq -\frac{1}{12} + \sum_{\substack{d>0\\ d\equiv 0,3 \pmod 4}} H(d)q^d$$

is a mock modular form of weight $3/2$ on $\Gamma_0(4)$.

After that, in 2002, Zagier [13] extended the result to cover a more general case with $m \geq 0$.

Theorem 2.2. [13, Theorem 5] *For* $m > 0$*, the generating function*

$$g_m(\tau) \coloneqq -\sum_{\kappa \mid m} \kappa q^{-\kappa^2} + 2\sigma_1(m) + \sum_{\substack{d>0\\ d\equiv 0,3 \pmod 4}} \mathbf{t}_m(d)q^d$$

is a weakly holomorphic modular form of weight $3/2$ *on* $\Gamma_0(4)$.

For simplicity, for $d \leq 0$, we put

$$\mathbf{t}_m(d) = \begin{cases} 2\sigma_1(m) & \text{if } d = 0,\\ -\kappa & \text{if } d = -\kappa^2,\\ 0 & \text{if otherwise.} \end{cases}$$

3. Zagier's proof of the Eichler–Selberg trace formula

The Eichler–Selberg trace formula establishes a connection between the Kronecker–Hurwitz class number $H(d)$ and the trace of the Hecke operator.

Theorem 3.1 (The Eichler–Selberg trace formula). *For $k \geq 2$, we have*

$$\mathrm{Tr}(T_n, S_{2k}) = -\frac{1}{2}\sum_{r\in\mathbb{Z}} p_{2k}(r,n)H(4n-r^2) - \lambda_{2k-1}(n),$$

where

- *S_{2k} is the space of holomorphic cusp forms of weight $2k$ on Γ,*
- *T_n is the n-th Hecke operator,*
- $p_k(r,n) = \sum_{0\leq j\leq \frac{k}{2}-1} (-1)^j \binom{k-2-j}{j} n^j r^{k-2-2j} = \mathrm{Coeff}_{X^{k-2}}\left(\frac{1}{1-rX+nX^2}\right)$,
- $\lambda_k(n) := \frac{1}{2}\sum_{d|n}\min(d,n/d)^k$.

In unpublished notes [11], Zagier gave a new proof of the Eichler–Selberg trace formula. This is recently revisited and published by Ono–Saad [10]. First, we review it.

The idea is based on a computation of $\pi_{\mathrm{hol}}([\mathcal{H},\theta]_\nu|U_4)$ in two ways. Here

- $\theta(z) = \sum_{n\in\mathbb{Z}} q^{n^2}$ is a holomorphic modular form of weight $1/2$ on $\Gamma_0(4)$.
- U-operator is defined by

$$(f|U_t)(\tau) := \frac{1}{t}\sum_{j=0}^{t-1} f\left(\frac{\tau+j}{t}\right),$$

  that is,

$$\left(\sum_m c_f(m,v)q^m\right)|U_t = \sum_m c_f\left(tm,\frac{v}{t}\right)q^m.$$

- $[f,g]_\nu$ is the Rankin–Cohen bracket defined by

$$[f,g]_\nu := \sum_{\substack{r,s\geq 0\\ r+s=\nu}} (-1)^r \frac{\Gamma(k+\nu)\Gamma(l+\nu)}{s!\Gamma(k+r)r!\Gamma(l+s)} D^r(f)D^s(g)$$

  for modular forms f,g of weight k,l, respectively, where $D := \frac{1}{2\pi i}\frac{\mathrm{d}}{\mathrm{d}\tau} = q\frac{\mathrm{d}}{\mathrm{d}q}$, (see [2, Section 5]).

- π_{hol} is the holomorphic projection, (see [1, Section 10]).

For the right-hand side, by applying Mertens' result [8], we have

$$\pi_{\mathrm{hol}}([\mathcal{H},\theta]_\nu|U_4) = [\mathcal{H}^+,\theta]_\nu|U_4 + 2\binom{2\nu}{\nu}\sum_{n=1}^{\infty}\lambda_{2\nu+1}(n)q^n.$$

By a direct calculation, we obtain

$$[\mathcal{H}^+,\theta]_\nu|U_4 = \binom{2\nu}{\nu}\sum_{n=0}^{\infty}\left(\sum_{r\in\mathbb{Z}} p_{2\nu+2}(r,n)H(4n-r^2)\right)q^n. \tag{3.1}$$

Therefore, we conclude that the n-th coefficient of $\pi_{\rm hol}([\mathcal{H},\theta]_\nu|U_4)$ is

$$\binom{2\nu}{\nu}\left(\sum_{r\in\mathbb{Z}} p_{2\nu+2}(r,n)H(4n-r^2)+2\lambda_{2\nu+1}(n)\right). \tag{3.2}$$

For the left-hand side, first, we recall the following.

Lemma 3.2 (Eichler–Zagier [4, Theorem 5.5], with some modification)**.** *The function* $[\mathcal{H},\theta]_\nu|U_4$ *is a modular form of weight* $2\nu+2$ *on* Γ*.*

In particular, $\pi_{\rm hol}([\mathcal{H},\theta]_\nu|U_4)$ becomes a holomorphic cusp form in $S_{2\nu+2}(\Gamma)$. It can be expressed as

$$\pi_{\rm hol}([\mathcal{H},\theta]_\nu|U_4)=\sum_{j=1}^{d} a_j f_j, \tag{3.3}$$

where f_j's are normalized Hecke eigenforms of $S_{2\nu+2}(\Gamma)$.

Lemma 3.3. *For any* $1\le j\le d$*, we have*

$$a_j=-2\binom{2\nu}{\nu}.$$

Idea of proof. By using expression of $\mathcal{H}(\tau)$ in terms of the Eisenstein series, (see [10, Section 2.2] or [5, Chapter 2]), we can compute

$$\begin{aligned}
a_j\langle f_j,f_j\rangle &= \langle \pi_{\rm hol}([\mathcal{H},\theta]_\nu|U_4),f_j\rangle\\
&= \langle [\mathcal{H},\theta]_\nu|U_4,f_j\rangle \quad \text{(by definition of } \pi_{\rm hol})\\
&= -2\binom{2\nu}{\nu}\frac{\pi}{3}\frac{(2\nu+1)!}{(4\pi)^{2\nu+2}}\sum_{n=1}^{\infty}\frac{c_{f_j}(n^2)}{(n^2)^{\nu+1}} \quad \text{(by unfolding argument)}\\
&= -2\binom{2\nu}{\nu}\langle f_j,f_j\rangle \quad \text{(Rankin–Selberg's method [2, Section 11.12])},
\end{aligned}$$

which concludes the proof. □

Thus, we have

$$\pi_{\rm hol}([\mathcal{H},\theta]_\nu|U_4)=-2\binom{2\nu}{\nu}\sum_{j=1}^{d} f_j.$$

Since

$$T_n f_j = c_{f_j}(n) f_j,$$

we conclude that the n-th coefficient of $\pi_{\rm hol}([\mathcal{H},\theta]_\nu|U_4)$ is

$$-2\binom{2\nu}{\nu}\mathrm{Tr}(T_n,S_{2\nu+2}). \tag{3.4}$$

Comparing (3.2) and (3.4) implies Theorem 3.1.

4. Main results

Inspired by Zagier's proof, we try to compute $[g_m,\theta]_\nu|U_4$. By a similar calculation as in (3.1), we have

$$[g_m,\theta]_\nu|U_4 = \binom{2\nu}{\nu}\sum_{n\gg-\infty}\sum_{r\in\mathbb{Z}} p_{2\nu+2}(r,n)\mathbf{t}_m(4n-r^2)q^n$$

and $[g_m,\theta]_\nu|U_4 \in M^!_{2\nu+2}$. For $\nu=0$, since $M^!_2(\Gamma)=\{0\}$, we have

$$\mathcal{G}_{m,0}(\tau) := [g_m,\theta]_0|U_4 - \sum_{-\frac{m^2}{4}\le n\le -1}\frac{1}{n}\left(\sum_{r\in\mathbb{Z}}\mathbf{t}_m(4n-r^2)\right)Dj_{-n}(\tau)=0. \tag{4.1}$$

For $\nu>0$, we see that

$$\mathcal{G}_{m,\nu}(\tau) := [g_m,\theta]_\nu|U_4 - \binom{2\nu}{\nu}\sum_{-\frac{m^2}{4}\le n\le 0}\sum_{r\in\mathbb{Z}} p_{2\nu+2}(r,n)\mathbf{t}_m(4n-r^2)P_{2\nu+2,n}(\tau) \tag{4.2}$$

is a holomorphic cusp form in $S_{2\nu+2}(\Gamma)$, where

$$P_{k,m}(\tau) := \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} q^m|_k\gamma$$

is the Poincaré series. In particular, $P_{k,0}(\tau)=E_k(\tau)$ is the Eisenstein series. In a similar manner to (3.3), it should be expressed as

$$\mathcal{G}_{m,\nu}(\tau) = \sum_{j=1}^{d} b_j f_j.$$

Example 4.1. Let $m=1$ and $\nu=0,1$. Since $S_2(\Gamma)=S_4(\Gamma)=\{0\}$, we have $\mathcal{G}_{m,\nu}(\tau)=0$, that is,

$$\begin{aligned}\mathcal{G}_{1,0}(\tau) &= [g_1,\theta]_0|U_4 = 0,\\ \mathcal{G}_{1,1}(\tau) &= [g_1,\theta]_1|U_4 + 4P_{4,0}(\tau) = 0.\end{aligned}$$

By comparing the n-th Fourier coefficients ($n\ge 1$) on both sides, we get the recursion formulas

$$\begin{aligned}\sum_{r\in\mathbb{Z}}\mathbf{t}_1(4n-r^2) &= 0,\\ \sum_{r\in\mathbb{Z}} r^2\mathbf{t}_1(4n-r^2) &= -480\sigma_3(n).\end{aligned}$$

As noted in [7], the traces $\mathbf{t}_1(d)$ can be calculated by the above formulas recursively without knowing anything about its original definition.

Example 4.2. Let $\nu = 5$. For $m = 1, 2, 3$, we have

$$\begin{aligned}
\mathcal{G}_{1,5}(\tau) &= [g_1, \theta]_5 | U_4 + 504 E_{12}(\tau) \\
&= -504 \cdot \left(-\frac{82104}{691} \right) \Delta(\tau), \\
\mathcal{G}_{2,5}(\tau) &= [g_2, \theta]_5 | U_4 + 504 \left(P_{12,-1}(\tau) + 2049 E_{12}(\tau) \right) \\
&= -504 \left(\frac{1746612}{691} - \alpha \right) \Delta(\tau), \\
\mathcal{G}_{3,5}(\tau) &= [g_3, \theta]_5 | U_4 + 504 \left(2049 P_{12,-2}(\tau) + 177148 E_{12}(\tau) \right) \\
&= -504 \left(\frac{3294976184}{691} - 2049\beta \right) \Delta(\tau),
\end{aligned}$$

where we have

$$\begin{aligned}
P_{12,-1}(\tau) &= \Delta(\tau)(j_2(\tau) + 24 j_1(\tau) + 324 + \alpha) = \frac{1}{q} + \alpha q + \cdots, \\
P_{12,-2}(\tau) &= \Delta(\tau)(j_3(\tau) + 24 j_2(\tau) + 324 j_1(\tau) + 3200 + \beta) = \frac{1}{q^2} + \beta q + \cdots,
\end{aligned}$$

with $\alpha = 1842.894\ldots$ and $\beta = 23274.075\ldots$. We observe that

$$\begin{aligned}
-\frac{82104}{691} &= -24 - \frac{65520}{691} = -24 + \frac{\Gamma(11)}{(4\pi)^{11}} \frac{1}{\|\Delta\|^2} \cdot (-33.383\ldots), \\
\frac{1746612}{691} - \alpha &= -24 \cdot 3 + \frac{\Gamma(11)}{(4\pi)^{11}} \frac{1}{\|\Delta\|^2} \cdot 266.439\ldots, \\
\frac{3294976184}{691} - 2049\beta &= -24 \cdot 4 + \frac{\Gamma(11)}{(4\pi)^{11}} \frac{1}{\|\Delta\|^2} \cdot (-1519.2\ldots),
\end{aligned}$$

where $\|\Delta\|^2 = \langle \Delta, \Delta \rangle = 0.0000010353\ldots$.

We can compare them with the values of the symmetrized shifted convolution Dirichlet series defined by

$$\widehat{D}(\Delta, m; 11) := \sum_{n=1}^{\infty} \frac{\tau(n)\tau(n+m)}{n^{11}} - \sum_{n=1}^{\infty} \frac{\tau(n)\tau(n-m)}{n^{11}},$$

(see also Hoffstein–Hulse [6]). As in Mertens–Ono [9], it is known that

$$\widehat{D}(\Delta, 1; 11) = -33.383\ldots, \quad \widehat{D}(\Delta, 2; 11) = 266.439\ldots, \quad \widehat{D}(\Delta, 3; 11) = -1519.2\ldots,$$

which suggest the equation

$$\mathcal{G}_{m,5}(\tau) = -2\binom{2\nu}{\nu} \left(-24\sigma_1(m) + \frac{\Gamma(11)}{(4\pi)^{11}} \frac{1}{\|\Delta\|^2} \widehat{D}(\Delta, m; 11) \right) \Delta(\tau).$$

Our main result gives an explicit formula for the general cases.

Theorem 4.3. *For any $\nu \geq 0$ and $m \geq 1$, we define $\mathcal{G}_{m,\nu}(\tau)$ by (4.1) and (4.2). Then we have*

$$\mathcal{G}_{m,\nu}(\tau) = -2\binom{2\nu}{\nu} \sum_{j=1}^{d} \left(-24\sigma_1(m) + \frac{\Gamma(2\nu+1)}{(4\pi)^{2\nu+1}} \frac{1}{\|f_j\|^2} \widehat{D}(f_j, m; 2\nu+1) \right) f_j,$$

where f_j's are normalized Hecke eigenforms of $S_{2\nu+2}(\Gamma)$.

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