A SET OF PRIME-REPRESENTING CONSTANTS

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Abstract. Let $\lfloor x \rfloor$ denote the integer part of x. In 1947, Mills constructed a real number $A > 1$ such that $\lfloor A^{3^k} \rfloor$ is a prime number for every $k \in \mathbb{N}$. Let $\mathcal{W}$ be the set of all such real numbers A. It is known that $\mathcal{W}$ is uncountable, nowhere dense, closed, and has Lebesgue measure 0. In this paper, we give a simple proof of this fact.

1. Introduction

Let $\mathbb{N}$ be the set of all positive integers, and $\lfloor x \rfloor$ denotes the integer part of x. In 1947, Mills showed the following theorem.

Theorem 1.1 ([8, THEOREM]). *There exists a real number $A > 1$ such that*

$$\lfloor A^{3^k} \rfloor \text{ is a prime number for every } k \in \mathbb{N}. \tag{1.1}$$

The following fact is already known. Before stating it, a subset X of $\mathbb{R}$ is *nowhere dense* if $X \cap U$ is not dense in U for all non-empty open subsets U of $\mathbb{R}$ in the sense of the Euclidean topology.

Theorem 1.2. *Let $\mathcal{W}$ be the set of all real numbers $A > 1$ satisfying* (1.1). *Then the set $\mathcal{W}$ is uncountable, nowhere dense, closed, and has Lebesgue measure* 0.

We note that the uncountability, nowhere denseness, and zero measure of $\mathcal{W}$ are consequences of [13, THEOREMS 5,6,7] and the closedness is deduced from [4, Lemma 3] or [12, Theorem 1.2].

After Mills' work, several mathematicians are interested in the existence of $A > 1$ such that $\lfloor A^{c^k} \rfloor$ is a prime number for every $k \in \mathbb{N}$, where c is a fixed positive real number. For instance, Kuipers [7] showed the existence of such A for every integer $c \geq 3$. Ansari [2] extended the range to real numbers $c > 77/29$; Niven [9] independently presented a similar extension, but it is for $c > 8/3 = 2.666\cdots$.

Wright [13] first considered a set of such numbers A, and he investigated its geometric properties. To exhibit his result, let $K > 1$ and $(D_k)_{k=0}^{\infty}$ be a sequence of positive real numbers. Suppose that $(\lambda_k)_{k=1}^{\infty}$ is a sequence of real functions satisfying that for all $k \in \mathbb{N}$

- $\lambda_k(x)$ is positive and continuous on $[D_{k-1}, \infty)$;
- $\lambda_k(x') - \lambda_k(x) > K(x' - x)$ for all $x' > x \geq D_{k-1}$.

We further define $\phi_k(x) = \lambda_k \circ \lambda_{k-1} \circ \cdots \lambda_1(x)$ for all $x \geq D_0$, where $f \circ g(x) = f(g(x))$. Let $\mathcal{B}$ be a subset of $\mathbb{N}$. Then, Wright studied the properties of

$$\mathcal{W}((\phi_k)_{k=1}^{\infty}) = \{A > 1 \colon \lfloor \phi_k(A) \rfloor \in \mathcal{B} \text{ for all } k \in \mathbb{N}\}.$$

He gave sufficient conditions on $\mathcal{B}$ and $(\lambda_k)_{k=1}^{\infty}$ to obtain $\mathcal{W}((\phi_k)_{k=1}^{\infty})$ is uncountable, nowhere dense, and has Lebesgue measure 0. It is hard to follow the proofs for beginners because his paper is highly generalized. Thus, by focusing only on the case as $\lambda_k(x) = x^3$, this paper aims to give a more accessible proof.

Deschamps [4] studied more details on the geometric properties of $\mathcal{W}((\phi_k)_{k=1}^\infty)$ in the case when $\lambda_1(x) = \lambda_2(x) = \cdots = \lambda(x)$, that is,

$$\phi_k(x) = \overbrace{\lambda \circ \cdots \circ \lambda}^{k}(x).$$

He gave sufficient conditions so that $\mathcal{W}((\phi_k)_{k=1}^\infty)$ is closed, totally disconnected, and has no isolated points. The author and Takeda [12] also gave a similar result when $\lambda_k(x) = x^{c_k}$ for every $k \in \mathbb{N}$ and $(c_k)_{k=1}^\infty$ is a sequence of integers satisfying suitable conditions.

We refer to [5] for more details on the early research of this topic. Recently, the author [10] showed that $\min \mathcal{W}$ is irrational. We refer to [1, 12] for the readers who want to know the arithmetic properties of elements in $\mathcal{W}$.

2. Uncountability

Throughout this paper, let $\mathcal{W}$ be as in Theorem 1.2, and $\mathcal{P}$ denotes the set of all prime numbers. This section aims to prove that $\mathcal{W}$ is uncountable. Before that, we show Mills' result (Theorem 1.1) for practicing. To prove it, we should apply a suitable result on prime gaps. Actually, Mills applied the following result given by Ingham.

Theorem 2.1 ([6]). *For every $\epsilon > 0$, there exists $x_0 = x_0(\epsilon) > 0$ such that for every real number $x \geq x_0$, we find a prime number p satisfying that $x \leq p \leq x + x^{5/8+\epsilon}$.*

We remark that Ingham asserted a stronger statement, that is, for every $\epsilon > 0$, there exists $x_0 > 0$ such that for every $x \geq x_0$, we find a prime number p satisfying that $x \leq p \leq x + x^{577/925+\epsilon}$. Baker, Harman, and Pintz [3] proposed the best-known result which states that we may replace $577/925 + \epsilon$ with $21/40$.

We choose $\epsilon = 1/24$ and let x_0 be as in Theorem 2.1. By this theorem and $15/8+3\epsilon = 2$, for every integer $n \geq x_0$ there exists $p \in \mathcal{P}$ such that

$$n^3 \leq p \leq n^3 + n^{15/8+3\epsilon} < n^3 + 3n^2 + 3n = (n+1)^3 - 1. \tag{2.1}$$

Proof of Theorem 1.1. Let p_1 be a sufficiently large prime number so that $p_1 \geq x_0^{1/3}$. By (2.1) with $n = p_1^3$, we find $p_2 \in \mathcal{P}$ such that $p_1^3 \leq p_2 < (p_1+1)^3 - 1$. Similarly, by (2.1) with $n = p_2^3$, we find $p_3 \in \mathcal{P}$ such that $p_2^3 \leq p_3 < (p_2+1)^3 - 1$. By iterating this argument (more precisely, by induction), we find a sequence $(p_k)_{k=1}^\infty$ of prime numbers such that

$$p_k^3 \leq p_{k+1} < (p_k+1)^3 - 1 \tag{2.2}$$

for every $k \in \mathbb{N}$. This leads to

$$p_1^{1/3^1} \leq p_2^{1/3^2} \leq p_3^{1/3^3} \leq \cdots < (p_3+1)^{1/3^3} < (p_2+1)^{1/3^2} < (p_1+1)^{1/3^1}, \tag{2.3}$$

and hence

$$\lim_{k\to\infty} p_k^{1/3^k} =: A \quad \text{and} \quad \lim_{k\to\infty} (p_k+1)^{1/3^k} =: A'$$

exist. In addition, $A \leq A'$ holds[1] by (2.3). Therefore, for every $k \in \mathbb{N}$, we have

$$p_k \leq A^{3^k} \leq A'^{3^k} < p_k + 1,$$

which implies that $p_k = \lfloor A^{3^k} \rfloor$ for every $k \in \mathbb{N}$. □

Theorem 2.2. *The set $\mathcal{W}$ is uncountable.*

[1] Actually, $A = A'$ holds, but $A \leq A'$ is enough for this proof.

The inequalities (2.1) are not strong enough to prove this theorem. For instance, we prepare the following auxiliary lemma.

Lemma 2.3. *There eixsts $x_1 > 0$ such that for all integers $n \geq x_1$, we find prime numbers $p(0)$ and $p(1)$ such that $n^3 \leq p(0) < p(1) < (n+1)^3 - 1$.*

Proof. Let $\epsilon = 1/24$, and let x_0 be as in Theorem 2.1. Let x_1 be a sufficiently large positive real number. Take an arbitrary $n \geq x_1$. We may assume that $n \geq x_1 \geq x_0$. Then, by Theorem 2.1 with $x = n^3$, there exists $p(0) \in \mathcal{P}$ such that

$$n^3 \leq p(0) \leq n^3 + n^2. \tag{2.4}$$

Furthermore, by Theorem 2.1 with $x = n^3 + n^2 + 1$, there exists $p(1) \in \mathcal{P}$ such that

$$n^3 + n^2 + 1 \leq p(1) \leq n^3 + n^2 + 1 + (n^3 + n^2 + 1)^{2/3}, \tag{2.5}$$

where $2/3 = 5/8 + 1/24 = 5/8 + \epsilon$. Furthermore, we observe that

$$(n^3 + n^2 + 1)^{2/3} = n^2(1 + n^{-1} + n^{-3})^{2/3} \leq 2n^2 \tag{2.6}$$

by replacing x_1 with larger one so that $(1 + x_1^{-1} + x_1^{-3})^{2/3} \leq 2$ if necessary. By combining (2.4), (2.5), and (2.6), we obtain

$$\begin{aligned} n^3 &\leq p(0) \leq n^3 + n^2 < n^3 + n^2 + 1 \\ &\leq p(1) \leq n^3 + 3n^2 + 1 < n^3 + 3n^2 + 3n = (n+1)^3 - 1, \end{aligned}$$

and hence $n^3 \leq p(0) < p(1) < (n+1)^3 - 1$. □

Proof of Theorem 2.2. Let x_1 be as in Lemma 2.3. Take prime numbers $p(0)$ and $p(1)$ with $x_1 \leq p(0) < p(1)$. For every $i \in \{0,1\}$, Lemma 2.3 implies that there exist prime numbers $p(i,0)$ and $p(i,1)$ such that

$$p(i)^3 \leq p(i,0) < p(i,1) < (p(i)+1)^3 - 1. \tag{2.7}$$

Thus, we obtain prime numbers $p(i_1, i_2)$ $((i_1, i_2) \in \{0,1\}^2)$.

For every $(i_1, i_2) \in \{0,1\}^2$, Lemma 2.3 implies that there exist prime numbers $p(i_1, i_2, 0)$ and $p(i_1, i_2, 1)$ such that

$$p(i_1, i_2)^3 \leq p(i_1, i_2, 0) < p(i_1, i_2, 1) < (p(i_1, i_2) + 1)^3 - 1. \tag{2.8}$$

Thus, we obtain prime numbers $p(i_1, i_2, i_3)$ $((i_1, i_2, i_3) \in \{0,1\}^3)$.

We now define $\mathbf{i}_k = (i_1, i_2, \ldots, i_k)$ for all $\mathbf{i} = (i_1, i_2, \ldots) \in \{0,1\}^{\mathbb{N}}$. By iterating the above argument, we obtain a set $\{(p(\mathbf{i}_k))_{k=1}^{\infty} \colon \mathbf{i} \in \{0,1\}^{\mathbb{N}}\}$ of sequences of prime numbers so that for every $\mathbf{i} \in \{0,1\}^{\mathbb{N}}$ and for every $k \in \mathbb{N}$, we have

$$p(\mathbf{i}_k)^3 \leq p(\mathbf{i}_k, 0) < p(\mathbf{i}_k, 1) < (p(\mathbf{i}_k) + 1)^3 - 1. \tag{2.9}$$

In a similar manner to the proof of Theorem 1.1, for every $\mathbf{i} \in \{0,1\}^{\mathbb{N}}$, there exists a real number $A(\mathbf{i}) > 1$ such that $\lfloor A(\mathbf{i})^{3^k} \rfloor = p(\mathbf{i}_k)$ for every $k \in \mathbb{N}$. Therefore, we have

$$\{A(\mathbf{i}) \colon \mathbf{i} \in \{0,1\}^{\mathbb{N}}\} \subseteq \mathcal{W}.$$

Since $\{0,1\}^{\mathbb{N}}$ is uncountable, it suffices to show that $A(\mathbf{i}) \neq A(\mathbf{j})$ for all $\mathbf{i}, \mathbf{j} \in \{0,1\}^{\mathbb{N}}$ with $\mathbf{i} \neq \mathbf{j}$. Take arbitrary $\mathbf{i}, \mathbf{j} \in \{0,1\}^{\mathbb{N}}$ with $\mathbf{i} \neq \mathbf{j}$, and let $m = \min\{k \in \mathbb{N} \colon \mathbf{i}_k \neq \mathbf{j}_k\}$.

Suppose that $m = 1$. Then, it is clear that $A(\mathbf{i}) \neq A(\mathbf{j})$ since

$$\lfloor A(\mathbf{i})^3 \rfloor = p(\mathbf{i}_1), \quad \lfloor A(\mathbf{j})^3 \rfloor = p(\mathbf{j}_1), \quad \text{and} \quad p(\mathbf{i}_1) \neq p(\mathbf{j}_1).$$

Suppose that $m \geq 2$. Then, by (2.9) with $k = m-1$ and the definition of m, we have

$$p(\mathbf{i}_{m-1}, 0) < p(\mathbf{i}_{m-1}, 1), \quad \mathbf{i}_{m-1} = \mathbf{j}_{m-1}, \quad \text{and} \quad \mathbf{i}_m \neq \mathbf{j}_m\ . \tag{2.10}$$

Therefore, we obtain $A(\mathbf{i}) \neq A(\mathbf{j})$ since there are distinct integers $i_m, j_m \in \{0,1\}$ such that

$$\lfloor A(\mathbf{i})^3 \rfloor = p(\mathbf{i}_{m-1}, i_m), \quad \lfloor A(\mathbf{j})^3 \rfloor = p(\mathbf{j}_{m-1}, j_m),$$
$$p(\mathbf{i}_{m-1}, i_m) = p(\mathbf{j}_{m-1}, i_m) \neq p(\mathbf{j}_{m-1}, j_m),$$

where we apply (2.10) to obtain the latter formula. □

3. Topological properties

In this section, we prove that $\mathcal{W}$ is nowhere dense and closed. It is also known that $\mathcal{W}$ has no isolated points (See [4, 12] for more details), but we do not give a proof of this fact in this paper.

Theorem 3.1. *The set $\mathcal{W}$ is nowhere dense.*

Proof. Take an arbitrary non-empty open set $U \subseteq \mathbb{R}$. We may assume that $\mathcal{W} \cap U \neq \emptyset$. Then, let $I = (\alpha, \beta)$ be an open interval such that $I \subseteq U$ and $1 \leq \alpha < \beta$. We take a sufficiently large $k \in \mathbb{N}$ so that $\beta^{3^k} - \alpha^{3^k} \geq 100$. Then, there exists a composite number M such that $\alpha^{3^k} < M < M+1 < \beta^{3^k}$.

Let $x = M^{1/3^k} \in I \subseteq U$, and let $\epsilon = \min(1, 3^{-k}(x+1)^{1-3^{-k}})$. Then, $(x, x+\epsilon) \cap \mathcal{W} = \emptyset$. Indeed, for all $A \in (x, x+\epsilon)$, we have

$$M = x^{3^k} < A^{3^k} < (x+\epsilon)^{3^k} < x^{3^k} + 3^k \epsilon (x+1)^{3^k - 1} \leq M + 1,$$

and hence $\lfloor A^{3^k} \rfloor \notin \mathcal{P}$, that is, $A \notin \mathcal{W}$. Therefore, $\mathcal{W} \cap U$ is not dense in U. □

Remark 3.2. For all $X \subseteq \mathbb{R}$, X is totally disconnected if X is nowhere dense. Thus, Theorem 3.1 yields that $\mathcal{W}$ is totally disconnected.

Theorem 3.3. *The set $\mathcal{W}$ is closed.*

Proof. Let $(A_j)_{j=1}^{\infty}$ be a sequence of $\mathcal{W}$ such that $\lim_{j\to\infty} A_j =: A$ exists. Suppose that there is a subsequence $(A_{j_r})_{r=1}^{\infty}$ such that $A_{j_1} \geq A_{j_2} \geq \cdots \geq A$. Then, for any fixed $k \in \mathbb{N}$, by the right-side continuity of the floor function, we observe that

$$\lfloor A^{3^k} \rfloor = \lfloor \lim_{r\to\infty} A_{j_r}^{3^k} \rfloor = \lim_{r\to\infty} \lfloor A_{j_r}^{3^k} \rfloor.$$

We note that $\lfloor A_{j_r}^{3^k} \rfloor \in \mathcal{P}$ since $A_{j_r} \in \mathcal{W}$ for all $r \in \mathbb{N}$. Therefore, $\lfloor A^{3^k} \rfloor \in \mathcal{P}$ since $\mathbb{Z}$ is a discrete topology (or $\mathcal{P}$ is closed).

We may suppose that there exists $j_0 > 0$ such that

$$1 < A_{j_0} \leq A_{j_0+1} \leq A_{j_0+2} \leq \cdots \leq A.$$

Take an arbitrary positive integer k. If $A^{3^k} \notin \mathbb{Z}$, then the continuity of the floor function leads to

$$\lfloor A^{3^k} \rfloor = \lfloor \lim_{j\to\infty} A_j^{3^k} \rfloor = \lim_{j\to\infty} \lfloor A_j^{3^k} \rfloor \in \mathcal{P}.$$

Suppose that $A^{3^k} \in \mathbb{Z}$. If $A_j = A$ for some $j \geq j_0$, then $(A^{3^k})^3 = A_j^{3^{k+1}}$, and so $(A^{3^k})^3 = \lfloor A_j^{3^{k+1}} \rfloor \in \mathcal{P}$. This is a contradiction since the left-hand side is a composite number. Thus, we may assume that $A_j < A$ for all $j \geq j_0$. Then, there exists $j \geq j_0$ such that

$$A^{3^{k+1}} = \lfloor A^{3^{k+1}} \rfloor = \lfloor A_j^{3^{k+1}} \rfloor + 1$$

by combining $A_j < A$ (for all $j \in \mathbb{N}$), $\lim_{j\to\infty} A_j = A$, and $A^{3^{k+1}} \in \mathbb{Z}$. Therefore, we have

$$(A^{3^k})^3 - 1 = \lfloor A_j^{3^{k+1}} \rfloor,$$

a contradiction since the left-hand side is a composite number but the right-hand side is a prime number. □

If a topological space X is non-empty, compact, totally disconnected, and has no isolated point, then X is homeomorphic to the middle third Cantor set. Therefore, the set $\mathcal{W} \cap [0, a]$ is homeomorphic to the middle third Cantor set for every sufficiently large $a \in \mathbb{R} \setminus \mathcal{W}$. See [4, 12] for more details.

4. Lebesgue measure

Let μ denote the Lebesgue measure on $\mathbb{R}$. In this section, we show the following theorem.

Theorem 4.1. *The set $\mathcal{W}$ has Lebesgue measure* 0.

Lemma 4.2. *Let $A \in \mathcal{W}$. Let $p_k = \lfloor A^{3^k} \rfloor$ for every $k \in \mathbb{N}$. Then for every $k \in \mathbb{N}$, we have $p_k^3 \le p_{k+1} < (p_k + 1)^3$.*

Proof. Take an arbitrary integer $k \in \mathbb{N}$. Since $p_k = \lfloor A^{3^k} \rfloor$, we have $p_k \le A^{3^k} < p_k + 1$, and hence $p_k^3 \le A^{3^{k+1}} < (p_k + 1)^3$. Since $p_k^3 \in \mathbb{N}$, we obtain $p_k^3 \le \lfloor A^{3^{k+1}} \rfloor < (p_k + 1)^3$, which implies that $p_k^3 \le p_{k+1} < (p_k + 1)^3$. □

Lemma 4.3. *For every $\epsilon > 0$, there exists $x_2 > 0$ such that for every $x \ge x_2$, we have*

$$\#\big([x^3, (x+1)^3) \cap \mathcal{P}\big) \le \left(\frac{3}{2} + \epsilon\right) x^2.$$

Proof. Let ϵ be a positive real number. Let x_2 be a sufficiently large real number depending only on ϵ. Take an arbitrary real number $x \ge x_2$. Then since $(x+1)^3 - x^3 = 3x^3 + 3x + 1$, we have

$$\#\big([x^3, (x+1)^3) \cap \mathcal{P}\big) \le \frac{3x^2 + 3x + 1}{2} + 1 \le \frac{3}{2}x^2(1 + 2x^{-1}) \le \frac{3}{2}(1 + 2x_2^{-1})x^2,$$

where the first inequality follows by counting odd numbers in $[x^3, (x+1)^3)$. By choosing $x_2 > 0$ as $x_2 > 3/\epsilon$, we complete the proof. □

Proof of Theorem 4.1. Let $\epsilon = 1/2$, and let x_2 be as in Lemma 4.3. The symbol p_j denotes a variable running over $\mathcal{P}$. For every $m \in \mathbb{N}$. By Lemma 4.2, we observe that

$$\begin{aligned}\mathcal{W} &= \{A > 1\colon \lfloor A^{3^k} \rfloor \in \mathcal{P} \text{ for all } k \in \mathbb{N}\}\\ &= \bigcup_{p_1 \in \mathcal{P}} \{A > 1\colon \lfloor A^3 \rfloor = p_1\} \cap \mathcal{W}\\ &= \bigcup_{p_1 \in \mathcal{P}} \bigcup_{p_2 \in [p_1^3, (p_1+1)^3)} \{A > 1\colon \lfloor A^3 \rfloor = p_1 \text{ and } \lfloor A^{3^2} \rfloor = p_2\} \cap \mathcal{W}.\end{aligned}$$

By iterating the above argument, for every $k \in \mathbb{N}$, we have

$$\mathcal{W} = \bigcup_{p_1 \in \mathcal{P}} \bigcup_{p_2 \in [p_1^3, (p_1+1)^3)} \cdots \bigcup_{p_k \in [p_{k-1}^3, (p_{k-1}+1)^3)} \{A > 1\colon \lfloor A^{3^j} \rfloor = p_j \text{ for every } j \in [1, k]\} \cap \mathcal{W}.$$

We note that $p_k \geq p_1^{3^k} \geq 2^{3^k}$ since $p_j^3 \leq p_{j+1}$ for every $j = m, m+1, \dots, k-1$. Thus, we take a positive integer m as $p_m \geq 2^{3^m} \geq x_2$. By the subadditivity of the Lebesgue measure, it suffices to show that for every fixed $(p_1, \dots, p_m) \in \mathcal{P}^m$ with $p_j \in [p_{j-1}^3, (p_{j-1}+1)^3)$ $(j = 2, 3, \dots, m)$, the Lebesgue measure of

$$\{A > 1 \colon \lfloor A^{3^j} \rfloor = p_j \text{ for every } j \in [1, m]\} \cap \mathcal{W}$$

is zero. Let $\mathcal{W}'$ be this set. Similarly, for every $k > m$, the set $\mathcal{W}'$ is

$$\bigcup_{p_{m+1} \in [p_m^3, (p_m+1)^3)} \cdots \bigcup_{p_k \in [p_{k-1}^3, (p_{k-1}+1)^3)} \{A > 1 \colon \lfloor A^{3^j} \rfloor = p_j \text{ for every } j \in [1, k]\} \cap \mathcal{W}$$

Here, for every $k > m$, we observe that

$$\begin{aligned}
&\mu\Big(\{A > 1 \colon \lfloor A^{3^j} \rfloor = p_j \text{ for all } j \in [1, k]\}\Big) \leq \mu\Big(\{A > 1 \colon \lfloor A^{3^k} \rfloor = p_k\}\Big) \\
&= \mu\Big([p_k^{1/3^k}, (p_k+1)^{1/3^k})\Big) = (p_k+1)^{1/3^k} - p_k^{1/3^k} \leq \frac{p_k^{1/3^k - 1}}{3^k},
\end{aligned}$$

where we apply $(x+1)^\alpha - x^\alpha \leq \alpha x^{\alpha-1}$ for all $x > 0$ at the last inequality. Thus, the subadditivity of the Lebesgue measure implies that $\mu(\mathcal{W}')$ is less than or equal to

$$\begin{aligned}
&\sum_{p_m^3 \leq p_{m+1} < (p_m+1)^3} \cdots \sum_{p_{k-1}^3 \leq p_k < (p_{k-1}+1)^3} \mu\Big(\{A > 1 \colon \lfloor A^{3^j} \rfloor = p_j \text{ for all } j \in [1, k]\}\Big) \\
&\leq \sum_{p_m^3 \leq p_{m+1} < (p_m+1)^3} \cdots \sum_{p_{k-1}^3 \leq p_k < (p_{k-1}+1)^3} \frac{1}{3^k} p_k^{1/3^k - 1} \\
&\leq \sum_{p_m^3 \leq p_{m+1} < (p_m+1)^3} \cdots \sum_{p_{k-2}^3 \leq p_{k-1} < (p_{k-2}+1)^3} 2p_{k-1}^2 \cdot \frac{1}{3^k} p_{k-1}^{1/3^{k-1} - 3},
\end{aligned}$$

where we apply Lemma 4.3 with $\epsilon = 1/2$ at the last inequality. Therefore, by iterating this calculation, we obtain

$$\mu(\mathcal{W}') \leq \left(\frac{2}{3}\right)^{k-m} \frac{1}{3^m} p_m^{1/3^m - 1}$$

for all integers $k > m$, and hence by taking $k \to \infty$, we conclude that $\mu(\mathcal{W}') = 0$. $\square$

Combining Theorems 2.2, 3.1, 3.3, and 4.1, we obtain Theorem 1.2.

In [11, Theorem 18], the author showed that the Hausdorff dimension of

$$\mathcal{W} \cap [p^{1/3}, (p+1)^{1/3})$$

is greater than or equal to $\left(1 + \frac{3}{p \log p}\right)^{-1}$ for every sufficiently large $p \in \mathcal{P}$. It is still open to verify that the dimension equals 1.

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References

[1] G. Alkauskas and A. Dubickas. Prime and composite numbers as integer parts of powers. *Acta Math. Hungar.*, 105(3):249–256, 2004.
[2] A. R. Ansari. On prime representing function. *Gaṇita*, 2:81–82, 1951.
[3] R. C. Baker, G. Harman, and J. Pintz. The difference between consecutive primes. II. *Proc. London Math. Soc. (3)*, 83(3):532–562, 2001.
[4] B. Deschamps. Une propriété topologique de certains ensembles de Mills. *Unif. Distrib. Theory*, 12(1):139–153, 2017.
[5] U. Dudley. History of a formula for primes. *Amer. Math. Monthly*, 76:23–28, 1969.
[6] A. E. Ingham. On the difference between consecutive primes. *The Quarterly Journal of Mathematics*, os-8(1):255–266, 1937.
[7] L. Kuipers. Prime-representing functions. *Nederl. Akad. Wetensch., Proc.*, 53:309–310 = Indagationes Math. 12, 57–58, 1950.
[8] W. H. Mills. A prime-representing function. *Bull. Amer. Math. Soc.*, 53:604, 1947.
[9] I. Niven. Functions which represent prime numbers. *Proc. Amer. Math. Soc.*, 2:753–755, 1951.
[10] K. Saito. Mills' constant is irrational. preprint (2024), available at arXiv:2404.19461.
[11] K. Saito. Prime-representing functions and Hausdorff dimension. *Acta Math. Hungar.*, 165(1):203–217, 2021.
[12] K. Saito and W. Takeda. Topological properties and algebraic independence of sets of prime-representing constants. *Mathematika*, 68(2):429–453, 2022.
[13] E. M. Wright. A class of representing functions. *J. London Math. Soc.*, 29:63–71, 1954.

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