Vertex Operators of the Elliptic Quantum Toroidal Algebra and the Elliptic Stable Envelopes

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1 Introduction

This is a review of the works given in [12]. The main result is a new formulation of the vertex operators of the elliptic quantum toroidal algebra (EQTA) $U_{t_1,t_2,p}(\mathfrak{gl}_{1,tor})$ by combining its representations and the notions of the elliptic stable envelopes (ESE) for the instanton moduli space $\mathcal{M}(n,r)$.

The EQTA $U_{t_1,t_2,p}(\mathfrak{gl}_{1,tor})$ is an elliptic quantum group associated with the toroidal algebra of type $\mathfrak{gl}_1$ [11]. The Hopf algebroid structure associated with the Drinfeld comultiplication allows us to construct two types of vertex operators, the type I and the type II dual, as intertwining operators of $U_{t_1,t_2,p}(\mathfrak{gl}_{1,tor})$-modules [11]. It turns out that they give a realization of the affine quiver W-algebra associated with the Jordan quiver varieties [8]. In addition, the same vertex operators realize the refined topological vertices [7], which are relevant to the calculation of the instanton partition functions of the 5d and 6d lift of the 4d $\mathcal{N}=2^*$ $U(M)$ gauge theory [13, 14]. However their relations to the elliptic stable envelopes [1] and to the vertex functions [15] of the corresponding quiver variety were missing. These relations have been observed in the case of the elliptic quantum group $U_{q,p}(\widehat{\mathfrak{sl}}_N)$ [9, 10] and expected to be possessed in the intertwining operators w.r.t. the standard comultiplication, which preserves the RLL-relation.

We here propose a new formulation of the vertex operators. We realize both the type I and the type II dual vertex operators as screened vertex operators, i.e. operator valued integrals with the ESE's for $\mathrm{E}_T(\mathcal{M}(n,r))$ as their integration kernels. We then make several checks on their consistency such as

- a derivation of the shuffle product formula of ESE's [2] by considering a composition of the vertex operators
- a construction of the K-theoretic vertex functions for $\mathcal{M}(n,r)$ as the highest to

highest expectation values of the corresponding vertex operators

- exchange relations among the vertex operators, whose coefficients are given by the elliptic instanton R-matrices defined as transition matrices of the ESE's for $\mathrm{Hilb}^n(\mathbb{C}^2)$
- a construction of the L-operator $L^+(u)$ satisfying the RLL-relation by combining the type I and the type II dual vertex operators
- exchange relations between the L-operator and the vertex operators.

The last relations indicates that our new vertex operators are the intertwining operators of the $U_{t_1,t_2,p}(\mathfrak{gl}_{1,tor})$-modules w.r.t. the standard comultiplication.

2 Elliptic Quantum Toroidal Algebra $U_{t_1,t_2,p}(\mathfrak{gl}_{1,tor})$

The elliptic quantum toriodal algebra $\mathcal{U}_{t_1,t_2,p}(\mathfrak{gl}_{1,tor})$ was introduced in [11]. The parameters $t_1, t_2, \hbar = t_1t_2$ in this paper correspond to $q^{-1}, t, t/q$ in [11], respectively.

2.1 Definition

Let us consider the Heisenberg algebras generated by $c,\ \Lambda_0,\ c^\perp,\ \Lambda_0^\perp,\ h,\ \alpha,\ P,\ Q$ satisfying the commutation relations

$$[c,\Lambda_0] = 1 = [c^\perp,\Lambda_0^\perp], \quad [h,\alpha] = 2 = [P,Q], \tag{2.1}$$

the others are zero. We set $\gamma = \hbar^{c^\perp/2},\ C = \hbar^{c/2},\ \mathfrak{z}^* = \hbar^P$ and $\mathfrak{z} = \hbar^{P+c}$. We call $\mathfrak{z}^*$ the dynamical parameter. Let $\mathbb{F}$ be the field of meromorphic functions of $\mathfrak{z}$ and $\mathfrak{z}^*$. We have

$$g(\mathfrak{z},\mathfrak{z}^*)e^{\Lambda_0-Q} = e^{\Lambda_0-Q}g(\mathfrak{z},\mathfrak{z}^*\hbar^{-2}), \quad g(\mathfrak{z},\mathfrak{z}^*)e^{-\Lambda_0} = e^{-\Lambda_0}g(\mathfrak{z}\hbar^{-2},\mathfrak{z}^*) \qquad \forall g(\mathfrak{z},\mathfrak{z}^*) \in \mathbb{F}.$$

Set

$$\kappa_m = -(1-t_1^m)(1-t_2^m)(1-\hbar^{-m}),$$
$$G^\pm(z) = (1-t_1^{\mp1}z)(1-t_2^{\mp1}z)(1-\hbar^{\pm1}z).$$

Definition 2.1. *The elliptic quantum toroidal algebra* $\mathcal{U} = U_{t_1,t_2,p}(\mathfrak{gl}_{1,tor})$ *is a topological associative algebra over* $\mathbb{F}[[p]]$ *generated by* $\alpha_m, x_n^\pm$*,* $(m \in \mathbb{Z}\backslash\{0\}, n \in \mathbb{Z})$ *and* $C, \gamma^{1/2}$*. Let* $x^\pm(z), \psi^\pm(z)$ *be the following generating functions*[1]*.*

$$x^\pm(z) := \sum_{n\in\mathbb{Z}} x_n^\pm z^{-n},$$

$$\psi^+(z) := C \exp\left(-\sum_{m>0} \frac{p^m}{1-p^m}\alpha_{-m}(\gamma^{-1/2}z)^m\right) \exp\left(\sum_{m>0} \frac{1}{1-p^m}\alpha_m(\gamma^{-1/2}z)^{-m}\right),$$

$$\psi^-(z) := C^{-1} \exp\left(-\sum_{m>0} \frac{1}{1-p^m}\alpha_{-m}(\gamma^{1/2}z)^m\right) \exp\left(\sum_{m>0} \frac{p^m}{1-p^m}\alpha_m(\gamma^{1/2}z)^{-m}\right).$$

We call them the elliptic currents. The defining relations are given by

$$\gamma^{1/2},\ C\ :\ \textit{central},$$

$$[\alpha_m, \alpha_n] = -\frac{\kappa_m}{m}(\gamma^m - \gamma^{-m})\gamma^{-m}\frac{1-p^m}{1-p^{*m}}\delta_{m+n,0},$$

$$[\alpha_m, x^+(z)] = -\frac{\kappa_m}{m}\frac{1-p^m}{1-p^{*m}}\gamma^{-m}z^m x^+(z) \quad (m \neq 0).$$

$$[\alpha_m, x^-(z)] = \frac{\kappa_m}{m}z^m x^-(z) \quad (m \neq 0),$$

$$[x^+(z), x^-(w)] = -\frac{(1-t_1)(1-t_2)}{(1-\hbar)}\left(\delta(\gamma^{-1}z/w)\psi^+(\gamma^{1/2}w) - \delta(\gamma z/w)\psi^-(\gamma^{-1/2}w)\right),$$

$$z^3 G^+(w/z)g(w/z;p^*)x^+(z)x^+(w) = -w^3 G^+(z/w)g(z/w;p^*)x^+(w)x^+(z),$$

$$z^3 G^-(w/z)g(w/z;p)^{-1}x^-(z)x^-(w) = -w^3 G^-(z/w)g(z/w;p)^{-1}x^-(w)x^-(z),$$

$$g(w/z;p^*)g(u/w;p^*)g(u/z;p^*)\left(\frac{w}{u}+\frac{w}{z}-\frac{z}{w}-\frac{u}{w}\right)x^+(z)x^+(w)x^+(u)$$
$$+\textit{permutations in } z, w, u = 0,$$

$$g(w/z;p)^{-1}g(u/w;p)^{-1}g(u/z;p)^{-1}\left(\frac{w}{u}+\frac{w}{z}-\frac{z}{w}-\frac{u}{w}\right)x^-(z)x^-(w)x^-(u)$$
$$+\textit{permutations in } z, w, u = 0,$$

where we set $p^* = p\gamma^{-2}$ *and*

$$g(z;s) = \exp\left(\sum_{m>0}\frac{\kappa_m}{m}\frac{s^m}{1-s^m}z^m\right) \in \mathbb{C}[[z]]$$

for $s = p, p^*$*. The dynamical parameters* $\mathfrak{z}, \mathfrak{z}^*$ *commute with* $\alpha_m, x_n^\pm$*.*

[1] Our $x^\pm(z)$ is $x^\pm(\gamma^{1/2}z)$ in [11].

It is convenient to set

$$\alpha'_m = \frac{1-p^{*m}}{1-p^m}\gamma^m \alpha_m \qquad (m \in \mathbb{Z}_{\neq 0}).$$

Through this paper, we treat $t_1, t_2, p, p^* = p\gamma^{-2}$ as generic complex numbers with $|t_1|, |t_2|, |p|, |p^*| < 1$. In particular, we have

$$g(z;p) = \frac{(pt_1^{-1}z;p)_\infty}{(pt_1 z;p)_\infty}\frac{(pt_2^{-1}z;p)_\infty}{(pt_2 z;p)_\infty}\frac{(p\hbar z;p)_\infty}{(p\hbar^{-1}z;p)_\infty},$$

where

$$(z;p)_\infty = \prod_{n=0}^{\infty}(1-zp^n) \qquad |z|<1.$$

2.2 Representations of $U_{t_1,t_2,p}(\mathfrak{gl}_{1,tor})$

Let $\mathcal{V}$ be a $\mathcal{U}$-module. For $(k,l) \in \mathbb{C}^2$, we say that $\mathcal{V}$ has level (k,l), if the central elements γ and C act as[2]

$$\gamma \cdot \xi = \hbar^{k/2}\xi, \qquad C\cdot \xi = \hbar^l \xi \qquad \forall \xi \in \mathcal{V}.$$

2.2.1 The level-$(1,N)$ representation

Let h, α satisfy $[h,\alpha]=1$ and commuting with the other generators. Define for $v \in \mathbb{C}^\times$

$$|0\rangle_v^{(1,N)} := v^\alpha e^{\Lambda_0^\perp} e^{N\Lambda_0} 1. \tag{2.2}$$

We assume $\gamma^{1/2}\cdot 1 = C \cdot 1 = e^{\pm h}\cdot 1 = 1$ and $e^Q \cdot 1 = e^Q 1$. One has

$$e^{\pm h}u^{\pm c}\cdot |0\rangle_v^{(1,N)} = v^{\pm 1}u^{\pm N}|0\rangle_{vu^{\pm 1}}^{(1,N)}, \quad \gamma\cdot |0\rangle_v^{(1,N)} = \hbar^{1/2}|0\rangle_v^{(1,N)}, \quad C\cdot |0\rangle_v^{(1,N)} = \hbar^{N/2}|0\rangle_v^{(1,N)}.$$

Let $\mathcal{F}_v^{(1,N)} = \mathbb{C}[\alpha_{-m}\ (m>0)]|0\rangle_v^{(1,N)}$ be a Fock module of the Heisenberg subalgebra $\{\alpha_m\ (m \in \mathbb{Z}_{\neq 0})\}$.

[2] We changed the definition of the level of representation from the one given in [11] so that our level (k,l) is the level $(k,-l)$ there. Note also our $C = \hbar^{c/2}$ is ψ_0^+ in [11].

Theorem 2.2. *The following gives a level $(1,N)$ representation of $\mathcal{U}$ on $\mathcal{F}_v^{(1,N)}$.*

$$x^+(z) = e^h (z^{-1}\hbar^{1/2})^c \exp\left\{-\sum_{n>0} \frac{\hbar^{n/2}}{1-\hbar^n}\alpha_{-n}z^n\right\} \exp\left\{\sum_{n>0} \frac{\hbar^{n/2}}{1-\hbar^n}\alpha_n z^{-n}\right\}, \tag{2.3}$$

$$x^-(z) = e^{-h} (z^{-1}\hbar^{1/2})^{-c} \exp\left\{\sum_{n>0} \frac{\hbar^{n/2}}{1-\hbar^n}\alpha'_{-n}z^n\right\} \exp\left\{-\sum_{n>0} \frac{\hbar^{n/2}}{1-\hbar^n}\alpha'_n z^{-n}\right\}, \tag{2.4}$$

$$\psi^+(\hbar^{1/4}z) = \hbar^{-c/2} \exp\left\{-\sum_{n>0} \frac{p^n}{1-p^n}\alpha_{-n}z^n\right\} \exp\left\{\sum_{n>0} \frac{1}{1-p^n}\alpha_n z^{-n}\right\}, \tag{2.5}$$

$$\psi^-(\hbar^{-1/4}z) = \hbar^{c/2} \exp\left\{-\sum_{n>0} \frac{1}{1-p^n}\alpha_{-n}z^n\right\} \exp\left\{\sum_{n>0} \frac{p^n}{1-p^n}\alpha_n z^{-n}\right\}. \tag{2.6}$$

2.2.2 The level-(0,-1) representation

For $u \in \mathbb{C}^\times$, let $\mathcal{F}_u^{(0,-1)}$ be a vector space spanned by $|\lambda\rangle_u$ ($\lambda \in \mathcal{P}$), where

$$\mathcal{P} = \{\lambda = (\lambda_1, \lambda_2, \cdots) \mid \lambda_i \geq \lambda_{i+1},\ \lambda_i \in \mathbb{Z}_{\geq 0},\ \lambda_l = 0 \text{ for sufficiently large } l\ \}.$$

We denote by $\ell(\lambda)$ the length of $\lambda \in \mathcal{P}$ i.e. $\lambda_{\ell(\lambda)} > 0$ and $\lambda_{\ell(\lambda)+1} = 0$. We also set $|\lambda| = \sum_{i\geq 1} \lambda_i$.

Theorem 2.3. *The following action gives a level-(0,-1) representation of $\mathcal{U}$ on $\mathcal{F}_u^{(0,-1)}$.*

$$x^+(z)|\lambda\rangle_u = a^+(p) \sum_{i=1}^{\ell(\lambda)+1} A^+_{\lambda,i}(p)\delta(u_i/z)|\lambda + \mathbf{1}_i\rangle_u, \tag{2.7}$$

$$x^-(z)|\lambda\rangle_u = a^-(p) \sum_{i=1}^{\ell(\lambda)} A^-_{\lambda,i}(p)\delta(t_1 u_i/z)|\lambda - \mathbf{1}_i\rangle_u, \tag{2.8}$$

$$\psi^+(z)|\lambda\rangle_u = \prod_{i=1}^{\ell(\lambda)} \frac{\theta(t_2^{-1}u_i/z)}{\theta(t_1 u_i/z)} \prod_{i=1}^{\ell(\lambda)+1} \frac{\theta(\hbar u_i/z)}{\theta(u_i/z)}|\lambda\rangle_u, \tag{2.9}$$

$$\psi^-(z)|\lambda\rangle_u = \prod_{i=1}^{\ell(\lambda)} \frac{\theta(t_2 z/u_i)}{\theta(t_1^{-1}z/u_i)} \prod_{i=1}^{\ell(\lambda)+1} \frac{\theta(\hbar^{-1}z/u_i)}{\theta(z/u_i)}|\lambda\rangle_u, \tag{2.10}$$

where

$$a^+(p) = (1-t)\frac{(p\hbar;p)_\infty (p/t_2;p)_\infty}{(p;p)_\infty (p/q;p)_\infty}, \quad a^-(p) = (1-t^{-1})\frac{(p/\hbar;p)_\infty (pt_2;p)_\infty}{(p;p)_\infty (pq;p)_\infty},$$

$$A^+_{\lambda,i}(p) = \prod_{j=1}^{i-1} \frac{\theta(t_2 u_i/u_j)\theta(\hbar^{-1}u_i/u_j)}{\theta(t_1^{-1}u_i/u_j)\theta(u_i/u_j)}, \quad A^-_{\lambda,i}(p) = \prod_{j=i+1}^{\ell(\lambda)} \frac{\theta(\hbar^{-1}u_j/u_i)}{\theta(u_j/u_i)} \prod_{j=i+1}^{\ell(\lambda)+1} \frac{\theta(t_2 u_j/u_i)}{\theta(t_1^{-1}u_j/u_i)}.$$

This is an elliptic analogue of a representation given in [5,6]. In [11] it is conjectured that this gives a geometric action of $\mathcal{U}$ on the equivariant elliptic cohomology of the Hilbert schemes $\bigoplus_{n\geq 0} \mathrm{E}_T(\mathrm{Hilb}^n(\mathbb{C}^2))$ under the identification of $|\lambda\rangle_u$ with the fixed point class $[\lambda]$ in $\bigoplus_{n\geq 0} \mathrm{E}_T(\mathrm{Hilb}^n(\mathbb{C}^2))$.

3 Elliptic Stable Envelopes for $\mathrm{E}_T(\mathcal{M}(n,r))$

The elliptic stable envelopes for the equivariant elliptic cohomology of the instanton moduli space $\mathrm{E}_T(\mathcal{M}(n,r))$ were constructed in [4,16].

3.1 The instanton moduli space $\mathcal{M}(n,r)$

Let $\mathcal{M}(n,r)$ be the moduli space of framed rank r torsion free sheaves $\mathcal{S}$ on $\mathbb{P}^2$ with $c_2(\mathcal{S}) = n$. One has a natural action of $G = GL(r) \times GL(2)$ on $\mathcal{M}(n,r)$. Let T be the maximal torus of G and set $A = T \cap GL(r)$. The parameters t_1, t_2 are identified with the generators of the character group of T/A. The rank 1 case is isomorphic to the Hilbert scheme of n-points on $\mathbb{C}^2$.

$$\mathcal{M}(n,1) \cong \mathrm{Hilb}^n(\mathbb{C}^2).$$

Let us consider the case $r = 1$, the Hilbert scheme $\mathcal{H}_n = \mathrm{Hilb}^n(\mathbb{C}^2)$, $A = \mathbb{C}^\times$. We denote the coordinate on A by u such that

$$t_1 = u\hbar^{1/2}, \qquad t_2 = u^{-1}\hbar^{1/2}.$$

There are a finite number of the A-fixed points of $\mathcal{H}_n$ labeled by partitions of n. Let

$$\mathcal{P}_n = \{\ \lambda \in \mathcal{P} \mid |\lambda| = n\ \}.$$

We regard $\lambda \in \mathcal{P}_n$ as a Young diagram with n boxes. For a box $\Box = (i,j) \in \lambda$, we deine

$$c_\Box := i - j, \quad h_\Box := i + j - 2, \quad \rho_\Box := c_\Box - \epsilon h_\Box$$

with $0 < \epsilon \ll 1$. We introduce a canonical ordering on the n boxes of λ by

$$a < b \ \Leftrightarrow\ \rho_a < \rho_b \qquad a, b \in \lambda$$

and define a bijection $\iota : \lambda \to [1,n]$ if $a \in \lambda$ is the $\iota(a)$-th box in this order. In the following we often denote the box a by $\square_{\iota(a)}$ or simply $\iota(a)$.

Let us consider the following presentation of the equivariant K-theory of $\mathcal{H}_n$.

$$\mathrm{K}_T(\mathcal{H}_n) = \mathbb{Z}[x_1^{\pm}, \cdots, x_n^{\pm}, t_1^{\pm}, t_2^{\pm}]^{\mathfrak{S}_n}/R,$$

where $\mathfrak{S}_n$ denotes the symmetrization in x_a's, and R the ideal of Laurent polynomials vanishing at all fixed points in $\mathcal{H}_n^T$. We often loosely use x_a as $x_{\iota(a)}$ for $a \in \lambda$ and vice versa. For $\square = (i,j) \in \lambda$, we set

$$\varphi^\lambda_\square = t_1^{-(j-1)} t_2^{-(i-1)} \in \mathrm{K}_T(pt).$$

The restriction of a K-theory class $f(x_1, \cdots, x_n, t_1, t_2)$ to a fixed point labeled by $\lambda \in \mathcal{P}_n$ is given by

$$i^*_\lambda f(x_1, \cdots, x_n, t_1, t_2) = f(\varphi^\lambda_{\square_1}, \cdots, \varphi^\lambda_{\square_n}, t_1, t_2).$$

Here $i_\lambda : \lambda \to \mathcal{H}_n$ denotes the canonical inclusion of a fixed point.

Let $\mathcal{V}$ be the rank n tautological bundle on $\mathcal{H}_n$. We present $\mathcal{V}$ as

$$\mathcal{V} = x_1 + \cdots + x_n$$

regarding $x_1, \cdots, x_n$ as the Chern roots of $\mathcal{V}$.

For $r \in \mathbb{Z}_{>0}$, fixed points in $\mathcal{M}(n,r)$ are lebeled by a r-tuple partition $\boldsymbol{\lambda} = (\lambda^{(1)}, \cdots, \lambda^{(r)})$ with $|\boldsymbol{\lambda}| = \sum_{i=1}^r |\lambda^{(i)}| = n$. We denote the Chern roots of the rank n tautological bundle $\mathcal{V}$ on $\mathcal{M}(n,r)$ by $\mathbf{x} = (x_1, \cdots, x_n)$ and a coordinate of $A = (\mathbb{C}^\times)^r$ by $\mathbf{u} = (u_1, \cdots, u_r)$. We define a canonical ordering on the n boxes of $\boldsymbol{\lambda}$ by extending the one for each partition $\lambda^{(i)}$ with adding the following condition, for $i < j$

$$a < b \ \Leftrightarrow \ a \in \lambda^{(i)}, b \in \lambda^{(j)}.$$

3.2 Elliptic stable envelopes

The elliptic stable envelopes for $\mathrm{E}_T(\mathcal{H}_n)$ was constructed in [16]. Let $T^{1/2} \in \mathrm{K}_T(\mathcal{H}_n)$ be a polarization of $\mathcal{H}_n$ satisfying

$$T\mathcal{H}_n = T^{1/2} + \hbar (T^{1/2})^\vee \in \mathrm{K}_T(\mathcal{H}_n).$$

For $\lambda \in \mathcal{P}_n$, let us set

$$S_\lambda^{Ell}(x_1, \cdots, x_n, u) := \frac{\prod_{\substack{a,b\in\lambda \\ \rho_a+1<\rho_b}} \theta(t_1 x_a/x_b) \prod_{\substack{a,b\in\lambda \\ \rho_a+1>\rho_b}} \theta(t_2 x_b/x_a) \prod_{\substack{a\in\lambda \\ \rho_a\le 0}} \theta(x_a/u) \prod_{\substack{a\in\lambda \\ \rho_a>0}} \theta(\hbar u/x_a)}{\prod_{\substack{a,b\in\lambda \\ \rho_a<\rho_b}} \theta(x_a/x_b)\theta(\hbar x_a/x_b)}. \quad (3.1)$$

Let $\mathbf{t}$ be a λ-tree, see Definition 1 in Section 4.2 of [16]. Namely, $\mathbf{t}$ is a rooted tree in a Young diagram λ with vertices corresponding to boxes of λ, edges connecting only adjacent boxes and the root at the box $(1,1) \in \lambda$. Let us set (formula (54) in [16]):

$$W_{T^{1/2}}^{Ell}(\mathbf{t}; x_1, \cdots, x_n, u, z) = (-1)^{\kappa_{\mathbf{t}}} \phi\Big(\frac{x_r}{u}, z^n (t_1 t_2)^{\mathsf{v}_r}\Big) \prod_{e\in\mathbf{t}} \phi\Big(\frac{x_{h(e)}\varphi_{t(e)}^{\lambda}}{x_{t(e)}\varphi_{h(e)}^{\lambda}}, z^{\mathsf{w}_e}(t_1 t_2)^{\mathsf{v}_e}\Big)$$

with

$$\phi(x, y) = \frac{\theta(xy)}{\theta(x)\theta(y)},$$

where the product runs over the edges of the tree $\mathbf{t}$ and $h(e) \in \lambda$, $t(e) \in \lambda$ denote the head and tail box of the edge e. And, $\kappa_{\mathbf{t}}, \mathsf{w}_e, \mathsf{v}_e \in \mathbb{Z}$ are certain integers computed from the tree in a combinatorial way, we refer to Sections 4.2-4.5 in [16] for definitions of these integers. The symbol z denotes the Kähler parameter in $\mathrm{E}_T(\mathcal{H}_n)$ [1]. In the below we identify the dynamical parameter $\mathfrak{z}$ with the Kähler parameter z. Finally, let Υ_λ be the set of λ-trees without ⅃-shaped subgraphs, see section 4.6 in [16]. Then, we have:

Theorem 3.1. *[16] The elliptic stable envelope of a fixed point $\lambda \in \mathcal{H}_n^A$ is given by*

$$\mathrm{Stab}_{\mathfrak{C},T^{1/2}}(\lambda; z) = \mathrm{Sym}\left(S_\lambda^{Ell}(x_1, \cdots, x_n, u) \sum_{\delta\in\Upsilon_\lambda} W_{T^{1/2}}^{Ell}(\mathbf{t}_\delta; x_1, \cdots, x_n, u; z) \right), \quad (3.2)$$

where the symbol Sym *stands for symmetrization over $x_1, \cdots, x_n$. The chamber $\mathfrak{C}$ is taken as a stability condition $t_1/t_2 > 0$.*

The elliptic stable envelope for $\mathrm{E}_T(\mathcal{M}(n,r))$ is constructed by taking the shuffle product [2] of those for the Hilbert schemes. Let λ', λ'' be two partitions with $|\lambda'| = n'$, $|\lambda''| = n''$ and consider the elliptic stable envelopes $\mathrm{Stab}_{T^{1/2},\mathfrak{C}'}(\lambda'; z')$ and $\mathrm{Stab}_{T^{1/2},\mathfrak{C}''}(\lambda''; z'')$ for $\mathrm{E}_T(\mathcal{H}_{n'})$ and $\mathrm{E}_T(\mathcal{H}_{n''})$ with the equivariant parameters u_1, u_2, respectively. Here one takes $\mathfrak{C}' = \mathfrak{C}''$ as $t_1/t_2 > 0$. We take the canonical ordering on the $n' + n''$ boxes in the double partition

(λ', λ'') as defined in Sec.3.1. Then the following $\mathrm{Stab}_{\mathfrak{C},T^{1/2}}((\lambda',\lambda'');z)$ gives the elliptic stable envelope for $\mathrm{E}_T(\mathcal{M}(n'+n'',2))$ with the chamber $\mathfrak{C}$ given by $|u_1| \ll |u_2|$.

$$
\begin{aligned}
&\mathrm{Stab}_{\mathfrak{C},T^{1/2}}((\lambda',\lambda'');z)\\
&= \mathrm{Sym}_{\{x_a\}_{a\in(\lambda',\lambda'')}} \left(\prod_{a\in\lambda',b\in\lambda''} \frac{\theta(t_1x'_a/x''_b)\theta(t_2x'_a/x''_b)}{\theta(x'_a/x''_b)\theta(\hbar x'_a/x''_b)} \prod_{b\in\lambda''}\theta(\hbar u_1/x''_b)\prod_{a\in\lambda'}\theta(x'_a/u_2) \right.\\
&\qquad\qquad \left. \times\ \mathrm{Stab}_{\mathfrak{C}',T^{1/2}}(\lambda';z'\hbar^{-1})\mathrm{Stab}_{\mathfrak{C}'',T^{1/2}}(\lambda'';z'') \right). \qquad (3.3)
\end{aligned}
$$

Here x'_a $(a\in\lambda')$ and x''_b $(b\in\lambda'')$ are the Chern roots for the tautological bundle on $\mathcal{H}_{n'}$ and $\mathcal{H}_{n''}$, respectively, and we set $\{x_a\}_{a\in(\lambda',\lambda'')} = \{x'_a\}_{a\in\lambda'} \cup \{x''_b\}_{b\in\lambda''}$. The formula (3.3) is called the shuffle product [2]. By taking the shuffle product r times, one obtains the elliptic stable envelopes $\mathrm{Stab}_{\mathfrak{C},T^{1/2}}(\boldsymbol{\lambda};\mathfrak{z})$ for $\mathrm{E}_T(\mathcal{M}(n,r))$. Here $\boldsymbol{\lambda} = (\lambda^{(1)},\cdots,\lambda^{(r)})$ denotes a r-tuple partition with $|\boldsymbol{\lambda}| = \sum_{i=1}^r |\lambda^{(i)}| = n$. An explicit formula for $\mathrm{Stab}_{\mathfrak{C},T^{1/2}}(\boldsymbol{\lambda};\mathfrak{z})$ is given in Proposition 3.2 in [12].

In the next sections, we use $\widehat{\mathrm{Stab}}_{\mathfrak{C},T^{1/2}}(\boldsymbol{\lambda};\mathfrak{z})$ defined by

$$
\mathrm{Stab}_{\mathfrak{C},T^{1/2}}(\boldsymbol{\lambda};\mathfrak{z}) = (-)^{\varepsilon(\boldsymbol{\lambda},r)}\Theta(T^{1/2})\widehat{\mathrm{Stab}}_{\mathfrak{C},T^{1/2}}(\boldsymbol{\lambda};\mathfrak{z}) \qquad (3.4)
$$

where $\varepsilon(\boldsymbol{\lambda},r) = \displaystyle\sum_{i=1}^{r} \sum_{\substack{a\in\boldsymbol{\lambda}\\ \rho_a>\rho_{r_i}}} 1$ and

$$
\Theta(T^{1/2}) = \prod_{\substack{a,b\in\boldsymbol{\lambda}\\ \rho_a\neq\rho_b}} \frac{\theta(t_1x_a/x_b)}{\theta(x_a/x_b)} \prod_{i=1}^{r}\prod_{a\in\boldsymbol{\lambda}}\theta(x_a/u_i).
$$

Proposition 3.2. *The shuffle product formula for* $\widehat{\mathrm{Stab}}_{\mathfrak{C},T^{1/2}}(\boldsymbol{\lambda};\mathfrak{z})$*'s is given by*

$$
\begin{aligned}
&\widehat{\mathrm{Stab}}_{\mathfrak{C},T^{1/2}}((\boldsymbol{\lambda}',\boldsymbol{\lambda}'');\mathfrak{z})\\
&= \mathrm{Sym}_{\{x_a\}_{a\in(\boldsymbol{\lambda}',\boldsymbol{\lambda}'')}} \left(\prod_{a\in\boldsymbol{\lambda}',b\in\boldsymbol{\lambda}''} \frac{\theta(x''_b/x'_a)\theta(t_2x'_a/x''_b)}{\theta(t_1x''_b/x'_a)\theta(\hbar x'_a/x''_b)} \prod_{i=1}^{r'}\prod_{b\in\boldsymbol{\lambda}''}\left(-\frac{\theta(\hbar u'_i/x''_b)}{\theta(x''_b/u'_i)}\right) \right.\\
&\qquad\qquad \left. \times\widehat{\mathrm{Stab}}_{\mathfrak{C}',T^{1/2}}(\boldsymbol{\lambda}';\mathfrak{z}\hbar^{-r''-2n''})\widehat{\mathrm{Stab}}_{\mathfrak{C}'',T^{1/2}}(\boldsymbol{\lambda}'';\mathfrak{z}) \right). \qquad (3.5)
\end{aligned}
$$

We also use the elliptic stable envelopes for $\mathrm{E}_T(\mathcal{M}(n,r))$ with the opposite polarization $T^{1/2}_{opp}=\hbar(T^{1/2})^\vee$, the elliptic nome p^* and the Kähler parameter $\mathfrak{z}^{*-1}$:

$$\mathrm{Stab}^*_{\mathfrak{C},T^{1/2}_{opp}}(\boldsymbol{\lambda};\mathfrak{z}^{*-1}):=\mathrm{Stab}_{\mathfrak{C},T^{1/2}}(\boldsymbol{\lambda};\mathfrak{z})\big|_{T^{1/2}\mapsto T^{1/2}_{opp},\ p\mapsto p^*,\ \mathfrak{z}\mapsto\mathfrak{z}^{*-1}} \tag{3.6}$$

and its hatted version defined by

$$\mathrm{Stab}^*_{\mathfrak{C},T^{1/2}_{opp}}(\boldsymbol{\lambda};\mathfrak{z}^{*-1})=(-)^{\varepsilon^*(\boldsymbol{\lambda},r)}\Theta(T^{1/2}_{opp})\widehat{\mathrm{Stab}}^*_{\mathfrak{C},T^{1/2}_{opp}}(\boldsymbol{\lambda};\mathfrak{z}^{*-1}), \tag{3.7}$$

where $\varepsilon^*(\boldsymbol{\lambda},r)=\sum_{i=1}^{r}\sum_{\substack{a\in\boldsymbol{\lambda}\\ \rho_a\le\rho_{r_i}}}1$ and

$$\Theta(T^{1/2}_{opp})=\prod_{\substack{a,b\in\boldsymbol{\lambda}\\ \rho_a\neq\rho_b}}\frac{\theta^*(t_2x_a/x_b)}{\theta^*(\hbar x_a/x_b)}\prod_{i=1}^{r}\prod_{a\in\boldsymbol{\lambda}}\theta^*(\hbar u_i/x_a).$$

4 Vertex Operators of $U_{t_1,t_2,p}(\mathfrak{gl}_{1,tor})$

We first define the basic vertex operators which correspond to $\mathrm{Hilb}^n(\mathbb{C}^2)$ and then construct the ones for general $\mathcal{M}(n,r)$ by composing the basic ones.

4.1 OPE of the elliptic currents

Let us consider the level $(1,N)$ representation given in Sec.2.2.1, on which $p^*=p\hbar^{-1}$. For the elliptic currents $x^+(u),x^-(v)$, one gets the following operator product expansion (OPE).

$$x^+(u)x^+(v)=<x^+(u)x^+(v)>^{sym}\frac{\theta^*(t_1v/u)\theta^*(\hbar u/v)}{\theta^*(v/u)\theta^*(t_2u/v)}:x^+(u)x^+(v):, \tag{4.1}$$

$$x^-(u)x^-(v)=<x^-(u)x^-(v)>^{sym}\frac{\theta(v/u)\theta(t_2u/v)}{\theta(t_1v/u)\theta(\hbar u/v)}:x^-(u)x^-(v):, \tag{4.2}$$

with

$$<x^+(u)x^+(v)>^{sym}=t_1^{-1}\frac{(p^*t_2v/u,p^*t_2u/v,v/u,u/v;p^*)_\infty}{(p^*\hbar v/u,p^*\hbar u/v,v/t_1u,u/t_1v;p^*)_\infty}, \tag{4.3}$$

$$<x^-(u)x^-(v)>^{sym}=t_1^{-1}\frac{(pt_1^{-1}v/u,pt_1^{-1}u/v,\hbar v/u,\hbar u/v;p)_\infty}{(pv/u,pu/v,t_2v/u,t_2u/v;p)_\infty}. \tag{4.4}$$

Here we set

$$(a_1,\cdots,a_M;p)_\infty=\prod_{i=1}^{M}(a_i;p)_\infty.$$

4.2 Vertex operators for $\mathrm{Hilb}^n(\mathbb{C}^2)$

Let us define

$$\Phi_\emptyset(u) := (-u)^{-\alpha} e^{-\Lambda_0} \ \exp\left\{-\sum_{m>0} \frac{1}{\kappa_m} \alpha'_{-m} (\hbar^{1/2} u)^m\right\} \exp\left\{\sum_{m>0} \frac{1}{\kappa_m} \alpha'_m (\hbar^{1/2} u)^{-m}\right\}, \quad (4.5)$$

$$\Psi^*_\emptyset(u) := (-u)^{\alpha} e^{\Lambda_0 - Q/2} \ \exp\left\{\sum_{m>0} \frac{1}{\kappa_m} \alpha_{-m} (\hbar^{1/2} u)^m\right\} \exp\left\{-\sum_{m>0} \frac{1}{\kappa_m} \alpha_m (\hbar^{1/2} u)^{-m}\right\}. \quad (4.6)$$

One can show the following commutation relations.

Proposition 4.1.

$$\Psi^*_\emptyset(u) x^+(v) = -\frac{\theta^*(v/u)}{\theta^*(\hbar u/v)} x^+(v) \Psi^*_\emptyset(u), \quad (4.7)$$

$$\Phi_\emptyset(u) x^-(v) = -\frac{\theta(\hbar u/v)}{\theta(v/u)} x^-(v) \Phi_\emptyset(u), \quad (4.8)$$

$$x^+(v)\Phi_\emptyset(u) = \Phi_\emptyset(u) x^+(v), \qquad x^-(v)\Psi^*_\emptyset(u) = \Psi^*_\emptyset(u) x^-(v), \quad (4.9)$$

$$\mathfrak{z}\Phi_\emptyset(u) = \hbar^{-1}\Phi_\emptyset(u)\mathfrak{z}, \qquad \Psi^*_\emptyset(u)\mathfrak{z}^* = \hbar\, \mathfrak{z}^* \Psi^*_\emptyset(u), \quad (4.10)$$

$$[x^\pm(x), \mathfrak{z}] = [\Psi^*_\emptyset, \mathfrak{z}] = 0, \qquad [x^\pm(x), \mathfrak{z}^*] = [\Phi_\emptyset(u), \mathfrak{z}^*] = 0. \quad (4.11)$$

Definition 4.2. *We define the type I $\Phi(u)$ and the type II dual $\Psi^*(u)$ vertex operators to be the following linear maps.*

$$\Phi(u) \ : \ \mathcal{F}^{(1,N)}_v \ \to \ \mathcal{F}^{(0,-1)}_u \widetilde{\otimes} \mathcal{F}^{(1,N+1)}_{-v/u},$$
$$\Psi^*(u) \ : \ \mathcal{F}^{(1,N)}_v \widetilde{\otimes} \mathcal{F}^{(0,-1)}_u \ \to \ \mathcal{F}^{(1,N-1)}_{-vu},$$

with

$$\Phi(u) = \sum_{\lambda \in \mathcal{P}} |\lambda\rangle_u \widetilde{\otimes} \Phi_\lambda(u),$$
$$\Phi_\lambda(u) = \int_{\mathcal{C}} \prod_{a\in\lambda} dx_a : \prod_{a\in\lambda} x^-(x_a) : \Phi_\emptyset(u) \prod_{\rho_a<\rho_b} < x^-(x_a) x^-(x_b) >^{sym} \ \widehat{\mathrm{Stab}}_{\mathfrak{C},T^{1/2}}(\lambda; \mathfrak{z}),$$

and

$$\Psi^*_\lambda(u)\xi = \Psi^*(u)(\xi \widetilde{\otimes} |\lambda\rangle_u), \qquad \forall \xi \in \mathcal{F}^{(1,N)}_v,$$
$$\Psi^*_\lambda(u) = \int_{\mathcal{C}^*} \prod_{a\in\lambda} dx_a \ \widehat{\mathrm{Stab}}^*_{\mathfrak{C},T^{1/2}_{opp}}(\lambda; \mathfrak{z}^{*-1}) \ : \prod_{a\in\lambda} x^+(x_a) : \Psi^*_\emptyset(u) \prod_{\rho_a<\rho_b} < x^+(x_a) x^+(x_b) >^{sym} .$$

The integration cycles $\mathcal{C}, \mathcal{C}^*$ are chosen appropriately depending on the situation of the application. We call $\Phi_\lambda(u)$ (resp. $\Psi^*_\lambda(u)$) with $\lambda \in \mathcal{P}_n$ the type I (resp. type II dual) vertex operator for $\mathrm{Hilb}^n(\mathbb{C}^2)$.

4.3 Vertex operators for $\mathcal{M}(n, r)$

Let λ', λ'' be two partitions with $|\lambda'| = n', |\lambda''| = n''$. Let us consider the following composition of the two type I vertex operators for the Hilbert schemes.

$$\begin{aligned}
&\Phi_{\lambda'}(u_1)\Phi_{\lambda''}(u_2) \\
&= \int_{\mathcal{C}} \prod_{a\in\lambda'} dx'_a \int_{\mathcal{C}} \prod_{b\in\lambda''} dx''_b : \prod_{a\in\lambda'} x^-(x'_a) : \Phi_\emptyset(u_1) \prod_{\rho_a<\rho_b} < x^-(x'_a)x^-(x'_b) >^{sym} \widehat{\mathrm{Stab}}_{\mathfrak{C}',T^{1/2}}(\lambda'; \mathfrak{z}) \\
&\quad \times : \prod_{b\in\lambda''} x^-(x''_b) : \Phi_\emptyset(u_2) \prod_{\rho_c<\rho_d} < x^-(x''_c)x^-(x''_d) >^{sym} \widehat{\mathrm{Stab}}_{\mathfrak{C}'',T^{1/2}}(\lambda''; \mathfrak{z}).
\end{aligned}$$

Here the chambers $\mathfrak{C}', \mathfrak{C}''$ are the same and taken as the stability condition $t_1/t_2 > 0$. We also assume $|u_1| \ll |u_2|$. In a similar way to the type A linear quiver case studied in [9], let us arrange the order of the elements in the integrand as follows.

1. Move $\widehat{\mathrm{Stab}}_{\mathfrak{C}',T^{1/2}}(\lambda'; \mathfrak{z})$ to the right of all operators.

2. Move $: \prod_{b\in\lambda''} x^-(x''_b) :$ to the left of $\Phi_\emptyset(u_1)$ by using the formula (4.8).

3. Make $: \prod_{a\in\lambda'} x^-(x'_a) :: \prod_{b\in\lambda''} x^-(x''_b) :$ totally normal ordered product by the formula
$$: \prod_{a\in\lambda'} x^-(x'_a) :: \prod_{b\in\lambda''} x^-(x''_b) := \prod_{a\in\lambda', b\in\lambda''} < x^-(x'_a)x^-(x''_b) > : \prod_{a\in(\lambda',\lambda'')} x^-(x_a) : .$$
Here we set $\{x_a\}_{a\in(\lambda',\lambda'')} = \{x'_a\}_{a\in\lambda'} \cup \{x''_b\}_{b\in\lambda''}$. We define the order of boxes in the different partitions by $\rho_a < \rho_b$ for $a \in \lambda', b \in \lambda''$.

4. Divide $< x^-(x'_a)x^-(x''_b) >$ into the symmetric and the non-symmetric parts as (4.2).

5. Symmetrize the integrand over $\{x_a\}_{a\in(\lambda',\lambda'')}$.

One thus obtains

$$\begin{aligned}
\Phi_{\lambda'}(u_1)\Phi_{\lambda''}(u_2) \;&=\; \int_{\mathcal{C}\times\mathcal{C}} \prod_{a\in(\lambda',\lambda'')} dx_a \;:\; \prod_{a\in(\lambda',\lambda'')} x^-(x_a) : \Phi_\emptyset(u_1)\Phi_\emptyset(u_2) \\
&\quad \times \prod_{\substack{a,b\in(\lambda',\lambda'') \\ \rho_a<\rho_b}} < x^-(x_a)x^-(x_b) >^{sym} \; \widehat{\mathrm{Stab}}_{\mathfrak{C},T^{1/2}}((\lambda', \lambda''); \mathfrak{z}),
\end{aligned}$$

where we set

$$\widehat{\mathrm{Stab}}_{\mathfrak{C},T^{1/2}}((\lambda',\lambda'');\mathfrak{z}) = \mathrm{Sym}_{\{x_a\}_{a\in(\lambda',\lambda'')}}\left(\prod_{a\in\lambda',b\in\lambda''}\frac{\theta(x''_b/x'_a)\theta(t_2x'_a/x''_b)}{\theta(t_1x''_b/x'_a)\theta(\hbar x'_a/x''_b)}\prod_{b\in\lambda''}\left(-\frac{\theta(\hbar u_1/x''_b)}{\theta(x''_b/u_1)}\right)\right.$$
$$\left.\times\widehat{\mathrm{Stab}}_{\mathfrak{C}',T^{1/2}}(\lambda';\mathfrak{z}\hbar^{-1})\widehat{\mathrm{Stab}}_{\mathfrak{C}'',T^{1/2}}(\lambda'';\mathfrak{z})\right).$$

Then it is remarkable that $\widehat{\mathrm{Stab}}_{\mathfrak{C},T^{1/2}}((\lambda',\lambda'');\mathfrak{z})$ coincides with the hatted version of the elliptic stable envelope for $\mathrm{E}_T(\mathcal{M}(n'+n'',2))$ given in (3.3). We hence regard the composition $\Phi_{\lambda'}(u_1)\Phi_{\lambda''}(u_2)$ as a vertex operator for $\mathcal{M}(n'+n'',2)$.

In general, one obtains the type I vertex operator for $\mathcal{M}(n,r)$ by composing the basic vertex operators repeatedly.

$$(\mathrm{id}\widetilde{\otimes}\cdots\widetilde{\otimes}\mathrm{id}\widetilde{\otimes}\Phi(u_1))\circ\cdots(\mathrm{id}\widetilde{\otimes}\Phi(u_{r-1}))\circ\Phi(u_r)$$
$$:\ \mathcal{F}^{(1,N)}_v\ \to\ \mathcal{F}^{(0,-1)}_{u_r}\widetilde{\otimes}\cdots\widetilde{\otimes}\mathcal{F}^{(0,-1)}_{u_1}\widetilde{\otimes}\mathcal{F}^{(1,N+r)}_{(-)^rv/u_1\cdots u_r}.$$

Defining the components $\Phi_{\boldsymbol{\lambda}}(u_1,\cdots,u_r)$ by

$$(\mathrm{id}\widetilde{\otimes}\cdots\widetilde{\otimes}\mathrm{id}\widetilde{\otimes}\Phi(u_1))\circ\cdots(\mathrm{id}\widetilde{\otimes}\Phi(u_{r-1}))\circ\Phi(u_r)$$
$$=\sum_{n\in\mathbb{Z}_{\geq0}}\sum_{\substack{\boldsymbol{\lambda}=(\lambda^{(1)},\cdots,\lambda^{(r)})\\|\boldsymbol{\lambda}|=n}}|\lambda^{(r)}\rangle_{u_r}\widetilde{\otimes}\cdots\widetilde{\otimes}|\lambda^{(1)}\rangle_{u_1}\widetilde{\otimes}\ \Phi_{\boldsymbol{\lambda}}(u_1,\cdots,u_r), \tag{4.12}$$

one finds

$$\Phi_{\boldsymbol{\lambda}}(u_1,\cdots,u_r) = \Phi_{\lambda^{(1)}}(u_1)\cdots\Phi_{\lambda^{(r)}}(u_r)$$
$$=\int_{\mathcal{C}^r}\prod_{a\in\boldsymbol{\lambda}}dx_a:\prod_{a\in\boldsymbol{\lambda}}x^-(x_a):\ \Phi_\emptyset(u_1)\cdots\Phi_\emptyset(u_r)\prod_{\substack{a,b\in\boldsymbol{\lambda}\\\rho_a<\rho_b}}<x^-(x_a)x^-(x_b)>^{sym}\ \widehat{\mathrm{Stab}}_{\mathfrak{C},T^{1/2}}(\boldsymbol{\lambda};\mathfrak{z}), \tag{4.13}$$

where $\widehat{\mathrm{Stab}}_{\mathfrak{C},T^{1/2}}(\boldsymbol{\lambda};\mathfrak{z})$ is the hatted version of the elliptic stable envelope for $\mathrm{E}_T(\mathcal{M}(n,r))$ satisfying the shuffle product formula (3.5). We hence regard $\Phi_{\boldsymbol{\lambda}}(u_1,\cdots,u_r)$ as the type I vertex operator for $\mathcal{M}(n,r)$.

We also have a similar construction for the type II dual vertex operators.

5 The $\mathbb{K}$-Theoretic Vertex functions for $\mathcal{M}(n,r)$

A vertex function for a quiver variety X is a generating function of counting quasi maps from $\mathbb{P}^1$ to X [15]. We show that the vacuum expectation value of the vertex operator constructed in the last section gives the K-theoretic vertex function for $X = \mathcal{M}(n,r)$.

Let λ, μ be two partitions with $|\lambda| = |\mu| = n$. There is a bijection ς from boxes in λ to those in μ defined by $\varsigma(a) = b \in \mu$ for $a \in \lambda$ if $\iota(a) = \iota(b)$. Here ι is defined in Sec.3.1. For a box $a = (i,j) \in \mu$, we set $\varphi^\mu_a = t_1^{-(j-1)} t_2^{-(i-1)}$ as before. For the Chern root x_a $(a \in \lambda)$ we take the Jackson integral

$$\int_0^{\varphi^\mu_{\varsigma(a)}} d_p x_a f(x_a) = (1-p)\varphi^\mu_{\varsigma(a)} \sum_{d\in\mathbb{N}} f(\varphi^\mu_{\varsigma(a)} p^d) p^d$$

in the vertex operators for $\mathrm{Hilb}^n(\mathbb{C}^2)$. Let $|0\rangle^{(1,N)}_v$ be the vacuum state (2.2) in $\mathcal{F}^{(1,N)}_v$ and ${}^{(1,N)}_{\ \ v}\langle 0|$ be the dual state satisfying

$${}^{(1,N)}_{\ \ v}\langle 0||0\rangle^{(1,N)}_v = 1.$$

One finds that the following normalized vacuum expectation value gives the K-theoretic vertex function for $\mathrm{Hilb}^n(\mathbb{C}^2)$.

$$\begin{aligned} V^\mu_\lambda(\hbar^{-1}u, \mathfrak{z}) &= \frac{1}{\mathcal{N}_\mu} {}^{(1,N-1)}_{\ -v/u}\langle 0|\Phi_\lambda(u)|0\rangle^{(1,N)}_v \\ &= \sum_{\mathbf{d}\in\mathbb{N}^n} (\hbar p^{-N-1}\mathfrak{z})^{-\sum_a d_a} \prod_{a\in\lambda} \frac{(\hbar\varphi^\mu_{\varsigma(a)}/u;p)_{d_a}}{(p\varphi^\mu_{\varsigma(a)}/u;p)_{d_a}} \prod_{\substack{a,b\in\lambda\\ \rho_a\neq\rho_b}} \frac{(p\varphi^\mu_{\varsigma(a)}/\varphi^\mu_{\varsigma(b)};p)_{d_a-d_b}(t_2\varphi^\mu_{\varsigma(a)}/\varphi^\mu_{\varsigma(b)};p)_{d_a-d_b}}{(pt_1^{-1}\varphi^\mu_{\varsigma(a)}/\varphi^\mu_{\varsigma(b)};p)_{d_a-d_b}(\hbar\varphi^\mu_{\varsigma(a)}/\varphi^\mu_{\varsigma(b)};p)_{d_a-d_b}}. \end{aligned}$$

Here we take $\mathcal{N}_\mu$ as the the specialization of the integrand of ${}_{-v/u}\langle 0|\Phi_\lambda(\hbar^{-1}u)|0\rangle^{(1,N)}_v$ to $x_a = \varphi^\mu_{\varsigma(a)}$ $(a \in \lambda)$. We also use the following quasi-periodicity of the ESE.

$$\widehat{\mathrm{Stab}}_{\mathfrak{C},T^{1/2}}(\lambda;\mathfrak{z})\Big|_{\substack{x_a=\varphi^\mu_{\varsigma(a)}p^{d_a}\\ (a\in\lambda)}} = \mathfrak{z}^{-\sum_{a\in\lambda} d_a} \times \widehat{\mathrm{Stab}}_{\mathfrak{C},T^{1/2}}(\lambda;\mathfrak{z})\Big|_\mu,$$

where

$$\widehat{\mathrm{Stab}}_{\mathfrak{C},T^{1/2}}(\lambda;\mathfrak{z})\Big|_\mu := \widehat{\mathrm{Stab}}_{\mathfrak{C},T^{1/2}}(\lambda;\mathfrak{z})\Big|_{\substack{x_a=\varphi^\mu_{\varsigma(a)}\\ (a\in\lambda)}}.$$

The special case $\mu = \lambda$, hence $\varsigma = \mathrm{id}$, of this expression coincides with the formula obtained geometrically in [17].

In the same way, the vertex function for $\mathcal{M}(n,r)$ is obtained from the vertex operator (4.13). One obtains the following result.

$$V^{\boldsymbol{\mu}}_{\boldsymbol{\lambda}}(u_1/\hbar,\cdots,u_r/\hbar,\mathfrak{z}) := \frac{1}{\mathcal{N}_{\boldsymbol{\mu}}}{}_{(-\hbar)^r v/u_1\cdots u_r}^{(1,N-r)}\langle 0|\Phi_{\boldsymbol{\lambda}}(u_1/\hbar,\cdots,u_r/\hbar)|0\rangle^{(1,N)}_v$$
$$= \sum_{\mathbf{d}\in\mathbb{N}^n}\prod_{i=1}^{r}(\hbar^{-(r-2)}p^{r-N-1}\mathfrak{z})^{-\sum_{a\in\lambda^{(i)}}d_a}\prod_{i=1}^{r}\prod_{a\in\boldsymbol{\lambda}}\frac{(\hbar\varphi^{\boldsymbol{\mu}}_{\varsigma(a)}/u_i;p)_{d_a}}{(p\varphi^{\boldsymbol{\mu}}_{\varsigma(a)}/u_i;p)_{d_a}}$$
$$\times\prod_{\substack{a,b\in\boldsymbol{\lambda}\\ \rho_a\neq\rho_b}}\frac{(p\varphi^{\boldsymbol{\mu}}_{\varsigma(a)}/\varphi^{\boldsymbol{\mu}}_{\varsigma(b)};p)_{d_a-d_b}(t_2\varphi^{\boldsymbol{\mu}}_{\varsigma(a)}/\varphi^{\boldsymbol{\mu}}_{\varsigma(b)};p)_{d_a-d_b}}{(pt_1^{-1}\varphi^{\boldsymbol{\mu}}_{\varsigma(a)}/\varphi^{\boldsymbol{\mu}}_{\varsigma(b)};p)_{d_a-d_b}(\hbar\varphi^{\boldsymbol{\mu}}_{\varsigma(a)}/\varphi^{\boldsymbol{\mu}}_{\varsigma(b)};p)_{d_a-d_b}}.$$

Here $\mathbf{d}=(d_a),\ a\in\boldsymbol{\lambda}=(\lambda^{(1)},\cdots,\lambda^{(r)})$. This agrees with the formula in [4].

6 Exchange Relations of the Vertex Operators

Let $(\mathcal{P}^2)_n = \{\boldsymbol{\alpha}=(\alpha',\alpha'')\in\mathcal{P}\times\mathcal{P}\mid|\boldsymbol{\alpha}|=|\alpha'|+|\alpha''|=n\ \}$. For $\boldsymbol{\alpha}=(\alpha',\alpha'')$, let $\bar{\boldsymbol{\alpha}}=(\alpha'',\alpha')$. We define the elliptic dynamical instanton R-matrix $R_{T^{1/2}}(u_1,u_2;\mathfrak{z})\in\mathrm{End}_{\mathbb{F}}(\mathcal{F}^{(0,-1)}_{u_1}\widetilde{\otimes}\mathcal{F}^{(0,-1)}_{u_2})$ as the following transition matrix of the elliptic stable envelopes. For $\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}\in(\mathcal{P}^2)_n$,

$$R_{T^{1/2}}(u_1,u_2;\mathfrak{z})|\beta'\rangle_{u_1}\widetilde{\otimes}|\beta''\rangle_{u_2} = \sum_{\boldsymbol{\alpha}\in(\mathcal{P}^2)_n}R_{T^{1/2}}(u_1,u_2;\mathfrak{z})^{\boldsymbol{\beta}}_{\boldsymbol{\alpha}}|\alpha'\rangle_{u_1}\widetilde{\otimes}|\alpha''\rangle_{u_2}, \tag{6.1}$$
$$R_{T^{1/2}}(u_1,u_2;\mathfrak{z})^{\boldsymbol{\beta}}_{\boldsymbol{\alpha}} = \mu(u_1/u_2)\bar{R}_{T^{1/2}}(u_1,u_2;\mathfrak{z})^{\boldsymbol{\beta}}_{\boldsymbol{\alpha}}, \tag{6.2}$$
$$\widehat{\mathrm{Stab}}_{\bar{\mathfrak{C}},T^{1/2}}(\bar{\boldsymbol{\alpha}};\mathfrak{z})\big|_{\bar{\boldsymbol{\gamma}}} = \sum_{\boldsymbol{\beta}\in(\mathcal{P}^2)_n}\widehat{\mathrm{Stab}}_{\mathfrak{C},T^{1/2}}(\boldsymbol{\beta};\mathfrak{z})\big|_{\boldsymbol{\gamma}}\ \bar{R}_{T^{1/2}}(u_1,u_2;\mathfrak{z})^{\boldsymbol{\beta}}_{\boldsymbol{\alpha}}. \tag{6.3}$$

Here $\mu(u)$ is a scalar function defined by

$$\mu(u_1/u_2)\Phi_{\emptyset}(u_1)\Phi_{\emptyset}(u_2) = \Phi_{\emptyset}(u_2)\Phi_{\emptyset}(u_1). \tag{6.4}$$

It is explicitly calculated by using (4.5) as

$$\mu(u) = \frac{\Gamma(\hbar u;t_1,t_2,p)}{\Gamma(pu;t_1,t_2,p)},$$

where $\Gamma(z;t_1,t_2,p)$ denotes the triple Gamma function defined by

$$\Gamma(z;t_1,t_2,p) = (z;t_1,t_2,p)_\infty(t_1t_2p/z;t_1,t_2,p)_\infty,$$
$$(z;t_1,t_2,p)_\infty = \prod_{m_1,m_2,m_3=0}^{\infty}(1-zt_1^{m_1}t_2^{m_2}p^{m_3}).$$

By definition, $R_{T^{1/2}}(u_1,u_2;\mathfrak{z})$ preserves the representation level w.r.t. $C=\hbar^{c/2}$:

$$[R_{T^{1/2}}(u_1,u_2;\mathfrak{z}),c^{(1)}+c^{(2)}]=0. \tag{6.5}$$

In addition, we assume the property

$$R_{T^{1/2}}(u_1,u_2;\mathfrak{z}\hbar^{c^{(1)}+c^{(2)}})=R_{T^{1/2}}(u_1,u_2;\mathfrak{z}). \tag{6.6}$$

We use this in Proposition 6.1, 6.3 and as a consistency condition in a derivation of the dynamical Yang-Baxter equation (6.12). See [12] for detail. Note also that $\widehat{\mathrm{Stab}}_{\mathfrak{C},T^{1/2}}(\boldsymbol{\alpha};\mathfrak{z})$ is depend on u_1,u_2 only through the chamber $\mathfrak{C}$: $u_1\ll u_2$ essentially. Hence for any $a\in\mathbb{C}^\times$

$$R_{T^{1/2}}(au_1,au_2;\mathfrak{z})=R_{T^{1/2}}(u_1,u_2;\mathfrak{z}). \tag{6.7}$$

Proposition 6.1. *The type I vertex operators satisfy the following exchange relation.*

$$\Phi_{\omega''}(u_2)\Phi_{\omega'}(u_1)=\sum_{\boldsymbol{\lambda}=(\lambda',\lambda'')\in(\mathcal{P}^2)_n}R_{T^{1/2}}(u_1,u_2;\mathfrak{z})_{\boldsymbol{\omega}}^{\boldsymbol{\lambda}}\ \Phi_{\lambda'}(u_1)\Phi_{\lambda''}(u_2),$$

where $\boldsymbol{\omega}=(\omega',\omega'')\in(\mathcal{P}^2)_n$.

Similarly, let us define $R^*_{T^{1/2}_{opp}}(u_1,u_2;\mathfrak{z}^{*-1})\in\mathrm{End}_{\mathbb{F}}(\mathcal{F}^{(0,-1)}_{u_1}\widetilde{\otimes}\mathcal{F}^{(0,-1)}_{u_2})$ as the following transition matrix of the elliptic stable envelopes $\widehat{\mathrm{Stab}}^*_{\mathfrak{C},T^{1/2}_{opp}}(\bullet;\mathfrak{z}^{*-1})$. For $\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}\in(\mathcal{P}^2)_n$,

$$R^*_{T^{1/2}_{opp}}(u_1,u_2;\mathfrak{z}^{*-1})_{\boldsymbol{\alpha}}^{\boldsymbol{\beta}}=\mu^*(u_1/u_2)\bar{R}^*_{T^{1/2}_{opp}}(u_1,u_2;\mathfrak{z}^{*-1})_{\boldsymbol{\alpha}}^{\boldsymbol{\beta}}, \tag{6.8}$$

$$\widehat{\mathrm{Stab}}^*_{\bar{\mathfrak{C}},T^{1/2}_{opp}}(\bar{\boldsymbol{\alpha}};\mathfrak{z}^{*-1})\big|_{\bar{\boldsymbol{\gamma}}}=\sum_{\boldsymbol{\beta}\in(\mathcal{P}^2)_n}\widehat{\mathrm{Stab}}^*_{\mathfrak{C},T^{1/2}_{opp}}(\boldsymbol{\beta};\mathfrak{z}^{*-1})\big|_{\boldsymbol{\gamma}}\ \bar{R}^*_{T^{1/2}_{opp}}(u_1,u_2;\mathfrak{z}^{*-1})_{\boldsymbol{\alpha}}^{\boldsymbol{\beta}}, \tag{6.9}$$

where $\mu^*(u)$ is a scalar function satisfying

$$\Psi^*_\emptyset(u_1)\Psi^*_\emptyset(u_2)=\mu^*(u_1/u_2)\Psi^*_\emptyset(u_2)\Psi^*_\emptyset(u_1). \tag{6.10}$$

Explicitly it is given by

$$\mu^*(u)=\frac{\Gamma(p^*\hbar u;t_1,t_2,p^*)}{\Gamma(u;t_1,t_2,p^*)}.$$

Proposition 6.2.

$$R_{T^{1/2}}(u_1,u_2;\mathfrak{z})={}^tR_{T^{1/2}_{opp}}(u_1,u_2;\mathfrak{z}^{-1}). \tag{6.11}$$

Proposition 6.3.

$$\Psi^*_{\omega'}(u_1)\Psi^*_{\omega''}(u_2) = \sum_{\boldsymbol{\lambda}=(\lambda',\lambda'')\in(\mathcal{P}^2)_n} \Psi^*_{\lambda''}(u_2)\Psi^*_{\lambda'}(u_1)R^*_{T^{1/2}}(u_1,u_2;\mathfrak{z}^*)^{\boldsymbol{\omega}}_{\boldsymbol{\lambda}},$$

where $\boldsymbol{\omega} = (\omega', \omega'') \in (\mathcal{P}^2)_n$.

Proposition 6.1 and the associativity for the composition of three type I vertex operators yield the following dynamical Yang-Baxter equation under the assumption (6.6).

$$R^{(12)}_{T^{1/2}}(u_1,u_2;\mathfrak{z}\hbar^{c^{(3)}})R^{(13)}_{T^{1/2}}(u_1,u_3;\mathfrak{z})R^{(23)}_{T^{1/2}}(u_2,u_3;\mathfrak{z}\hbar^{c^{(1)}})$$
$$= R^{(23)}_{T^{1/2}}(u_2,u_3;\mathfrak{z})R^{(13)}_{T^{1/2}}(u_1,u_3;\mathfrak{z}\hbar^{c^{(2)}})R^{(12)}_{T^{1/2}}(u_1,u_2;\mathfrak{z}). \tag{6.12}$$

Similarly, Proposition 6.3 and the associativity for the composition of the type II dual vertex operators yield the same dynamical Yang-Baxter equation for $R^*_{T^{1/2}}(u_1,u_2;\mathfrak{z}^*)$.

Finally, the type I and the type II dual vertex operators exchange by a scalar function.

Proposition 6.4. *In the level* $(1, N)$ *representation, one has*

$$\Phi_\lambda(u)\Psi^*_\mu(v) = \chi(u/v)\Psi^*_\mu(v)\Phi_\lambda(u) \qquad \forall\lambda,\mu\in\mathcal{P}_n,$$
$$\chi(u) = \frac{1}{\Gamma(\hbar^{1/2}u;t_1,t_2)} = \Gamma(\hbar^{1/2}/u;t_1,t_2).$$

Here $\Gamma(z;t_1,t_2)$ *denotes the elliptic Gamma function given by*

$$\Gamma(z;t_1,t_2) = \frac{(t_1t_2/z;t_1,t_2)_\infty}{(z;t_1,t_2)_\infty}, \qquad (z;t_1,t_2)_\infty = \prod_{m_1,m_2=0}^{\infty}(1-zt_1^{m_1}t_2^{m_2}).$$

7 L-operator of $U_{t_1,t_2,p}(\mathfrak{gl}_{1,tor})$

Combining the type I and the type II dual vertex operators, we construct the L-operator L^+ satisfying the RLL-relation. We then derive the exchange relations between L^+ and the vertex operators, which can be regarded as the intertwining relations w.r.t. the standard comultiplication Δ.

7.1 L-operator on $\mathcal{F}_{\bullet}^{(1,N)}$

Let $\sigma^{op} : \xi\widetilde{\otimes}\eta \to \eta\widetilde{\otimes}\xi$ and consider the following composition of the type I and type II vertex operators.

$$\mathcal{F}_u^{(0,-1)}\widetilde{\otimes}\mathcal{F}_{\hbar^{1/2}v}^{(1,N)} \stackrel{\sigma^{op}}{\longrightarrow} \mathcal{F}_{\hbar^{1/2}v}^{(1,N)}\widetilde{\otimes}\mathcal{F}_u^{(0,-1)} \stackrel{\Phi(\hbar^{1/2}u)\widetilde{\otimes}\mathrm{id}}{\longrightarrow} \mathcal{F}_{\hbar^{1/2}u}^{(0,-1)}\widetilde{\otimes}\mathcal{F}_{-v/u}^{(1,N+1)}\widetilde{\otimes}\mathcal{F}_u^{(0,-1)} \stackrel{\mathrm{id}\widetilde{\otimes}\Psi^*(u)}{\longrightarrow} \mathcal{F}_{\hbar^{1/2}u}^{(0,-1)}\widetilde{\otimes}\mathcal{F}_v^{(1,N)}.$$

Hence we have the operator

$$L^+(u) := g(\mathrm{id}\widetilde{\otimes}\Psi^*(u)) \circ (\Phi(\hbar^{1/2}u)\widetilde{\otimes}\mathrm{id})\sigma^{op} \ : \ \mathcal{F}_u^{(0,-1)}\widetilde{\otimes}\mathcal{F}_{\hbar^{1/2}v}^{(1,N)} \to \mathcal{F}_{\hbar^{1/2}u}^{(0,-1)}\widetilde{\otimes}\mathcal{F}_v^{(1,N)}$$

for $N \in \mathbb{Z}$, $v \in \mathbb{C}^\times$. Here we set $g = (\hbar; t_1, t_2)_\infty$. Define the components of $L^+(u)$ by

$$L^+(u) \cdot |\nu\rangle_u\widetilde{\otimes}\xi \ = \ \sum_\mu |\mu\rangle_{\hbar^{1/2}u}\widetilde{\otimes}L^+_{\mu\nu}(u)\xi,$$

for $|\mu\rangle_u\widetilde{\otimes}\xi \in \mathcal{F}_u^{(0,-1)}\widetilde{\otimes}\mathcal{F}_{\hbar^{1/2}v}^{(1,N)}$. One finds

$$L^+_{\mu\nu}(u) = g\Psi^*_\nu(u)\Phi_\mu(\hbar^{1/2}u). \tag{7.1}$$

Now let us consider the following elliptic dynamical R-matrices.

$$R^+_{T^{1/2}}(u, v; \mathfrak{z})^{\boldsymbol{\beta}}_{\boldsymbol{\alpha}} := \rho^+(u/v)\bar{R}_{T^{1/2}}(u, v; \mathfrak{z})^{\boldsymbol{\beta}}_{\boldsymbol{\alpha}}, \tag{7.2}$$

where $\bar{R}_{T^{1/2}}$ is given in (6.3) and $R^{+*}_{T^{1/2}} = R^+_{T^{1/2}}|_{p\mapsto p^*}$.

Proposition 7.1. *The L^+ operator satisfies the following relation.*

$$\sum_{\mu',\nu'} R^+_{T^{1/2}}(u, v; \mathfrak{z})^{\mu'\nu'}_{\mu\nu} L^+_{\mu'\mu''}(u)L^+_{\nu'\nu''}(v) = \sum_{\mu',\nu'} L^+_{\nu\nu'}(v)L^+_{\mu\mu'}(u)R^{+*}_{T^{1/2}}(u, v; \mathfrak{z}^*)^{\mu''\nu''}_{\mu'\nu'}.$$

7.2 Intertwining relations

Proposition 7.2. *The type I and the type II vertex operators satisfy the following relations.*

$$\Phi_\nu(\hbar^{1/2}v)L^+_{\mu\mu''}(u) = \sum_{\mu'\nu'} R^+_{T^{1/2}}(u, v; \mathfrak{z})^{\mu'\nu'}_{\mu\nu} L^+_{\mu'\mu''}(u)\Phi_{\nu'}(\hbar^{1/2}v), \tag{7.3}$$

$$L^+_{\mu\mu''}(u)\Psi^*_{\nu''}(v) = \sum_{\mu'\nu'} \Psi^*_{\nu'}(v)L^+_{\mu\mu'}(u)R^{+*}_{T^{1/2}}(u, v; \mathfrak{z}^*)^{\mu''\nu''}_{\mu'\nu'}. \tag{7.4}$$

Assuming the existence of the universal L-operator $\mathcal{L}^+(u) \in \mathrm{End}_{\mathbb{F}}(\mathcal{F}_{\bullet}^{(0,-1)})\widetilde{\otimes}\,\mathcal{U}$, which coincides with $R^+_{T^{1/2}}(u, v; \mathfrak{z}^*)$ for $\mathcal{U} = \mathcal{F}_v^{(0,-1)}$ and with $L^+(u)$ for $\mathcal{U} = \mathcal{F}_{\bullet}^{(1,N)}$, one can define a comultiplication Δ as a matrix tensor product of $\mathcal{L}^+(u)$. Then the relations in Proposition 7.2 turn out to be the intertwining relations w.r.t. Δ.

References

[1] M.Aganagic and A.Okounkov, Elliptic Stable Envelopes, *J. Amer. Math. Soc.* **34** (2021), no. 1, 79–133.

[2] T.M.Botta, Shuffle Products for Elliptic Stable Envelopes of Nakajima Varieties, (2022), arXiv:2104.00976

[3] H.Dinkins, Elliptic Stable Envelopes of Affine Type A Quiver Varieties, *Int. Math. Res. Not.* (2023), no.15, 13426–13476, arXiv:2107.09569

[4] H.Dinkins, Exotic Quantum Difference Equations and Integral Solutions, PhD Thesis, University of North Carolina at Chapel Hill, 2022, arXiv:2205.01596

[5] B.Feigin, E. Feigin, M.Jimbo, T.Miwa and E.Mukhin, Quantum continuous $\mathfrak{gl}_\infty$: Semi-infinite construction of representations *Kyoto J.Math.* **51** (2011) 337–364.

[6] B.Feigin and A.Tsymbaliuk, Heisenberg action in the equivariant K-theory of Hilbert schemes via shuffle algebra, *Kyoto J.Math.* **51** (2011) 831–854.

[7] T.Hollowood, A.Iqbal and C.Vafa, Matrix Models, Geometric Engineering and Elliptic Genera, J.High Energy Phys. (2008) 069.

[8] T.Kimura and V.Pestun, Quiver W-algebras, *Lett.Math.Phys.* **108** (2018), 1351–1381.

[9] H.Konno, Elliptic Weight Functions and Elliptic q-KZ Equation, *Journal of Integrable Systems* **2** (2017) 1-43. doi: 10.1093/integr/xyx011.

[10] H.Konno, Elliptic Stable Envelopes and Finite-Dimensional Representations of Elliptic Quantum Group, *Journal of Integrable Systems* **3** (2018) 1-43. doi: 10.1093/integr/xyy012.

[11] H.Konno and K.Oshima, Elliptic Quantum Toroidal Algebra $U_{q,t,p}(\mathfrak{gl}_{1,tor})$ and Affine Quiver Gauge Theories, *Lett.Math.Phys.* **113** (2023) Paper No.32, 64pp.

[12] H.Konno and A.Smirnov, Elliptic Quantum Toroidal Algebra $U_{t_1,t_2,p}(\mathfrak{gl}_{1,tor})$, Vertex Operators and L-operators, arXiv:2406.00964

[13] N.Nekrasov, Seiberg-Witten Prepotential From Instanton Counting, *Adv. Theor. Math. Phys.* **7** (2004), 831–864.

[14] N.Nekrasov, BPS/CFT Correspondence: Non-perturbative Dyson-Schwinger Equations and qq-Characters, *J.High Energy Phys.* **03** (2016), 181.

[15] A.Okounkov, Lectures on K-theoretic Computations in Enumerative Geometry. (2015), arXiv: 1512.07363.

[16] A. Smirnov, Elliptic stable envelope for Hilbert scheme of points in the plane, *Selecta Math.* **26** (2020) Paper No.3, 57pp.

[17] A. Smirnov and H.Dinkins, Characters of Tangent Spaces at Torus Fixed Points and $3d$-Mirror Symmetry, *Lett. Math. Phys.* **110** (2020) 2337–2352.

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