TROPICAL HYPERPLANE ARRANGEMENTS AND COMBINATORIAL MUTATIONS OF THE MATCHING FIELD POLYTOPES OF GRASSMANNIANS

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1. Introduction

A *Grassmannian* $\mathrm{Gr}(k,n)$ is the set of all subspaces of a given dimension k in $\mathbb{K}^n$, where $\mathbb{K}$ is a field.

Toric degenerations are a tool in algebraic geometry that allows us to use polyhedral geometry to study varieties. One of the motivations for toric degenerations of Grassmannians comes from mirror symmetry [5]. Therefore, we study toric degenerations of Grassmannians. In particular, we focus on toric degeneration by using SAGBI bases.

SAGBI bases were independently introduced by Kapur and Madlener [7] and Robbiano and Sweedler [9]. SAGBI means Subalgebra Analogue of Gröbner Bases for Ideals. SAGBI bases answer the subalgebra membership problem similar to Gröbner bases to the ideal membership problem. What matters to us is that toric degenerations of Grassmannians by matching fields are characterized as those for which the Plücker coordinates form a SAGBI basis.

We need the concept of monomial orderings to define a SAGBI basis. To give this, we use matching fields. Given integers k and n, a *matching field* denoted by $\Lambda(k,n)$, or Λ when there is no confusion, is a choice of permutation $\Lambda(I) \in S_k$ for each $I \in \mathbf{I}_{k,n} = \{I \subset [n] : |I| = k\}$, where $[n] = \{1, 2, \dots, n\}$. We say that a matching field Λ has a toric degeneration if Plücker coordinates form a SAGBI basis for the subalgebra they generate, with respect to the monomial ordering associated with Λ. See Section 2.2 for more details. Matching fields were born to study the Newton polytope of the product of all maximal minors of a matrix of indeterminates $X = (x_{ij})$ by Sturmfels and Zelevinsky [10]. A matching field is called coherent if a matrix induces the matching field. (For more details, see Definition 2.1.) A matching field polytope is defined as the convex hull of the exponent vectors corresponding to the initial terms of maximal minors of a matrix of indeterminates $X = (x_{ij})$ by matching fields by Mohammadi and Shaw [8]. The most famous example of matching fields is the diagonal matching field. This chooses the diagonal term as the initial for each minor. Namely, the diagonal matching field sends every $I \in I_{k,n}$ to $\mathrm{id} \in S_k$.

Mohammadi and Shaw introduced a more generalized class of matching fields, block diagonal matching fields [8]. The image of this matching field is $\{\mathrm{id},\ (1\ 2)\}$. Block diagonal matching fields were studied by many researchers [2, 3, 4, 6]. Note we still do not know the complete answer to which block diagonal matching field has toric degenerations.

Checking if a coherent matching field gives toric degeneration is a difficult problem. However, in $\mathrm{Gr}(2,n)$, solving this problem is easy because all the coherent matching fields are the same as the diagonal matching field up to symmetry.

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How about $\mathrm{Gr}(3,n)$? In the case of $\mathrm{Gr}(3,n)$, it is proved in [8, Theorem 1.3] that a non-hexagonal $3\times n$ matching field whose ideal is quadratically generated provides a toric degeneration of $\mathrm{Gr}(3,n)$. At present, no non-hexagonal $3\times n$ matching fields whose ideal is not quadratically generated are known.

To study which matching field has toric degenerations for $\mathrm{Gr}(3,n)$, we use the combinatorial mutation. Combinatorial mutation was introduced by Akhtar, Coates, Galkin, and Kasprzyk [1]. Combinatorial mutation was applied by Clarke, Higashitani, and Mohammadi in the context of Grassmannians [2]. It is proved in [2, Theorem 1] that if $P_\Lambda, P_{\Lambda'}$ are matching field polytopes related by a combinatorial mutation, then Λ has a toric degeneration if and only if Λ' does.

However, checking combinatorial mutation equivalence is difficult because it is quite sensitive. So, we establish a new tool to solve the problems by using tropical hyperplane arrangements. This way can solve the problem, "literally at a glance".

Our main theorem is as follows.

Theorem 3.2. *We consider* $\mathrm{Gr}(3,n)$. *Assume that there are two matching fields* Λ, Λ', *both of which have the same tropical hyperplane arrangements except for shifting two adjacent tropical lines,* i *and* j, *and that condition* $(*)$ *is satisfied. Then, the pair of matching field polytopes* P_Λ, $P_{\Lambda'}$ *can be obtained from one another by a combinatorial mutation.*

For $(*)$, see Section 3. As a corollary of this theorem, we obtain there is a sequence of combinatorial mutation to the diagonal matching field for any block diagonal matching field. See Remark 3.5.

We finish the introduction with an outline of the paper. Section 2.1 fixes notations for the Grassmannians. In Section 2.2, we introduce matching fields and the associated polytope. In Section 2.3, we introduce the combinatorial mutation. Finally, our main theorem is proved in Section 3 and an example of our main theorem is also given there.

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2. Preliminaries

2.1. **Grassmannians and Plücker coordinates.** A *Grassmannian* $\mathrm{Gr}(k,n)$ is the set of all subspaces of a given dimension k in $\mathbb{K}^n$, where $\mathbb{K}$ is a field. For any subset $I=\{i_1,\dots,i_k\}\subset[n]=\{1,2,\dots,n\}$, and the $k\times n$ indeterminants matrix X, let X_I be the submatrix with rows $1,\dots,k$ and columns $i_1,\dots,i_k$. Let $\mathbf{I}_{k,n}=\{I\subset[n]:|I|=k\}$. The kernel of the map

$$\varphi\colon\ \mathbb{K}[P_I : I\in\mathbf{I}_{k,n}]\to\mathbb{K}[x_{ij}:1\le i\le k, 1\le j\le n]\quad\text{with}\quad P_I\mapsto\det(X_I)$$

is called the *Plücker ideal.* The *Plücker coordinates* are the minors $\det(X_I)$, and the corresponding *Plücker variable* is denoted by P_I. We identify Plücker coordinates with Plücker variables.

2.2. **Matching fields and their associated ideals.** We use many definitions from [2, 8]. Given integers k and n, a *matching field*, denoted by $\Lambda(k,n)$, or Λ when there is no confusion, is a choice of permutation $\Lambda(I)\in S_k$ for each $I\in\mathbf{I}_{k,n}$. We think of the permutation $\Lambda(I)$ as inducing a new ordering on the elements of I.

We call a $k\times 1$ tableau where the s-th entry is $i_{\Lambda(I)(s)}$ the *tableaux presentation* of $\Lambda(I)$. (See Example 2.5.)

Let $X=(x_{ij})$ be a $k\times n$ matrix of indeterminates. To every $I=\{i_1,\cdots,i_k\}\in\mathbf{I}_{k,n}$ with $i_1<\cdots<i_k$ and $\sigma=\Lambda(I)$, let $\mathbf{x}_{\Lambda(I)}:=x_{1i_{\sigma(1)}}x_{2i_{\sigma(2)}}\cdots x_{ki_{\sigma(k)}}$. The *matching field ideal* J_Λ is defined as the kernel of the monomial map

$$(1)\qquad \varphi_\Lambda\colon\ \mathbb{K}[P_I]\to\mathbb{K}[x_{ij}]\quad\text{with}\quad P_I\mapsto\operatorname{sgn}(\Lambda(I))\mathbf{x}_{\Lambda(I)},$$

where $\operatorname{sgn}(\Lambda(I))$ denotes the signature of the permutation $\Lambda(I)$ for each $I \in \mathbf{I}_{k,n}$.

Definition 2.1. A matching field Λ is *coherent* if there exists a $k \times n$ matrix $M = (m_{ij})$ with $m_{ij} \in \mathbb{R}$ such that for every $I \in \mathbf{I}_{k,n}$ the initial of the Plücker coordinate $\det(X_I) \in \mathbb{K}[x_{ij}]$ is $\text{in}_M(\det(X_I)) = \varphi_\Lambda(P_I)$, where $\text{in}_M(\det(X_I))$ is the sum of all terms in $\det(X_I)$ of the lowest weight and the weight of a monomial $x_{1i_1} \cdots x_{ki_k}$ is $m_{1i_1} + \cdots + m_{ki_k}$. In this case, we say that the matrix M *induces the matching field* Λ. We let $\mathbf{w}_M$ be the weight vector on the variables P_I induced by the entries m_{ij} of the weight matrix M on the variables x_{ij}. More precisely, the weight of each variable P_I is defined as the minimum weight of the terms of the corresponding minor of M, and it is called *the weight induced by* M.

In general, a matrix that induces a coherent matching field does not necessarily have the first row consisting of all zeros. However, we may assume that a matrix that induces a coherent matching field satisfies this without loss of generality.

Example 2.2. Consider the matching field $\Lambda(3,6)$ which assigns to each subset I the identity permutation. Consider the following matrix:

$$M = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 6 & 5 & 4 & 3 & 2 & 1 \\ 11 & 9 & 7 & 5 & 3 & 1 \end{pmatrix}.$$

The weights induced by M on the variables $P_{123}, P_{124}, \ldots, P_{456}$ are $12, 10, \ldots, 3$ respectively. Thus, for each $I = \{i,j,k\}$ we have that $\text{in}_M(\det(X_I)) = x_{1i}x_{2j}x_{3k}$ for $1 \le i < j < k \le 6$. Therefore, the matrix M induces $\Lambda(3,6)$.

Notice that each initial term $\text{in}_M(P_I)$ arises from the leading diagonal. Such matching fields are called *diagonal*.

As a generalization of the diagonal matching field, we define a *block diagonal matching field*.

Definition 2.3 ([8, Definition 4.1]). Given k, n and $0 \le \ell \le n$, we define the block diagonal matching field $\mathcal{B}_\ell$ as the map from $\mathbf{I}_{k,n}$ to S_k such that

$$\mathcal{B}_\ell(I) = \begin{cases} \text{id} & \text{if } |I| = 1 \text{ or } |I \cap [\ell]| \ne 1, \\ (12) & \text{otherwise.} \end{cases}$$

Given a matching field Λ, we associate to it a polytope P_Λ. The vertices of the polytope are in one-to-one correspondence with the tableaux of the matching field. In fact, reading the vertices of the polytope uniquely defines the matching field.

Definition 2.4 ([2, Definition 6]). Fix k and n. We take $\mathbb{R}^{k \times n}$ to be the vector space of $k \times n$ matrices with canonical basis $\{e_{i,j} : 1 \le i \le k, 1 \le j \le n\}$ where $e_{i,j}$ is the matrix with a 1 in row i and column j and zeros everywhere else. Given a matching field Λ, for each $I = \{i_1, \ldots, i_k\} \in \mathbf{I}_{k,n}$ with $1 \le i_1 < \cdots < i_k \le n$ we set $v_{I,\Lambda} := \sum_{j=1}^k e_{j, i_{\Lambda(I)(j)}}$. Then the *matching field polytope* is

$$P_\Lambda = \text{Conv}\left\{ v_{I,\Lambda} : \ I \in \mathbf{I}_{k,n} \right\}.$$

For notation we often write the tuple $(i_{\Lambda(I)(1)}, i_{\Lambda(I)(2)}, \ldots, i_{\Lambda(I)(k)})$ for the vector $v_{I,\Lambda}$.

Example 2.5. Consider $\mathrm{Gr}(3,6)$. A block diagonal matching field $\mathcal{B}_2$ is induced by a matrix $\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 6 & 5 & 4 & 3 \\ 11 & 9 & 7 & 5 & 3 & 1 \end{pmatrix}$. The tableaux presentation of $\mathcal{B}_2(\{2,3,5\})$ is $\boxed{\begin{matrix} 3 \\ 2 \\ 5 \end{matrix}}$. The corresponding vertex $v_{\{2,3,5\},\mathcal{B}_2}$ of the matching field polytope $P_{\mathcal{B}_2}$ is $\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$.

Similarly, the tableaux presentations of $\mathcal{B}_2(\{1,4,6\}), \mathcal{B}_2(\{2,4,5\})$, and $\mathcal{B}_2(\{1,3,6\})$ are $\boxed{\begin{matrix} 4 \\ 1 \\ 6 \end{matrix}}$, $\boxed{\begin{matrix} 4 \\ 2 \\ 5 \end{matrix}}$, and $\boxed{\begin{matrix} 3 \\ 1 \\ 6 \end{matrix}}$, respectively. Then we notice that the equality $\begin{array}{|c|c|} \hline 3 & 4 \\ 2 & 1 \\ 5 & 6 \\ \hline \end{array} = \begin{array}{|c|c|} \hline 4 & 3 \\ 2 & 1 \\ 5 & 6 \\ \hline \end{array}$ holds as row-wise equal tableaux. This equality means a relation of vertices of the matching field polytope: $v_{\{2,3,5\},\mathcal{B}_2} + v_{\{1,4,6\},\mathcal{B}_2} = v_{\{2,4,5\},\mathcal{B}_2} + v_{\{1,3,6\},\mathcal{B}_2}$.

2.3. **Tropical maps and combinatorial mutations.** Let $M = (m_{ij}) \in \mathbb{R}^{k\times n}$ be a weight matrix. For each $1 \le j \le n$ consider the piecewise linear function $F_j : \mathbb{R}^{k-1} \to \mathbb{R}$, given by

$$F_j((x_2, \cdots, x_k)) = \max\{m_{1j}, m_{2j} + x_2, \dots, m_{kj} + x_k\}.$$

A $k \times n$ weight matrix $M \in \mathbb{R}^{k\times n}$, whose first row consists of zeros, produces an arrangement of tropical hyperplanes $\mathcal{A} = \{H_1, \dots, H_n\}$ defined by the functions $F_1, \dots, F_n$, whose coefficients come from M. To each $k-1$ dimensional cell τ of the complement of the arrangement $\mathcal{A}$ in $\mathbb{R}^{k-1}$ there is an associated *covector* $c_\tau \in \mathcal{P}[n]^k$, where $\mathcal{P}[n]$ denotes the power set of n. The i-th entry of the covector c_τ is a subset $S_i \subset [n]$ corresponding to the collection of hyperplanes in $\mathcal{A}$ which intersect the ray $x + tv_i$ for $x \in \tau^\circ$ and $t \ge 0$, where τ° is the interior of τ, and we let the vectors $v_i = -e_{i-1}$ for $i = 2, \dots, k-1$, and $v_1 = (1, \dots, 1)$. The *coarse covector* of a cell is simply the vector which records the sizes of the subsets of the covector. For example, the tropical hyperplane arrangement of the matrix inducing the diagonal matching field is as in Figure 1. Each tropical hyperplane is labelled by the numbering of the columns of the matrix in Example 2.2.

The following proposition can be extracted from [8, Proposition 3.4]. This proposition is a bridge between tropical geometry and initial degeneration.

Proposition 2.6 ([8, Proposition 3.4])**.** *Let M be a $k \times n$ weight matrix, and let Λ be the coherent matching field induced by M. Assume that for any $I = \{i_1, \dots, i_k\} \in \mathbf{I}_{k,n}$ with $i_1 < \cdots < i_k$, the collection of hyperplanes $\{H_i\}_{i\in I}$ intersects properly. Then*

$$\mathrm{in}_M(\det(X_I)) = \mathrm{sgn}(\Lambda(I)) x_{1c_1} x_{2c_2} \dots x_{kc_k},$$

where $(\{c_1\}, \dots, \{c_k\})$ is the covector of the unique cell with coarse covector $(1, 1, \dots, 1)$ such that $i_{\Lambda(I)(j)} = c_j$ for $j = 1, \dots, k$.

Tropical hyperplanes we deal with are the fan that has the three rays, i.e. $k = 3$. Depending on the directions, we name every ray *x-ray* $-e_1$, *y-ray* $-e_2$, and *diagonal ray* $e_1 + e_2$. We call the intersection of those three rays the *central point*. The ray that has the inverse direction of a

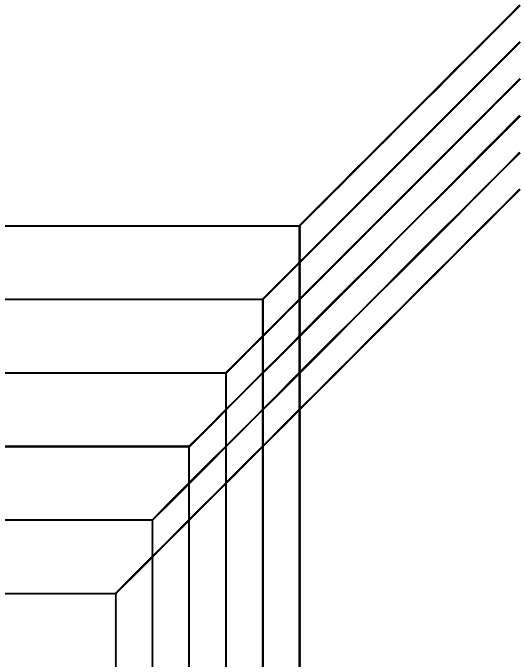

FIGURE 1. An arrangement of tropical hyperplanes $H_1, \cdots, H_6$ corresponding to the diagonal matchingfield of Gr(3,6)

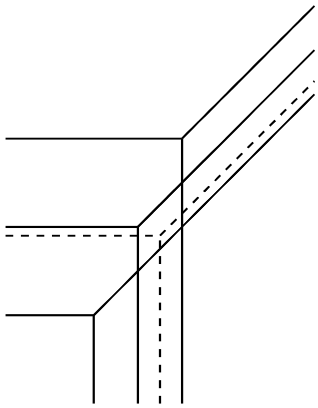

FIGURE 2. This covector is $(\{j\}, \{i\}, \{k\})$. Therefore, we obtain the ordering j, i, k.

diagonal ray is the *anti-diagonal ray*. Similarly, *anti-x-ray* and *anti-y-ray* are defined. See Figure 3.

We begin by fixing two lattices $N = \mathbb{Z}^d$ and its dual $M = \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$. We take $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$ and similarly $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$. We fix the standard inner product $\langle \cdot, \cdot \rangle : N_{\mathbb{R}} \times M_{\mathbb{R}} \to \mathbb{R}$ given by evaluation $\langle v, u \rangle := u(v)$ for $v \in N_{\mathbb{R}}$ and $u \in M_{\mathbb{R}}$.

Definition 2.7. Let w be a primitive lattice point of M and $F \subset w^{\perp} \subset N_{\mathbb{R}}$ a lattice polytope, where $w^{\perp} = \{v \in N_{\mathbb{R}} | \langle v, w \rangle = 0\}$. The *tropical map* defined by w and F is given by

$$\varphi_{w,F} : M_{\mathbb{R}} \to M_{\mathbb{R}}, \quad u \mapsto u - u_{\min} w,$$

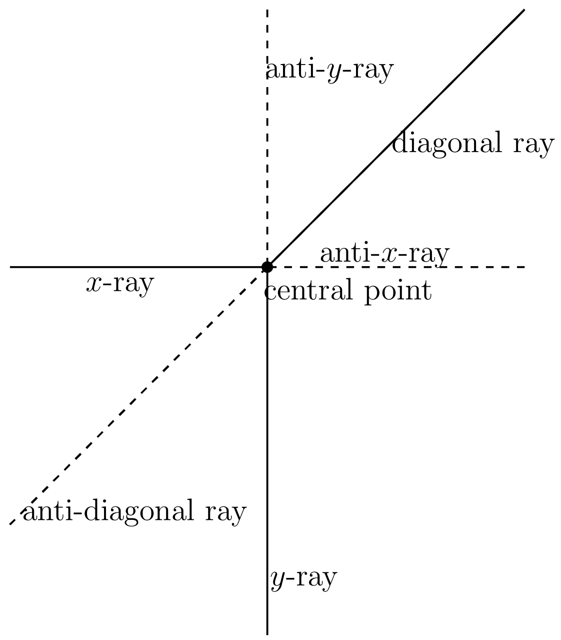

FIGURE 3. The name of rays

where $u_{\min} := \min\{\langle u, p\rangle : p \in F\}$.
Let $P \subset M_{\mathbb{R}}$ be a lattice polytope that contains the origin and suppose that $\varphi_{w,F}(P)$ is convex. Then we say that the polytope $\varphi_{w,F}(P)$ is a *combinatorial mutation* of P.

To check that the tropical map induces a combinatorial mutation, the following lemma is useful.

Lemma 2.8 ([2, Lemma 3])**.** *Let $P \subset M_{\mathbb{R}}$ be a lattice polytope and $f \in N$ a non-zero vector. Suppose that $P \subset \{x \in M_{\mathbb{R}} : -1 \le \langle f, x\rangle \le 1\}$. Write $P_+ = \{x \in P : 0 \le \langle f, x\rangle \le 1\}$ and $P_- = \{x \in P : -1 \le \langle f, x\rangle \le 0\}$. Let $w \in M$ be a non-zero vector such that $f \in w^\perp$ and define $F = \mathrm{Conv}\{0, f\}$. Let $\varphi = \varphi_{w,F}$ be the tropical map and note that $\varphi(P) = \varphi(P_+) \cup \varphi(P_-)$ is the union of two polytopes that intersect along a common face. Let $V(\varphi(P))$ be the vertices of $\varphi(P)$. Then, there is no vertex of $\varphi(P)$ that is not the image of any vertex of P if and only if for any vertices $u, v \in V(P)$ with $\langle f, u\rangle = -1$, $\langle f, v\rangle = 1$, there exist $t, t' \in P$ with $\langle f, t\rangle = \langle f, t'\rangle = 0$ such that $t + t' = u + v$.*

Example 2.9. Consider Gr(3, 6).

$$w = \begin{pmatrix} 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, F = \mathrm{conv}(\mathbf{0}, \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix})$$

Consider the matching field polytope corresponding to $M = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 6 & 5 & 4 & 3 & 4.5 & 1 \\ 11 & 9 & 7 & 5 & 3 & 1 \end{pmatrix}$.

Then, vertices that is moved by $\varphi_{w,F}$ are only one vertex corresponding to 536.

$$\varphi_{w,F}(\begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}) = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

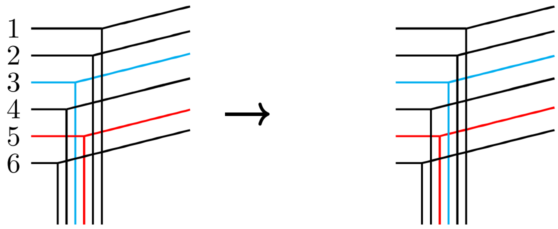

FIGURE 4. Example 2.9

3. Main results

Consider the arrangement of tropical hyperplanes associated with the coherent matching field of $\mathrm{Gr}(3,n)$. Let M be a $3\times n$ weight matrix and let $\{H_1,\cdots,H_n\}$ be the tropical lines associated with M. By abuse of notation, we identify H_i with i. We say that two tropical lines i and j are *adjacent* to each other if their y-rays are adjacent. Fix two adjacent tropical lines i and j and assume the x-coordinate of the central point of j is larger than that of i.

We define six colored regions. At first, assume the y-coordinate of the central point of j is larger than that of i.

- The red region is bounded by the anti-diagonal ray and y-ray of i.
- The gray region is bounded by the anti-diagonal ray, anti-y-ray of i, and x-ray of j.
- The brown region is bounded by the anti-y ray of i, and x-ray of j.
- The blue region is the area bounded by the anti-x-ray and y-ray extending from the intersection of the diagonal ray of i and the y-ray of j.
- The green region is the area bounded by the anti-x-ray, anti-y-ray extending from the intersection of the diagonal ray of i and the y-ray of j, and the diagonal ray of j.
- The yellow region is bounded by the anti-y-ray and diagonal ray of j.

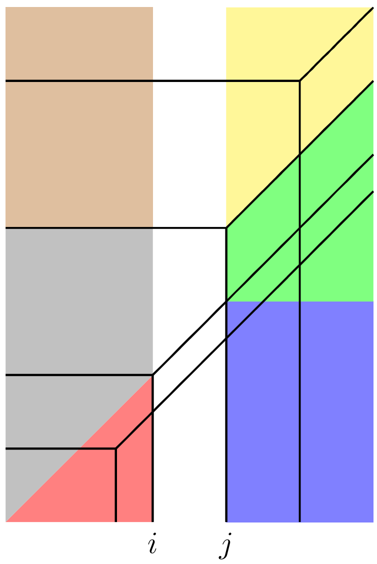

FIGURE 5. Six colored regions in the case where j is higher than i.

Next, assume the y-coordinate of the central point of i is larger than that of j.

- The red region is the area bounded by the anti-diagonal ray and y-ray extending from the intersection of the y-ray of i and the x-ray of j.

- The gray region is bounded by the anti-diagonal ray and anti-y-ray extending from the intersection of the y-ray of i and the x-ray of j, and x-ray of i.
- The brown region is bounded by the anti-y ray and x-ray of i.
- The blue region is bounded by the anti-x-ray and y-ray of j.
- The green region is bounded by the anti-x-ray, anti-y-ray of j, and the diagonal ray of i.
- The yellow region is bounded by the anti-y-ray of j and diagonal ray of i.

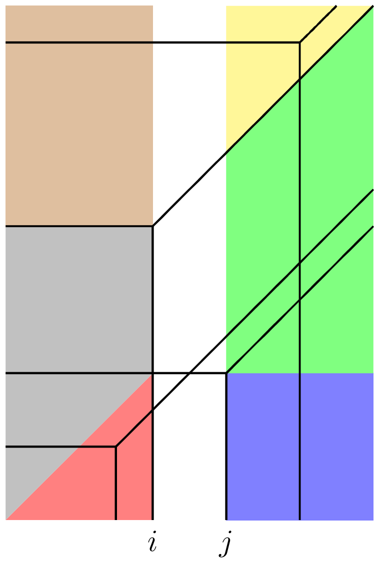

FIGURE 6. Six colored regions in the case where i is higher than j.

We say that a tropical line is contained in the colored region if its central point is contained in the colored region. We define the condition $(*)$ as follows.

$$(*)\begin{cases}\text{The red region is not empty.} \cdots (a)\\ \text{The blue and brown regions are empty.} \cdots (b)\\ \text{Either the yellow or green region is non-empty.} \cdots (c)\\ \text{There are at least two elements in the red and purple regions.} \cdots (d)\end{cases}$$

Note that if the condition (a) in $(*)$ is false, then nothing happens after we exchange i and j, so it is natural to assume (a).

Our main result uses the following w and F.

Definition 3.1. Let $w_{(i,j)}$ be the $3 \times n$ matrix such that all the columns are zero vectors except for the columns corresponding to i and j, the column corresponding to i is $(1, -1, 0)^T$, and the column corresponding to j is $(-1, 1, 0)^T$. Namely,

$$w_{(i,j)} := \begin{pmatrix} 0 & \cdots & 0 & \overset{i}{1} & 0 & \cdots & 0 & \overset{j}{-1} & 0 & \cdots 0 \\ 0 & \cdots & 0 & -1 & 0 & \cdots & 0 & 1 & 0 & \cdots 0 \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots 0 \end{pmatrix} \in \mathbb{R}^{3 \times n}.$$

The lattice polytope $F_{(i,j)}$ in $\mathbb{R}^{3\times n}$ is defined by the convex hull of 0 and f, where 0 denotes the zero matrix and f is the matrix such that the columns corresponding to green or yellow regions are $(1, -1, 0)^T$, and the columns corresponding to i or j are $(0, -1, 0)^T$ and the remaining columns are all zero vectors. (Here, that a column corresponds to a colored region means that the corresponding tropical hyperplane is contained in the colored region.) Namely,

$$F_{(i,j)} := \mathrm{Conv}\,\{0,\ f\} \subset \mathbb{R}^{3\times n}$$

$$\text{where } f := \begin{array}{c} \\ \left(\begin{array}{ccc|cc|ccc|ccc} & \textcircled{1} & & i & j & & \textcircled{2} & & & \textcircled{3} & \\ 0 & \cdots & 0 & 0 & 0 & 1 & \cdots & 1 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & -1 & -1 & -1 & \cdots & -1 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{array}\right) \end{array} \in \mathbb{R}^{3\times n}.$$

Remark that $F_{(i,j)} \subset w_{(i,j)}^{\perp}$. We associate the columns corresponding to the tropical line that belongs to the red region with ①. Similarly, we associate the columns corresponding to the green or yellow region with ②, the gray region with ③.

We say that we *shift* i to the left of j if we make the second row in the i-th column of M bigger than the second row in the j-th column of M. This means that we shift the central point of the tropical line corresponding to i to the left of that of the tropical line corresponding to j along the x-axis. See Figure 7.

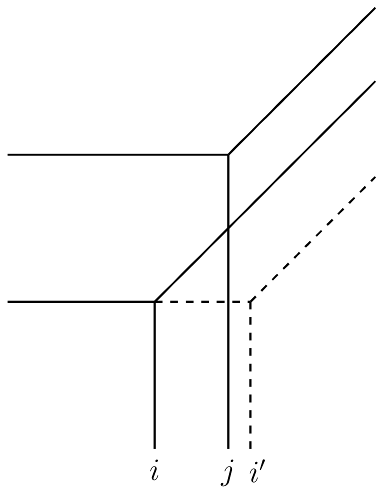

FIGURE 7. i' is obtained by shifting i to the right of j.

Theorem 3.2. *We consider* $\mathrm{Gr}(3,n)$*. Assume that there are two matching fields* $\Lambda,\ \Lambda'$*, both of which have the same tropical hyperplane arrangements except for shifting two adjacent tropical lines,* i *and* j*, and that condition* $(*)$ *is satisfied. Then, the pair of matching field polytopes* $P_\Lambda,\ P_{\Lambda'}$ *can be obtained from one another by a combinatorial mutation.*

Proof. Look for a paper to be published in the future. □

Remark 3.3. If we drop (c) and (d) in $(*)$, then a piecewise linear map $\varphi_{w,F}$ becomes a linear map. In particular, this becomes a unimodular transformation, i.e. a linear transformation by a unimodular matrix. So, we can still discuss a sequence of combinatorial mutations.

Example 3.4. We check that there is a sequence of combinatorial mutations from the block diagonal matching field $\mathcal{B}_2$ to the diagonal matching field. We want to move the red line to the dashed line of Figure 8. See [8, Figure 2] about tropical hyperplane arrangements of block diagonal matching fields. First, remark the condition (a) in $(*)$ is false on the left figure, then nothing happens after we exchange i and j in each step. Therefore, nothing is to be checked.

Next, see the second figure. Check the condition ($*$) for the red line and the left adjacent line. Then, the two lines are swapped. Repeating these operations for 4 times, we get the right figure. For exapmle, the second case of the sequence of the combinatorial mutation, $w_{(i,,j)}$ and $F_{(i,,j)}$ are the following:

$$w_{(i,j)} := \begin{pmatrix} 0 & 0 & 0 & \overset{i}{1} & \overset{j}{-1} & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad F_{(i,j)} := \mathrm{Conv}\left\{ 0, \left(\begin{array}{c|cc|cc|c} \overset{①}{0} & 0 & \overset{③}{0} & \overset{i}{0} & \overset{j}{0} & \overset{②}{1} \\ 0 & 0 & 0 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right) \right\}.$$

The number in the column of the matices corresponds to the number counting from the right in the second figure.

Remark 3.5. By repeating the above operation, any block diagonal matching field polytopes can be obtained from the diagonal matching field polytope by a sequence of combinatorial mutations. This is one of the main results of [2].

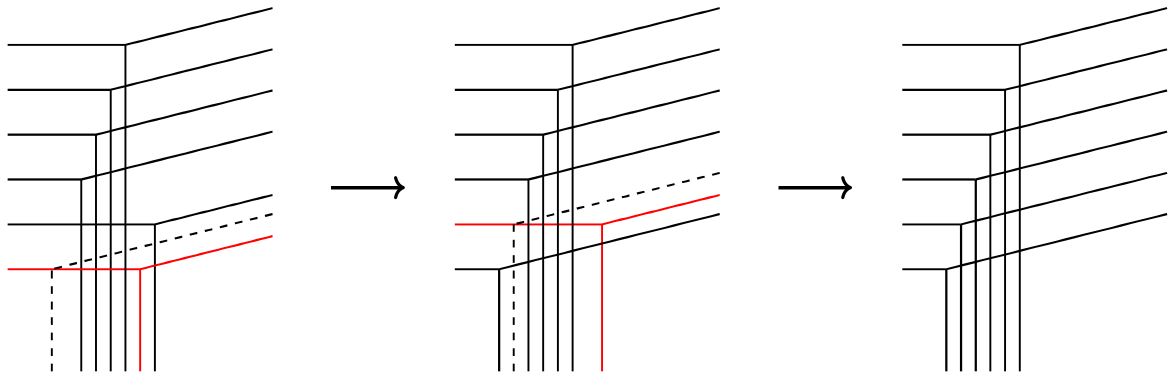

FIGURE 8. The example of combinatorial mutations from the block diagonal matching field $\mathcal{B}_2$ to the diagonal matching field. (The y-axis is scaled down by $\frac{1}{4}$.)

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