Geometric deformations of cuspidal S_1 singularities

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Abstract

We consider a one parameter deformation of a cuspidal S_1 singularity and its differential geometric properties. For that, we give a form representing the deformation using only diffeomorphisms on the source and isometries of the target.

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1 Introduction

Singularities are deformed and turn into various singularities. The cuspidal $S_1^{\pm}$ singularities correspond to the appearance/ disappearance of cuspidal cross caps. Therefore, it is natural to include the deformation when we study such singularities In this paper, we give a normal form for the deformations of the $S_k^{\pm}$ $(k \in \boldsymbol{Z}_{\geq 0})$ singularities using a diffeomorphism-germ on the source space and an isometry-germ on the target space. Furthermore, using this form, we investigate the differential geometric properties of the deformations of the cuspidal $S_1^{\pm}$ singularities.

The cuspidal $S_k^{\pm}$ $(k \in \boldsymbol{Z}_{\geq 0})$ *singularities* are map-germs $\mathcal{A}$-equivalent to the map-germ defined by $(u, v) \mapsto (u, v^2, v^3(u^{k+1}+v^2))$ at the origin. The cuspidal S_0 singularity is also called a *cuspidal cross cap*. Here, two map-germs $f_1 : (\boldsymbol{R}^2, 0) \to (\boldsymbol{R}^3, 0)$ and $f_2 : (\boldsymbol{R}^2, 0) \to (\boldsymbol{R}^3, 0)$ are *$\mathcal{A}$-equivalent* if there exist a coordinate change of source space φ and a coordinate change of target space ψ such that $f_2 = \psi \circ f_1 \circ \varphi^{-1}$. A map-germ $f : (\boldsymbol{R}^2, 0) \to (\boldsymbol{R}^3, 0)$ is called a *frontal* if there exist a normal vector field $\nu : (\boldsymbol{R}^2, 0) \to \boldsymbol{R}^3$ along f such that $\langle df(X), \nu \rangle = 0$ holds for any $p \in (\boldsymbol{R}^2, 0)$ and $X \in T_p\boldsymbol{R}^2$. This ν is called the *unit normal vector field.*

2 Normal forms of deformations of cuspidal $S_k^{\pm}$ singularities

Definition 2.1. A map-germ $f : (\boldsymbol{R}^2 \times \boldsymbol{R}, 0) \to (\boldsymbol{R}^3, 0)$ is a *deformation of* g : $(\boldsymbol{R}^2, 0) \to (\boldsymbol{R}^3, 0)$, if it is smooth and $f(u, v, 0) = g(u, v)$ and $f(0, 0, s) = (0, 0, 0)$.

In this definition, the parameter $s \in \boldsymbol{R}$ as the third component of the source space is called the *deformation parameter*. We define an equivalence relation between two deformations preserving the deformation parameters.

Definition 2.2. Let $f_1, f_2 : (\boldsymbol{R}^2 \times \boldsymbol{R}, 0) \to (\boldsymbol{R}^3, 0)$ be deformations of g. Then f_1 and f_2 are *equivalent as deformations* if there exist orientation preserving diffeomorphism-germs $\varphi : (\boldsymbol{R}^2 \times \boldsymbol{R}, 0) \to (\boldsymbol{R}^2 \times \boldsymbol{R}, 0)$ with the form

$$\varphi(u, v, s) = (\varphi_1(u, v, s), \varphi_2(u, v, s), \varphi_3(s)) \quad \left(\frac{d\varphi_3}{ds}(0) > 0\right) \tag{2.1}$$

and $\psi : (\boldsymbol{R}^3, 0) \to (\boldsymbol{R}^3, 0)$ such that $\psi \circ f_1 \circ \varphi^{-1}(u, v, s) = f_2(u, v, s)$ holds.

The above form on φ, namely φ_3 is defined depending only on s, implies that we allow change of parameter of deformation itself and prevent affecting the other parameters to the parameter of deformation. Since the third component of the source space is the deformation parameter, therefore this definition means an $\mathcal{A}$-equivalence that preserves the deformation parameters is defined.

Example 2.3. Let ${f_s}^{\pm}$ be a deformation of cuspidal $S_1^{\pm}$ singularities defined by

$${f_s}^{\pm} : (\boldsymbol{R}^2 \times \boldsymbol{R}, 0) \ni (u, v, s) \mapsto (u, v^2, v^3(u^2 \pm v^2) + sv^3) \in (\boldsymbol{R}^3, 0).$$

We show the deformation of ${f_s}^{+}$ in Figure 1 and ${f_s}^{-}$ in Figure 2.

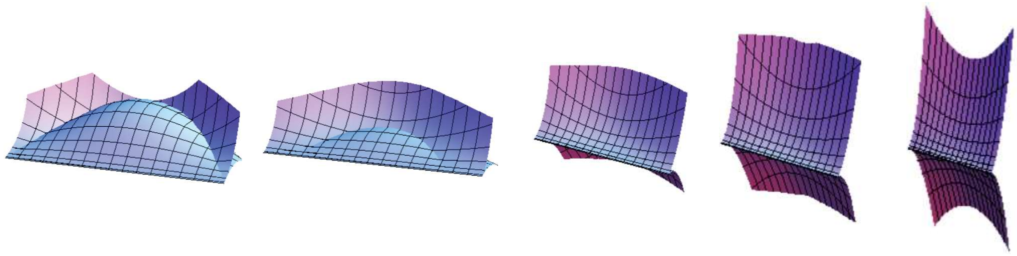

Figure 1: Deformations of cuspidal S_1^+ singularity (from left to right $f_{-1}^+, f_{-1/2}^+, f_0^+, f_{1/2}^+$ and f_1^+)

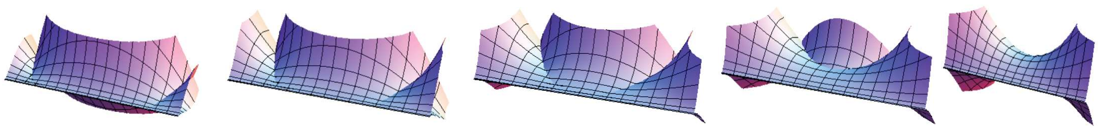

Figure 2: Deformations of cuspidal S_1^- singularity (from left to right $f_{-1}^-, f_{-1/2}^-, f_0^-, f_{1/2}^-$ and f_1^-)

Theorem 2.4. *Let $f : (\boldsymbol{R}^2 \times \boldsymbol{R}, 0) \to (\boldsymbol{R}^3, 0)$ be a deformation of $g : (\boldsymbol{R}^2, 0) \to (\boldsymbol{R}^3, 0)$ such that the 2-jet of g is $\mathcal{A}$-equivalent to $(u, v^2, 0)$ or (u, v^2, uv). Then there exist an orientation preserving diffeomorphism-germ $\varphi : (\boldsymbol{R}^2 \times \boldsymbol{R}, 0) \to (\boldsymbol{R}^2 \times \boldsymbol{R}, 0)$ with the form*

(2.1), $T \in SO(3)$ *and the functions* $f_{21}, f_{31} \in C^\infty(1,1), f_{24}, f_{33}, f_{34} \in C^\infty(2,1), f_{32} \in C^\infty(3,1)$ *such that*

$$\begin{aligned} {f_{\mathrm{n}1}}^s &= T \circ f \circ \varphi(u,v,s) \\ &= (u, u^2 f_{21}(u) + v^2 + usf_{24}(u,s), \\ &\qquad u^2 f_{31}(u) + v^2 f_{32}(u,v,s) + vf_{33}(u,s) + usf_{34}(u,s)), \end{aligned} \tag{2.2}$$

where $f_{32}(0,0,0) = f_{33}(0,0) = 0$. *If the* 2*-jet of* g *is* $\mathcal{A}$*-equivalent to* $(u, v^2, 0)$*, then* $(f_{33})_u(0,0) = 0$ *holds, and if it is* $\mathcal{A}$*-equivalent to* (u, v^2, uv)*, then* $(f_{33})_u(0,0) \neq 0$ *holds.*

In Theorem 2.4, the given f and ${f_{\mathrm{n}1}}^s$ are equivalent as deformations (Definition 2.2), and they have the same differential geometric properties.

For the frontality of the form ${f_{\mathrm{n}1}}^s$, we have the following theorem.

Theorem 2.5. *A map-germ* ${f_{\mathrm{n}1}}^s$ *is frontal for any* s *if and only if* $f_{33}(u,s) = 0$ *holds identically.*

Corollary 2.6. *Let* $f : (\boldsymbol{R}^2 \times \boldsymbol{R}, 0) \to (\boldsymbol{R}^3, 0)$ *be a deformation of* $g : (\boldsymbol{R}^2, 0) \to (\boldsymbol{R}^3, 0)$ *such that the* 2*-jet of* g *is* $\mathcal{A}$*-equivalent to* $(u, v^2, 0)$ *or* (u, v^2, uv) *and* f *is frontal for any* s*. Then there exist an orientation preserving diffeomorphism-germ* $\varphi : (\boldsymbol{R}^2 \times \boldsymbol{R}, 0) \to (\boldsymbol{R}^2 \times \boldsymbol{R}, 0)$ *with the form* (2.1), $T \in SO(3)$ *and the functions* $f_{21}, f_{31} \in C^\infty(1,1), f_{24}, f_{34} \in C^\infty(2,1), f_{32} \in C^\infty(3,1)$ *such that*

$$\begin{aligned} {f_{\mathrm{n}2}}^s &= T \circ f \circ \varphi(u,v,s) \\ &= (u, u^2 f_{21}(u) + v^2 + usf_{24}(u,s), \\ &\qquad u^2 f_{31}(u) + v^2 f_{32}(u,v,s) + usf_{34}(u,s)), \end{aligned} \tag{2.3}$$

where

$$f_{32}(u,v,s) = c_0(u,s) + vc_1(u,s) + v^2 c_2(u,v^2,s) + v^3 c_3(u,v^2,s) \tag{2.4}$$

and $f_{32}(0,0,0) = c_0(0,0) = 0$. *Furthermore,* $f(u,v,0) = g(u,v)$ *is a cuspidal* S_k^+ *singularity (respectively, cuspidal* S_k^- *singularity) if and only if* $\partial^i c_1/\partial u^i(0,0,0) = 0$ $(i = 1, \dots, k)$ *and* $\partial^{k+1} c_1/\partial u^{k+1}(0,0,0) c_3(0,0,0) > 0$ *(respectively,* < 0*) hold* $(k \in \boldsymbol{Z}_{\geq 0})$*. If* $(dc_1/ds)(0,0) \neq 0$*, then one can further reduce* $c_1(0,s) = s$*.*

In Corollary 2.6, the given f and ${f_{\mathrm{n}2}}^s$ are equivalent as deformations (Definition 2.2), and they have the same differential geometric properties.

We remark that normal forms for the cuspidal $S_k^\pm$ singularities itself is given in [2]. If $(dc_1/ds)(0,0) \neq 0$, then ${f_{\mathrm{n}2}}^s$ is a generic deformation. In what follows, we assume $(dc_1/ds)(0,0) \neq 0$ and $c_1(0,s) = s$ in ${f_{\mathrm{n}2}}^s$. When $k = 1$, the form ${f_{\mathrm{n}2}}^s$ is called the *normal form of the deformations of a cuspidal* $S_1^\pm$ *singularity.* We see the set of singular points $S_1({f_{\mathrm{n}2}}^s)$ of ${f_{\mathrm{n}2}}^s$ is

$$S_1({f_{\mathrm{n}2}}^s) = \{(u,v) \,|\, v = 0\}.$$

We set $S_2({f_{\mathrm{n}2}}^s)$ to be the set of singular points that are not cuspidal edge. Then it holds that

$$S_2({f_{\mathrm{n}2}}^s) = \{(u,v) \,|\, v = 0, c_1(u,s) = 0\}.$$

The uniqueness of the normal form holds as in the following sense.

Definition 2.7. A function $g(u,s)$ is said to be *regular of order* k *in* s if $\partial g/\partial s(0,0) = \cdots = \partial^{k-1}g/\partial s^{k-1}(0,0) = 0$, $\partial^k g/\partial s^k(0,0) \neq 0$. A function $g(u,s)$ is *regular of finite order in* s if there exists k such that g is regular of order k in s.

Proposition 2.8. *Let* $f^s_{\mathrm{n}x} : (\boldsymbol{R}^2 \times \boldsymbol{R}, 0) \to (\boldsymbol{R}^3, 0)$ $(x = 1, 2)$ *be a map-germ given by the form* (2.2) *or* (2.3) *satisfying that* f_{24} *or* f_{34} *is regular of finite order in* s*. If there exist orientation preserving diffeomorphism-germ* $\varphi : (\boldsymbol{R}^2 \times \boldsymbol{R}, 0) \to (\boldsymbol{R}^2 \times \boldsymbol{R}, 0)$ *with the form of Definition* 2.2 *and* $T \in SO(3)$ *such that*

$$T \circ f^s_{\mathrm{n}x} \circ \varphi(u,v,s) = f^s_{\mathrm{n}x}(u,v,s).$$

Then for any $k \in \boldsymbol{Z}$*, it holds that* $\hat{u} = u$ *and* $j^k\hat{v}(0) = j^k v(0)$ *and* $j^k\hat{s}(0) = j^k s(0)$*, where* $\varphi(u,v,s) = (\hat{u},\hat{v},\hat{s})$*.*

3 Geometry on deformations of cuspidal S_1 singularities

In this section, we consider geometry on the case of $k = 1$ in Corollary 2.6. We set $f = {f_{\mathrm{n}2}}^s$ (see (2.3)). Here, we assume $c_1(0,s) = s$

3.1 Description of singular point

To obtain the location of the singular point, we set $s = -\tilde{s}^2$, since $S_2(f) \neq \emptyset$ is equivalent to $s \leq 0$. If $(u,v) \in S_2(f)$, then $v = 0$ and u depends on $\tilde{s}$. We set this function $u(\tilde{s})$. Since $c_1(0,0) = (c_1)_u(0,0) = 0$ in (2.3), we set

$$c_1(u,s) = s + usd_1(s) + u^2 d_2(s) + u^3 d_3(s) + u^4 d_4(u,s). \tag{3.1}$$

Rewriting f_{32} as (2.4) and (3.1), then we have the following theorem.

Theorem 3.1. *If* $(u,v) \in S_2(f)$ *and* $d_2(0) > 0$*, then* $v = 0$ *holds, and the function* $u(\tilde{s})$ *can be expanded as follows*:

$$\begin{aligned} u(\tilde{s}) = {} & \frac{1}{d_{20}}\tilde{s} + \frac{1}{2d_{20}^4}\Big(d_1(0)d_{20}^2 - d_3(0)\Big)\tilde{s}^2 \\ & + \frac{1}{8d_{20}^7}\Big(\big(d_1^2(0) + 4(d_2)_s(0)\big)d_{20}^4 \\ & - 2\big(3d_1(0)d_3(0) + 2d_4(0,0)\big)d_{20}^2 + 5d_3^2(0)\Big)\tilde{s}^3 + O(4), \end{aligned}$$

where $d_2(0) = d_{20}^2$*. If* $u(\tilde{s}) \neq 0$*,* f *at* $(u(\pm\tilde{s}), 0)$ *are both the cuspidal cross caps.*

If $d_2(0) < 0$, the same calculation can be done by setting $d_2(0) = -d_{20}^2$, and we obtain the same results.

3.2 Self-intersection curves

In this section we focus on self-intersection curves of deformations of cuspidal S_1 singularities. In particular, deformations of the geodesic curvature and the normal curvature of self-intersection curves are considered.

The self-intersection curves are determined by $f(u_1, v_1) = f(u_2, v_2)$ in general. In our case, by looking the first component of ${f_{\mathrm{n}2}}^s$, we see $u_1 = u_2$, we rewrite as u. Focusing on the second component, $v_1^2 = v_2^2$ holds. If we set $v = v_1 = -v_2$, then the self-intersection curves are determined by $i(u, v, s) = c_1(u, s) + v^2 c_3(u, v^2, s) = 0$ from the third component. Since $i_u(u, 0, s) = sd_1(s) + 2ud_2(s) + 3u^2 d_3(s) + 4u^3 d_4(u, s) + u^4 (d_4)_u(u, s)$ and by the assumption $d_2(0) > 0$, it holds that $i_u(u, 0, s) \neq 0$ for small $s \neq 0$ and $u \neq 0$. By the implicit function theorem, there exists a function $u(v, s)$ such that $i(u(v, s), v, s) = c_1(u(v, s), s) + v^2 c_3(u(v, s), v^2, s) = 0$ holds for any v and $s \neq 0$.

Theorem 3.2. *Let $\kappa_g(v, \tilde{s})$ be the geodesic curvature and $\kappa_n(v, \tilde{s})$ be the normal curvature of self-intersection curves $v \mapsto f(u(v, \tilde{s}), v, -\tilde{s}^2)$. Then these curvatures can be expanded as follows*

$$\begin{aligned}\kappa_g(0, \tilde{s}) &= \pm\frac{(-2f_{21}(0)c_3(0,0,0) + (c_1)_{uu}(0,0))}{c_3(0,0,0)} + O_{\tilde{s}}(1),\\ \kappa_n(0, \tilde{s}) &= 2f_{31}(0) + O_{\tilde{s}}(1).\end{aligned}$$

In particular, both of the geodesic curvature and the normal curvature are bounded for fixed $\tilde{s} \neq 0$.

When $s = 0$, only cuspidal S_1^- singularity has self-intersection curves. Therefore, we assume $(c_1)_{uu}(0, 0)c_3(0, 0, 0) < 0$.

Theorem 3.3. *If $(c_1)_{uu}(0, 0)c_3(0, 0, 0) < 0$, then the geodesic curvature κ_g and the normal curvature κ_ν of the each branch of self-intersection curves are bounded and they are*

$$\kappa_g = \frac{2f_{21}(0)c_3(0,0,0) - (c_1)_{uu}(0,0)}{c_3(0,0,0)}, \quad \kappa_\nu = 2f_{31}(0).$$

Furthermore, these coincide with the absolute value of the limits of the geodesic curvature and the limits of the normal curvature of the self-intersection curves on the case of $\tilde{s} \neq 0$ respectively.

3.3 Geometric invariants of deformation

In [1], the bias and the secondary cuspidal curvature are defined for non-degenerate singular point which is not a cuspidal edge. The bias measures the bias of a curve around the singular point, and the secondary cuspidal curvature measures sharpness of 5/2-cusp. We calculate these invariants for deformations of cuspidal $S_1^\pm$ singularities.

Definition 3.4. [1, Definition 3.7] Let $f : (\boldsymbol{R}^2, 0) \to (\boldsymbol{R}^3, 0)$ be a frontal satisfying $j^2 f(0) = (u, v^2, 0)$ such that f at $v = 0$ is not a cuspidal edge. We take a coordinate system (u, v) satisfying the set of singular point of f $S(f) = \{v = 0\}$. Then there exists a vector field $\tilde{\eta}|_{\{v=0\}}$ such that

$$\tilde{\eta} f(u, 0) = 0, \quad \langle f_u, \tilde{\eta}^2 f\rangle(u, 0) = \langle f_u, \tilde{\eta}^3 f\rangle(u, 0) = 0.$$

Then there exists ℓ such that $\tilde{\eta}^3 f(0,0) = \ell\tilde{\eta}^2 f(0,0)$. Using this vector field $\tilde{\eta}$ and ℓ the *bias* r_b and the *secondary cuspidal curvature* r_c are defined by

$$r_b = \left.\frac{|f_u|^2 \det(f_u, \tilde{\eta}^2 f, \tilde{\eta}^4 f)}{|f_u \times \tilde{\eta}^2 f|^3}\right|_{(u,v)=(0,0)},$$

$$r_c = \left.\frac{|f_u|^{5/2} \det(f_u, \tilde{\eta}^2 f, 3\tilde{\eta}^5 f - 10\ell\tilde{\eta}^4 f)}{|f_u \times \tilde{\eta}^2 f|^{7/2}}\right|_{(u,v)=(0,0)}.$$

Theorem 3.5. *Let $f : (\boldsymbol{R}^2, 0) \to (\boldsymbol{R}^3, 0)$ be a deformation of cuspidal S_1 singularities. Then the bias r_b and the secondary cuspidal curvature r_c of f at $(u(\pm\tilde{s}), 0, -\tilde{s}^2)$ can be expanded as a function of $\tilde{s}$ as follows:*

$$r_b = 6c_2(0,0,0) + \frac{6(-2f_{21}(0)(c_0)_u(0,0) + (c_2)_u(0,0,0))}{d_{20}}\tilde{s} + O_{\tilde{s}}(2),$$

$$r_c = 45\sqrt{2}c_3(0,0,0) + \frac{45\sqrt{2}(c_3)_u(0,0,0)}{d_{20}}\tilde{s} + O_{\tilde{s}}(2).$$

3.4 Geometry on trajectory of singular points

We give a geometric meaning of the lowest order coefficients $f_{24}(0,0)$ and $f_{34}(0,0)$ including the deformation parameters. The trajectory of the singular points $S_2({f_{\mathrm{n}2}}^{-\tilde{s}^2})$ for the deformation of the cuspidal $S_1^{\pm}$ singularities $f = {f_{\mathrm{n}2}}^s$ is a space curve passing through the origin. It is parameterized by

$$\gamma(\tilde{s}) := {f_{\mathrm{n}2}}^{-\tilde{s}^2}(u(\tilde{s}), 0),$$

where $u(\tilde{s})$ is given in Theorem 3.1. Then the curvature κ of γ as a space curve at $\tilde{s} = 0$ satisfies $\kappa = 2(f_{21}^2(0) + f_{31}^2(0))^{1/2}$. Moreover, if $f_{21}^2(0) + f_{31}^2(0) \neq 0$, then

$$f_{24}(0,0) = \frac{\kappa\tau f_{31}(0) - f_{21}(0)d_{20}\kappa' + 3\kappa\dfrac{df_{31}}{du}(0)}{3\kappa d_{20}^2}$$

and

$$f_{34}(0,0) = -\frac{\kappa\tau f_{21}(0) + f_{31}(0)d_{20}\kappa' - 3\kappa\dfrac{df_{31}}{du}(0)}{3\kappa d_{20}^2}$$

hold, where τ is the torsion of γ, and $\kappa' = d\kappa/d\tilde{s}$.

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