Legendrian dualities and their applications

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Dedicated to Professor Donghe Pei on the occasion of his 60th birthday

and Professor Weizhi Sun on the occasion of his 70th birthday

Abstract

This paper presents a review of Legendrian dualities. These dualities were initially developed by S. Izumiya and subsequently generalized by S. Izumiya and the author. They serve as fundamental tools for researching the extrinsic differential geometry of submanifolds in non-flat spaces.

Keywords: Legendrian duality; non-flat space; front; caustic; semi-Euclidean space.

1 Introduction

S. Izumiya demonstrated a theorem named Legendrian dualities for pseudo spheres in Minkowski space in [7, 8]. This theorem has since become a fundamental tool for studying the extrinsic differential geometry of submanifolds immersed in these pseudo spheres from the perspective of singularity theory (cf., [3, 4, 5, 9, 10, 11, 12]). Subsequently, S. Izumiya and the author of this paper developed similar Legendrian dualities between pseudo spheres in general semi-Euclidean spaces [2]. In this paper, we will provide a review of these dualities and present some aspects of their applications, though this review may be somewhat limited.

We first prepare basic notions on semi-Euclidean space. Let $\mathbb{R}^{n+1} = \{(x_1, \ldots, x_{n+1}) \mid x_i \in \mathbb{R}, i = 1, \ldots, n+1\}$ be an $(n+1)$-dimensional vector space. For any vectors $\boldsymbol{x} = (x_1, \ldots, x_{n+1})$, $\boldsymbol{y} = (y_1, \ldots, y_{n+1})$ in $\mathbb{R}^{n+1}$, the *pseudo scalar product* of $\boldsymbol{x}$ and $\boldsymbol{y}$ is defined by $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = -\sum_{i=1}^{r} x_i y_i + \sum_{i=r+1}^{n+1} x_i y_i$. The space $(\mathbb{R}^{n+1}, \langle , \rangle)$ is called *semi-Euclidean* $(n+1)$-*space with index* r and denoted by $\mathbb{R}^{n+1}_r$. We say that a vector $\boldsymbol{x}$ in $\mathbb{R}^{n+1}_r \setminus \{\boldsymbol{0}\}$ is *spacelike*, *null* or *timelike* if $\langle \boldsymbol{x}, \boldsymbol{x} \rangle > 0, = 0$ or < 0 respectively. The norm of the vector $\boldsymbol{x} \in \mathbb{R}^{n+1}_r$ is defined by $\|\boldsymbol{x}\| = \sqrt{|\langle \boldsymbol{x}, \boldsymbol{x} \rangle|}$. We have the following three kinds of pseudo-spheres in $\mathbb{R}^{n+1}_r$: The *pseudohyperbolic* n*-space with idex* $r-1$ is defined by

$$H^n_{r-1} = \{\boldsymbol{x} \in \mathbb{R}^{n+1}_r \mid \langle \boldsymbol{x}, \boldsymbol{x} \rangle = -1\},$$

the *pseudo* n*-sphere with idex* r by

$$S^n_r = \{\boldsymbol{x} \in \mathbb{R}^{n+1}_r \mid \langle \boldsymbol{x}, \boldsymbol{x} \rangle = 1\}$$

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and the (*open*) *nullcone* by

$$\Lambda^n = \{\boldsymbol{x} \in \mathbb{R}^{n+1}_r \setminus \{\boldsymbol{0}\} | \langle \boldsymbol{x}, \boldsymbol{x} \rangle = 0 \ \}.$$

In relativity theory $\mathbb{R}^{n+1}_1$ is called *Minkoski* $(n+1)$*-space*, S^n_1 *de Sitter* n*-space* and H^n_1 is *Anti de Sitter* n*-space* which is denoted by AdS^n. These are the Lorentzian space forms. Moreover, H^n_0 is called *hyperbolic* n*-space* and S^n_0 is the Euclidean unit sphere which are the Riemannian space forms.

2 Legendrian dualities

In [7], S. Izumiya has shown the basic duality theorem which is the fundamental tool for the study of spacelike hypersurfaces in Minkowski pseudo-spheres and then we generalize the similar dualities in semi-Euclidean space in [2]. We now introduce these dualities as follows:

(1) (a) $H^n_{r-1} \times S^n_r \supset \Delta_1 = \{(\boldsymbol{v}, \boldsymbol{w}) \mid \langle \boldsymbol{v}, \boldsymbol{w} \rangle = 0 \ \}$,
(b) $\pi_{11} : \Delta_1 \longrightarrow H^n_{r-1}, \pi_{12} : \Delta_1 \longrightarrow S^n_r$,
(c) $\theta_{11} = \langle d\boldsymbol{v}, \boldsymbol{w} \rangle | \Delta_1$, $\theta_{12} = \langle \boldsymbol{v}, d\boldsymbol{w} \rangle | \Delta_1$.
(2) (a) $H^n_{r-1} \times \Lambda^n \supset \Delta_2 = \{(\boldsymbol{v}, \boldsymbol{w}) \mid \langle \boldsymbol{v}, \boldsymbol{w} \rangle = -1 \ \}$,
(b) $\pi_{21} : \Delta_2 \longrightarrow H^n_{r-1}, \pi_{22} : \Delta_2 \longrightarrow \Lambda^n$,
(c) $\theta_{21} = \langle d\boldsymbol{v}, \boldsymbol{w} \rangle | \Delta_2$, $\theta_{22} = \langle \boldsymbol{v}, d\boldsymbol{w} \rangle | \Delta_2$.
(3) (a) $\Lambda^n \times S^n_r \supset \Delta_3 = \{(\boldsymbol{v}, \boldsymbol{w}) \mid \langle \boldsymbol{v}, \boldsymbol{w} \rangle = 1 \ \}$,
(b) $\pi_{31} : \Delta_3 \longrightarrow \Lambda^n, \pi_{32} : \Delta_3 \longrightarrow S^n_r$,
(c) $\theta_{31} = \langle d\boldsymbol{v}, \boldsymbol{w} \rangle | \Delta_3$, $\theta_{32} = \langle \boldsymbol{v}, d\boldsymbol{w} \rangle | \Delta_3$.
(4) (a) $\Lambda^n \times \Lambda^n \supset \Delta_4 = \{(\boldsymbol{v}, \boldsymbol{w}) \mid \langle \boldsymbol{v}, \boldsymbol{w} \rangle = -2 \ \}$,
(b) $\pi_{41} : \Delta_4 \longrightarrow \Lambda^n, \pi_{42} : \Delta_4 \longrightarrow \Lambda^n$,
(c) $\theta_{41} = \langle d\boldsymbol{v}, \boldsymbol{w} \rangle | \Delta_4$, $\theta_{42} = \langle \boldsymbol{v}, d\boldsymbol{w} \rangle | \Delta_4$.

Here, $\pi_{i1}(\boldsymbol{v}, \boldsymbol{w}) = \boldsymbol{v}$, $\pi_{i2}(\boldsymbol{v}, \boldsymbol{w}) = \boldsymbol{w}$, $\langle d\boldsymbol{v}, \boldsymbol{w} \rangle = -w_0 dv_0 + \sum_{i=1}^n w_i dv_i$ and $\langle \boldsymbol{v}, d\boldsymbol{w} \rangle = -v_0 dw_0 + \sum_{i=1}^n v_i dw_i$ are one-forms on $\mathbb{R}^{n+1}_r \times \mathbb{R}^{n+1}_r$.

We remark that $\theta_{i1}^{-1}(0)$ and $\theta_{i2}^{-1}(0)$ define the same tangent hyperplane field over Δ_i which is denoted by K_i. The basic duality theorem is the following theorem:

Theorem 2.1 *Under the same notations as the previous paragraph, each* (Δ_i, K_i) $(i = 1, 2, 3, 4)$ *is a contact manifold and both of* π_{ij} $(j = 1, 2)$ *are Legendrian fibrations. Moreover those contact manifolds are contact diffeomorphic each other.*

We now explain the situation by a "mandala of Legendrian dualities" as the following commutative diagram:

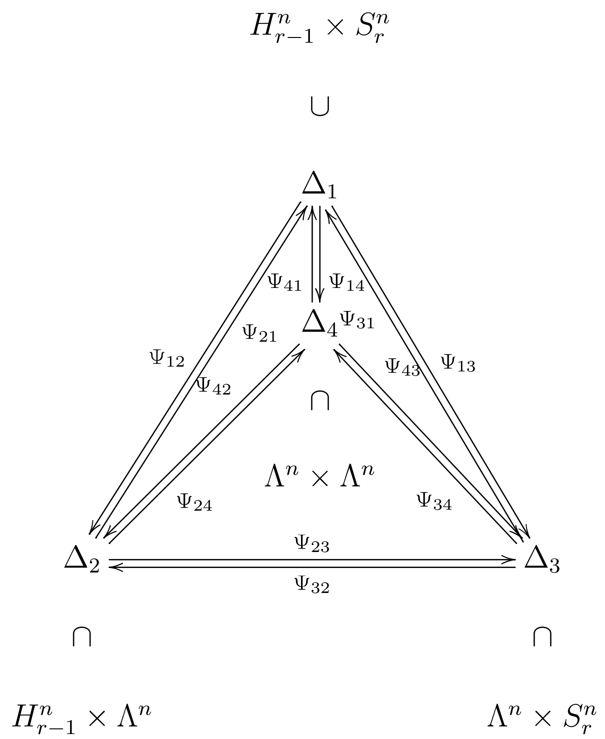


Figure 1. The Mandala of Legendrian Dulaities

We can also consider the following two extra double fibrations:

(5) (a) $S^n_r \times S^n_r \supset \Delta_5 = \{(\boldsymbol{v}, \boldsymbol{w}) \mid \langle \boldsymbol{v}, \boldsymbol{w} \rangle = 0\ \}$,
(b) $\pi_{51} : \Delta_5 \longrightarrow S^n_r, \pi_{52} : \Delta_5 \longrightarrow S^n_r$,
(c) $\theta_{51} = \langle d\boldsymbol{v}, \boldsymbol{w} \rangle | \Delta_5$, $\theta_{52} = \langle \boldsymbol{v}, d\boldsymbol{w} \rangle | \Delta_5$.

(6) (a) $H^n_{r-1} \times H^n_{r-1} \supset \Delta_6 = \{(\boldsymbol{v}, \boldsymbol{w}) \mid \langle \boldsymbol{v}, \boldsymbol{w} \rangle = 0\ \}$,
(b) $\pi_{61} : \Delta_6 \longrightarrow H^n_{r-1}, \pi_{62} : \Delta_6 \longrightarrow H^n_{r-1}$,
(c) $\theta_{61} = \langle d\boldsymbol{v}, \boldsymbol{w} \rangle | \Delta_6$, $\theta_{62} = \langle \boldsymbol{v}, d\boldsymbol{w} \rangle | \Delta_6$.

We have the following theorem.

Theorem 2.2 *Under the same notations as the above, each* (Δ_i, K_i) $(i = 5, 6)$ *is a contact manifold and both of* π_{ij} $(j = 1, 2)$ *are Legendrian fibrations.*

We can show that (Δ_5, K_5) (respectively, (Δ_6, K_6)) is locally diffeomorphic to the projective cotangent bundle $\pi : PT^*H^n_{r-1} \longrightarrow H^n_{r-1}$ (respectively, $\pi : PT^*S^n_r \longrightarrow S^n_r$) which sends K_i to the canonical contact structure. We remark that these contact manifolds (Δ_j, K_j) $(j = 5, 6)$ are not canonically contact diffeomorphic to (Δ_i, K_i) $(i = 1, 2, 3, 4)$. Therefore we cannot add these contact manifolds to the mandala of Legendrian dualities. By definition, S^n_0 is a unit sphere in Euclidean space $\mathbb{R}^{n+1}_0$, so that (Δ_5, K_5) is the well known classical spherical duality in this case. Finally we remark that $\Delta_6 = \emptyset$ in $H^n_0 \times H^n_0$.

3 Applications

3.1 Linear Weingarten surfaces

A surface M immersed in different ambient 3-space is called a *Weigarten surfaces*, if the principal curvatures κ_1, κ_2 of M satisfy $f(\kappa_1, \kappa_2) = 0$ or the Gauss curvature K and the mean curvature H satisfy $g(K, H) = 0$, where f and g are smooth functions. Moreover, if there exist constants a, b and c, such that the smooth functions f and g are linear, namely $a\kappa_1 + b\kappa_2 = c$ or $aK + bH = c$, we term M a *linear Weigarten surfaces*. Furthermore, a linear Weigarten surface is also called a *Bryant type* provided that $a + b \neq 0$, briefly*BLW-surface*. We would like to point out that when a linear Weingarten surface of Bryant type is immersed in the de Sitter 3-space, the resulting surface is termed a *linear Weingarten surface of Bianchi type*, briefly, *BLW-surface*. We note that the theory of linear Weingarten surfaces has a long standing research history. Nevertheless, from our point of view, we only reference the research results achieved by Gálvez, Martínez and Milán in [6] and those by Aledo and Espinar in [1] as follows.

Theorem 3.1 ([6], Theorem 2 ii) *Let V be a noncompact, simply connected domain. Fix a meromorphic map $A : V \to SL(2, \mathbb{C})$ satisfying*

$$A^{-1}dA = \begin{pmatrix} 0 & \omega \\ dh & 0 \end{pmatrix},$$

where h is a meromorphic function and ω a holomorphic one-form. If

$$\sigma = (a+b)\left((1+\varepsilon\|h\|^2)^2\|\omega\|^2 - \frac{(1-\varepsilon)^2\|dh\|^2}{(1+\varepsilon\|h\|^2)^2}\right)$$

is positive definite then $f = A(\Omega_+)A^*$ is a linear Weingarten surface. Where,

$$\Omega_\pm = \begin{pmatrix} \frac{(1\pm\varepsilon^2\|h\|^2}{1+\varepsilon\|h\|^2} & \mp\varepsilon\bar{h} \\ \mp\varepsilon h & \pm(1+\varepsilon\|h\|^2) \end{pmatrix}, \ \varepsilon = \frac{a}{a+b}, \ 1+\varepsilon\|h\|^2 > 0.$$

We remark that the hyperbolic Gauss map g of f is given by $g = A(\Omega_-)A^*$. Aledo and Espinar in [1] showed that g is a BLW-surface in S^3_1. Moreover, from the Legendrian duality viewpoint, we can show that f and g are Δ_1 dual to each other. Consequently, we arrive at the following assertion proposed by Izumiya and Saji.

Theorem 3.2 ([12], Theorem 5.2) *Let $\mathcal{L}_1 : U \to \Delta_1$ be a Legendrian immersion. Suppose that both of $\pi_{11} \circ \mathcal{L}_1 : U \to H^3_0$ and $\pi_{12} \circ \mathcal{L}_1 : U \to S^3_1$ are immersions. Then $\pi_{11} \circ \mathcal{L}_1 = f$ is a linear Weingarten surface of Bryant type if and only if $\pi_{12} \circ \mathcal{L}_1 = g$ is a linear Weingarten surface of Bianchi type.*

This implies that the dual surface of a linear Weingarten surface in H^3_0 is also a linear Weingarten surface in S^3_1 and vice versa.

3.2 The evolutes of fronts in hyperbolic 2-space and de Sitter 2-space

In this subsection, we review the differential geometry of smooth curves in hyperbolic 2-space and de Sitter 2-space. For details, see [5].

3.2.1 The evolutes of fronts in hyperbolic 2-space

We firstly introduce the differential geometry of curves in H^2_0. Let $\boldsymbol{\gamma}_h : I \to H^2_0$ be a smooth curve. We call $\boldsymbol{\gamma}_h$ the *spacelike frontal* in H^2_0, if there exists a smooth mapping $\boldsymbol{\gamma}^d_h : I \to S^2_1$, such that the pair $(\boldsymbol{\gamma}_h, \boldsymbol{\gamma}^d_h) : I \to \Delta_1$ satisfies $(\boldsymbol{\gamma}_h(t), \boldsymbol{\gamma}^d_h(t))^*\theta = 0$ for all $t \in I$. Here θ is a canonical contact 1-form on Δ_1. The condition $(\boldsymbol{\gamma}_h(t), \boldsymbol{\gamma}^d_h(t))^*\theta = 0$ is equivalent to $\langle \dot{\boldsymbol{\gamma}}_h(t), \boldsymbol{\gamma}^d_h(t)\rangle = 0$, for all $t \in I$. We call $(\boldsymbol{\gamma}_h, \boldsymbol{\gamma}^d_h)$ the *spacelike Legendrian curve* in Δ_1. Moreover, if $(\boldsymbol{\gamma}_h, \boldsymbol{\gamma}^d_h)$ is an immersion, we call $\boldsymbol{\gamma}_h$ the *spacelike front* in H^2_0 and $(\boldsymbol{\gamma}_h, \boldsymbol{\gamma}^d_h)$ the *spacelike Legendrian immersion* in Δ_1. Let $\boldsymbol{\gamma}^s_h(t) = \boldsymbol{\gamma}_h(t) \wedge \boldsymbol{\gamma}^d_h(t) \in S^2_1$. We have a moving frame $\{\boldsymbol{\gamma}_h, \boldsymbol{\gamma}^d_h, \boldsymbol{\gamma}^s_h\}$ which called the *hyperbolic Legendrian Frenet frame* of $\mathbb{R}^3_1$ along $\boldsymbol{\gamma}_h$. By the standard arguments, we have the following *hyperbolic Legendrian Frenet-Serret type formula*:

$$\begin{pmatrix} \dot{\boldsymbol{\gamma}}_h(t) \\ \dot{\boldsymbol{\gamma}}^d_h(t) \\ \dot{\boldsymbol{\gamma}}^s_h(t) \end{pmatrix} = \begin{pmatrix} 0 & 0 & m_h(t) \\ 0 & 0 & n_h(t) \\ m_h(t) & -n_h(t) & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{\gamma}_h(t) \\ \boldsymbol{\gamma}^d_h(t) \\ \boldsymbol{\gamma}^s_h(t) \end{pmatrix},$$

where $m_h(t) = \langle \dot{\boldsymbol{\gamma}}_h(t), \boldsymbol{\gamma}^s_h(t)\rangle$ and $n_h(t) = \langle \dot{\boldsymbol{\gamma}}^d_h(t), \boldsymbol{\gamma}^s_h(t)\rangle$. We call the pair (m_h, n_h) the *spacelike hyperbolic Legendrian curvature* of spacelike Legendrian curve $(\boldsymbol{\gamma}_h, \boldsymbol{\gamma}^d_h)$.

Example 3.3 *Let $\boldsymbol{\gamma}_h$ be a regular curve in H^2_0 with the hyperbolic geodesic curvature κ_h. If we take $\boldsymbol{\gamma}^d_h = \boldsymbol{e}_h$, then $(\boldsymbol{\gamma}_h, \boldsymbol{\gamma}^d_h)$ is a spacelike Legendrian curve with the spacelike hyperbolic Legendrian curvature $(-||\dot{\boldsymbol{\gamma}}_h||, ||\dot{\boldsymbol{\gamma}}_h||\kappa_h)$. In fact, it is a spacelike Legendrian immersion. Moreover, by a straightforward calculation, we have $n_h(t) = |m_h(t)|\kappa_h(t)$ for all $t \in I$. In this case, we have $n_h(t) = 0$ if and only if $\kappa_h(t) = 0$.*

Example 3.4 *Let $\boldsymbol{\gamma}_h : I \to H^2_0$ be $\boldsymbol{\gamma}_h(t) = (\sqrt{1+t^4+t^6}, t^2, t^3)$. It is obviously that the origin is the singular point of $\boldsymbol{\gamma}_h$. We assume that*

$$\boldsymbol{\gamma}^d_h(t) = \frac{1}{\sqrt{4+9t^2+t^6}}\left(t^3\sqrt{1+t^4+t^6}, t^5+3t, t^6-2\right).$$

By a straightforward calculation, we have

$$\boldsymbol{\gamma}^s_h(t) = \frac{\sqrt{1+t^4+t^6}}{\sqrt{t^6+9t^2+4}}\left(\frac{3t^4+2t^2}{\sqrt{1+t^4+t^6}}, 2, 3t\right)$$

and $(\boldsymbol{\gamma}_h, \boldsymbol{\gamma}^d_h)$ is a spacelike Legendrian immersion with the spacelike hyperbolic Legendrian curvature (m_h, n_h), where

$$\begin{aligned} m_h(t) &= \frac{t\sqrt{4+9t^2+t^6}}{\sqrt{1+t^4+t^6}}, \\ n_h(t) &= \frac{t^{10}+15t^6+10t^4+6}{(4+9t^2+t^6)\sqrt{1+t^4+t^6}}. \end{aligned}$$

We call $\boldsymbol{\gamma}_h$ the hyperbolic 3/2-cusp, see Figure 2.

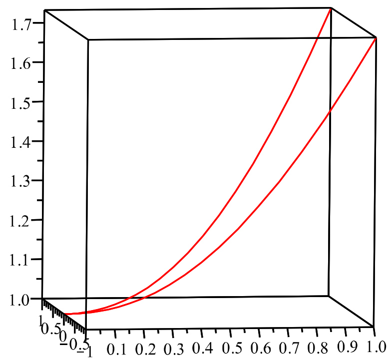


Figure 2. hyperbolic 3/2-cusp

We now consider the geometric meanings of evolutes of spacelike fronts in H^2_0. Let $(\boldsymbol{\gamma}_h, \boldsymbol{\gamma}^d_h) : I \to \Delta_1$ be a spacelike Legendrian immersion with the spacelike hyperbolic Legendrian curvature (m_h, n_h) which satisfies $n_h^2(t) \neq m_h^2(t)$ for all $t \in I$. We define a mapping $\mathcal{E}_v(\boldsymbol{\gamma}_h) : I \to \mathbb{R}^3_1$ by

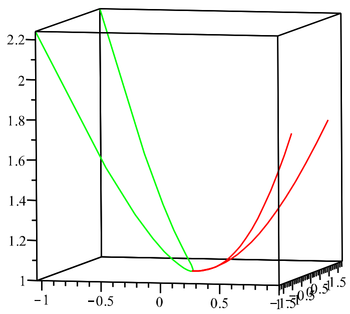
$$\mathcal{E}_v(\boldsymbol{\gamma}_h)(t) = \pm \frac{1}{\sqrt{|n_h^2(t) - m_h^2(t)|}} \big(n_h(t)\boldsymbol{\gamma}_h(t) - m_h(t)\boldsymbol{\gamma}^d_h(t)\big)$$

and call it the *totally evolute* of $\boldsymbol{\gamma}_h$ in $\mathbb{R}^3_1$. We remark that if $n_h^2(t) > m_h^2(t)$, then $\mathcal{E}_v(\boldsymbol{\gamma}_h)(t) \in H^2_0$. In this case, we denote it by $\mathcal{E}^h_v(\boldsymbol{\gamma}_h)$ and call it the *hyperbolic evolute* of $\boldsymbol{\gamma}_h$. Moreover, if $n_h^2(t) < m_h^2(t)$, then $\mathcal{E}_v(\boldsymbol{\gamma}_h)(t) \in S^2_1$. We rewrite it as $\mathcal{E}^d_v(\boldsymbol{\gamma}_h)$ and call it the *de Sitter evolute* of $\boldsymbol{\gamma}_h$.

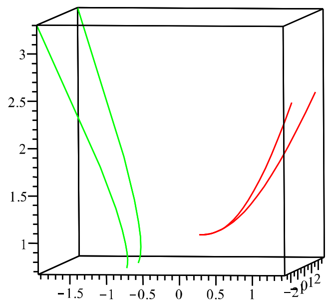
Example 3.5 *Let $\boldsymbol{\gamma}_h : I \to H^2(-1)$ be $\boldsymbol{\gamma}_h(t) = (\sqrt{1+t^4+t^6}, t^2, t^3)$ be the hyperbolic 3/2-cusp defined in the Example 3.4. By a straightforward calculation, we have*

$$\mathcal{E}_v(\boldsymbol{\gamma}_h)(t) = \pm \frac{\big(6(1+t^4+t^6)^{3/2},\ 3t^8+6t^6-27t^4-6t^2,\ 6t^9+8t^7+24t^3+8t\big)}{\sqrt{|(t^{10}+15t^6+10t^4+6)^2 - t^2(4+9t^2+t^6)^3|}},$$

see Figure 3.



hyperbolic evolute of the hyperbolic 3/2-cusp



de Sitter evolute of the hyperbolic 3/2-cusp

Figure 3

3.2.2 The evolutes of fronts in de Sitter 2-space

We now consider the differential geometry of spacelike curves in S^2_1. Suppose $\boldsymbol{\gamma}_d : I \to S^2_1$ is a spacelike curve at any regular points $t \in I$, namely, $\dot{\boldsymbol{\gamma}}_d(t)$ is a spacelike vector at the regular points. We call $\boldsymbol{\gamma}_d$ the *spacelike frontal* in S^2_1 if there exists a smooth mapping $\boldsymbol{\gamma}^h_d : I \to H^2_0$ such that the pair $(\boldsymbol{\gamma}^h_d, \boldsymbol{\gamma}_d) : I \to \Delta_1$ satisfies $(\boldsymbol{\gamma}^h_d(t), \boldsymbol{\gamma}_d(t))^*\theta = 0$ for all $t \in I$. The condition $(\boldsymbol{\gamma}^h_d(t), \boldsymbol{\gamma}_d(t))^*\theta = 0$ is equivalent to $\langle \dot{\boldsymbol{\gamma}}_d(t), \boldsymbol{\gamma}^h_d(t) \rangle = 0$, for all $t \in I$. We call $(\boldsymbol{\gamma}_d, \boldsymbol{\gamma}^h_d)$ the *spacelike Legendrian curve* in Δ_1. Moreover, if $(\boldsymbol{\gamma}_d, \boldsymbol{\gamma}^h_d)$ is an immersion, we call $\boldsymbol{\gamma}_d$ the *spacelike front* in S^2_1 and $(\boldsymbol{\gamma}_d, \boldsymbol{\gamma}^h_d)$ the *spacelike Legendrian immersion* in Δ_1. Let $\boldsymbol{\gamma}^s_d(t) = \boldsymbol{\gamma}_d(t) \wedge \boldsymbol{\gamma}^h_d(t) \in S^2_1$. We have a moving frame $\{\boldsymbol{\gamma}_d, \boldsymbol{\gamma}^h_d, \boldsymbol{\gamma}^s_d\}$ which called the *spacelike de Sitter Legendrian Frenet frame* of $\mathbb{R}^3_1$ along $\boldsymbol{\gamma}_d$. By the standard arguments, we have the following *spacelike de Sitter Legendrian Frenet-Serret type formula*:

$$\begin{pmatrix} \dot{\boldsymbol{\gamma}}_d(t) \\ \dot{\boldsymbol{\gamma}}^h_d(t) \\ \dot{\boldsymbol{\gamma}}^s_d(t) \end{pmatrix} = \begin{pmatrix} 0 & 0 & m_d(t) \\ 0 & 0 & n_d(t) \\ -m_d(t) & n_d(t) & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{\gamma}_d(t) \\ \boldsymbol{\gamma}^h_d(t) \\ \boldsymbol{\gamma}^s_d(t) \end{pmatrix},$$

where $m_d(t) = \langle \dot{\boldsymbol{\gamma}}_d(t), \boldsymbol{\gamma}^s_d(t) \rangle$ and $n_d(t) = \langle \dot{\boldsymbol{\gamma}}^h_d(t), \boldsymbol{\gamma}^s_d(t) \rangle$. We call the pair (m_d, n_d) the *spacelike de Sitter Legendrian curvature* of the spacelike Legendrian curve $(\boldsymbol{\gamma}_d, \boldsymbol{\gamma}^h_d)$. Under the assumption $n^2_d(t) \neq m^2_d(t)$ for all $t \in I$, we define a mapping $\mathcal{E}_v(\boldsymbol{\gamma}_d) : I \to \mathbb{R}^3_1$ by

$$\mathcal{E}_v(\boldsymbol{\gamma}_d)(t) = \pm \frac{1}{\sqrt{|n^2_d(t) - m^2_d(t)|}} \left(n_d(t)\boldsymbol{\gamma}_d(t) - m_d(t)\boldsymbol{\gamma}^h_d(t) \right)$$

and call it the *totally evolute* of $\boldsymbol{\gamma}_d$ in $\mathbb{R}^3_1$. We remark that if $n^2_d(t) < m^2_d(t)$, then $\mathcal{E}_v(\boldsymbol{\gamma}_d)(t) \in H^2_0$. In this case, we denote it by $\mathcal{E}^h_v(\boldsymbol{\gamma}_d)$ and call it the *hyperbolic evolute* of $\boldsymbol{\gamma}_d$. Moreover, if $n^2_d(t) > m^2_d(t)$, then $\mathcal{E}_v(\boldsymbol{\gamma}_d)(t) \in S^2_1$. We rewrite it as $\mathcal{E}^d_v(\boldsymbol{\gamma}_d)$ and call it the *de Sitter evolute* of $\boldsymbol{\gamma}_d$.

We now show the relationships between the totally evolute $\mathcal{E}_v(\boldsymbol{\gamma}_h)$ of a spacelike front $\boldsymbol{\gamma}_h$ in H^2_0 and the totally evolute $\mathcal{E}_v(\boldsymbol{\gamma}_d)$ of a spacelike front $\boldsymbol{\gamma}_d$ in S^2_1 as flollows.

Theorem 3.6 ([5], Theorem 5.1) *Suppose that $(\boldsymbol{\gamma}_h, \boldsymbol{\gamma}_d) : I \to \Delta_1$ is a spacelike Legendrian immersion with the spacelike hyperbolic Legendrian curvature (m_h, n_h) which satisfies $m^2_h(t) \neq n^2_h(t)$ for all $t \in I$. Then we have:*

(i) *If $m^2_h(t) < n^2_h(t)$, then $\mathcal{E}^h_v(\boldsymbol{\gamma}_h)(t) = \mathcal{E}^h_v(\boldsymbol{\gamma}_d)(t)$.*

(ii) *If $m^2_h(t) > n^2_h(t)$, then $\mathcal{E}^d_v(\boldsymbol{\gamma}_h)(t) = \mathcal{E}^d_v(\boldsymbol{\gamma}_d)(t)$.*

This assertion implies that when $\boldsymbol{\gamma}_h$ and $\boldsymbol{\gamma}_d$ are Δ_1-dual to each other, they share identical evolutes.

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