Phase singularity and criticality

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1 Introduction

This is a survey article on the results appeared in [1]. Recalling the results, we provide several observations on related topics of singularity theory.

The notion of "phase" is very important in various area of mathematical analysis and physical sciences related to phenomena of oscillations and waves. The phase phenomena are described by "phase functions". We mean by a "phase singularity" a zero point of a phase function, and by a "phase criticality"its critical point. See also related papers [2, 5, 6, 7, 19, 20]. Remark that also phase singularities appeared in solutions of differential equations are studied in the paper [1]. See also [15, 14].

In the present survey paper we discuss only on properties of phase singular points and phase critical points from the universal mathematical viewpoints, which are independent from physical settings.

2 Phase singularity and criticality

Let us denote by $\mathbb{C}$ the plane of complex numbers and write a complex number as

$$u + iw = re^{i\theta},$$

u, w being the real part and the imaginary part respectively, while r, θ are the modulus (or the amplitude) and the argument (or the phase), respectively.

Let us consider a complex valued C^∞ scalar function,

$$\Psi : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{C}, \quad \Psi(x, y, t) = u(x, y, t) + iw(x, y, t),$$

on the x-y-plane, depending on the "time" (or any other real single parameter) t.

If $\Psi(x, y, t) \neq 0$ at a point (x, y) and at a moment t, we can write as

$$\Psi(x, y, t) = r(x, y, t)e^{i\theta(x,y,t)},$$

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2020 *Mathematics Subject Classification*: Primary 58K40, Secondary 58K35, 42B20, 32S50 uniquely with $r(x,y,t) > 0$ and $\theta(x,y,t) \mod 2\pi$.

We are concerned with the zero-locus ("dislocation locus") at a moment $t = t_0$,

$$\{(x,y) \mid \Psi(x,y,t_0) = 0\} = \{(x,y) \mid u(x,y,t_0) = 0, w(x,y,t_0) = 0\}.$$

Outside of the zero-locus, the *phase function* $\theta(x,y,t_0)$ is well-defined mod. 2π, and it may have *critical points* where

$$\frac{\partial \theta}{\partial x}(x,y,t_0) = \frac{\partial \theta}{\partial y}(x,y,t_0) = 0.$$

3 Classification of phase singularities on the plane

Definition 3.1 Two function-germs $\Psi : (\mathbb{R}^2 \times \mathbb{R}, (x_0, y_0, t_0)) \to (\mathbb{C}, 0)$ and $\Psi' : (\mathbb{R}^2 \times \mathbb{R}, (x_0', y_0', t_0')) \to (\mathbb{C}, 0)$, at moments t_0 and t_0', are called *radially equivalent* if the diagram

$$\begin{array}{ccc} (\mathbb{R}^2, (x_0, y_0)) & \xrightarrow{\Psi|_{t=t_0}} & (\mathbb{C}, 0) \\ \sigma \downarrow \cong & & \cong \downarrow \tau \\ (\mathbb{R}^2, (x_0', y_0')) & \xrightarrow[\Psi'|_{t=t_0'}]{} & (\mathbb{C}, 0) \end{array}$$

commutes, for diffeomorphism-germ σ and τ of form

$$\tau(u,w) = (\rho(u,w)(au + bw), \rho(u,w)(cu + dw)),$$

$\rho(u,w) > 0, a, b, c, d \in \mathbb{R}, ad - bc \neq 0$.

Remark 3.2 Note that the *radial equivalence* is weaker than the *right-equivalence* and is stronger than the *right-left-equivalence*([3, 4, 13, 16, 21, 22]).

The radial equivalence on complex valued functions of real variables preserves both the dislocation locus, that is the *phase singularity*, and the critical locus of phase functions, that is the *phase criticality*.

Theorem 3.3 ([1]) *For a generic complex valued function $\Psi(x,y,t)$, the map-germ of $\Psi|_{t=t_0}(x,y)$ around any point (x_0,y_0) at any moment t_0 in the dislocation locus where $\Psi(x_0,y_0,t_0) = 0$, is radically equivalent at t_0 around (x_0,y_0) to one of the followings $(\mathbb{R}^2, 0) \to (\mathbb{C}, 0)$:*

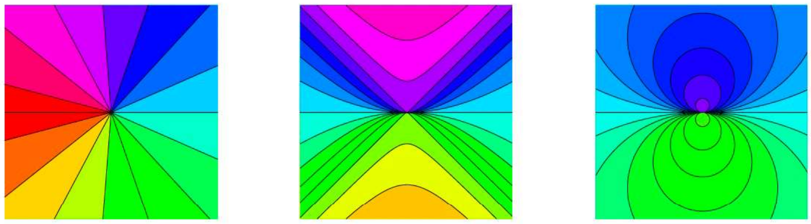
$R : \psi(x,y) = x + iy$, the regular singularity,

$H : \psi(x,y) = x^2 - y^2 + iy$, the hyperbolic singularity,

or to

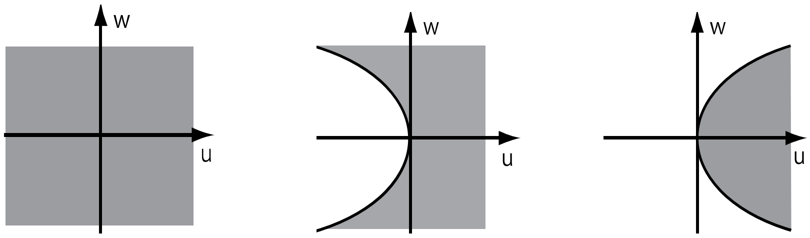
$E : \psi(x,y) = x^2 + y^2 + iy$, the elliptic singularity,

around the origin $(0,0)$. $(i = \sqrt{-1}.)$



R **H** **E**

The black lines of the above three pictures indicate the “equi-phase” contour lines. The images of the models R, H, E are depicted as follows:



Remark 3.4 Under the *right-left equivalences*, the both singularities H and E are equivalent to the fold singularity $(\mathbb{R}^2, 0) \to (\mathbb{C}, 0) = (\mathbb{R}^2, 0),\ (x, y) \mapsto (x^2, y)$.

Remark 3.5 The proof of Theorem 3.3 is based on the results appeared in [12, 9]. See [1] for details.

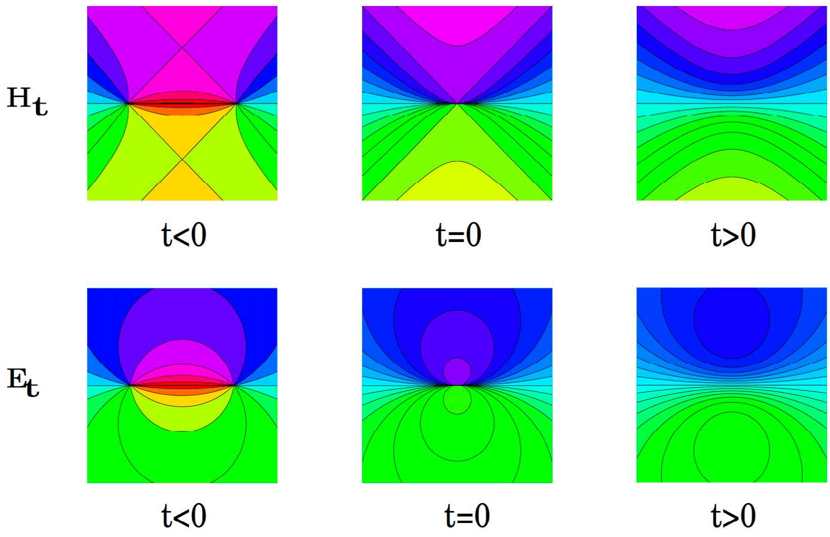
Theorem 3.6 ([1]) *The phase singularities arising from fold singularities are classified into*

$$\psi_m(x, y) = x^2 \pm y^m + iy, \quad m = 2, 3, 4, \ldots,$$

under radial transformations (and diffeomorphisms on the source), provided the discriminant curve has a contact with the tangent radial line at the origin in a finite multiplicity m.

Remark 3.7 Also generic bifurcations of ψ_m under radial transformations can be studied. For instance, for the case $m = 2$, the generic 1-parameter bifurcations of hyperbolic and elliptic singularity are given by

$$H_t : \Psi(x, y, t) = x^2 - y^2 + t + iy, (t \in \mathbb{R}),$$
$$E_t : \Psi(x, y, t) = x^2 + y^2 + t + iy, (t \in \mathbb{R}).$$



4 Topology of phase singularity and criticality

Let us recall the following works (see [18, 17]) to analyse the bifurcations of phase singularities and phase criticalities.

Let, in general, $f : (\mathbb{R}^n \times \mathbb{R}, (0,0)) \to (\mathbb{R}^n, 0)$, $f_t(x) := f(x,t)$, $(x,t) \in (\mathbb{R}^n \times \mathbb{R}, (0,0))$, be a 1-parameter family of map-germs.

We consider the *bifurcation diagram* of f,

$$D(f) := \{(x,t) \in (\mathbb{R}^n \times \mathbb{R}, 0) \mid f(x,t) = 0\},$$

with the projection $\mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$, $(x,t) \mapsto t$.

Denote by $b_+(f)$ (resp. $b_-(f)$) the number of branches located on $t > 0$, (resp. $t < 0$), of $f^{-1}(0)$ emanated from 0.

Define $Jf(x,t) := \det \left(\frac{\partial f_i}{\partial x_j}\right)_{1 \le i,j \le n} (x,t)$. Note that, if $Jf(0) \neq 0$, then there is no bifurcation by the implicit function theorem. So we suppose $Jf(0) = 0$.

Theorem 4.1 ([18]) *Let $f : (\mathbb{R}^n \times \mathbb{R}, 0) \to (\mathbb{R}^n, 0)$ be a $\mathcal{K}$-finite map-germ with $Jf(0) = 0$. Then*

$$\begin{aligned} b_+(f) + b_-(f) &= 2\deg(f, tJf), \\ b_+(f) - b_-(f) &= 2\deg(f, Jf). \end{aligned}$$

Here $(f, tJf), (f, Jf) : (\mathbb{R}^{n+1}, 0) \to (\mathbb{R}^{n+1}, 0)$, are defined by

$$(f, tJf)(x,t) := (f(x,t), tJf(x,t)), \quad (f, Jf)(x,t) := (f(x,t), Jf(x,t)),$$

respectively, and " deg *" means the mapping degree around* $0 \in \mathbb{R}^{n+1}$.

See [11] as a related work.

Let $F : (\mathbb{R}^m, 0) \to (\mathbb{R}^m, 0)$ be a $\mathcal{K}$-finite C^∞ map-germ. We set

$$Q(F) := \mathcal{E}_m / \langle F_1, \dots, F_m \rangle_{\mathcal{E}_m},$$

which is an $\mathbb{R}$-algebra with $\dim_{\mathbb{R}} Q(F) < \infty$.

Denote by JF the Jacobian determinant of F. Let $\alpha : Q(F) \to \mathbb{R}$ be an $\mathbb{R}$-linear form with $\alpha(JF) > 0$. Then we define a symmetric $\mathbb{R}$-bilinear form Φ_α : $Q(F) \times Q(F) \to \mathbb{R}$ by $\Phi_\alpha(u, v) := \alpha(uv), (u, v \in Q(F))$.

Theorem 4.2 ([10]) $\deg(F) = \mathrm{sign}(\Phi_\alpha)$.

Here "sign" means the *signature*;

(positive eigenvalues) - # (negative eigenvalues).

5 Bifurcations of phase singular loci and critical loci

We count the numbers of positive and negative branches of phase singularities and phase critical loci by using Nishimura-Fukuda-Aoki formula (Theorem 4.1) and Eisenbud-Levine formula (Theorem 4.2).

Let us consider a one-parameter family of *complex-valued* C^∞-functions $f : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{C} = \mathbb{R}^2$,

$$\Psi(x, y, t) = u(x, y, t) + i\, w(x, y, t).$$

We are able to apply Nishimura-Fukuda-Aoki formula to $f = \Psi$, for the study of bifurcations of *phase singularity.*

Moreover we set $\Gamma := (\gamma_1, \gamma_2) : (\mathbb{R}^2 \times \mathbb{R}, 0) \to \mathbb{R}^2$, where

$$\gamma_1(x, y, t) := \begin{vmatrix} u & w \\ \frac{\partial u}{\partial x} & \frac{\partial w}{\partial x} \end{vmatrix}, \quad \gamma_2(x, y, t) := \begin{vmatrix} u & w \\ \frac{\partial u}{\partial y} & \frac{\partial w}{\partial y} \end{vmatrix}.$$

Note that $\Gamma^{-1}(0)$ give union of phase singular loci and phase critical loci around $(0, 0)$.

If $\Gamma(0, 0, 0) \neq 0$, then the phase critical locus is empty. Hence we suppose $\Gamma(0, 0, 0) = 0$.

Remark that $\Gamma^{-1}(0)$ consists of the phase singular loci $\Psi^{-1}(0)$ and the phase critical loci of Ψ which are located outside of $\Psi^{-1}(0)$, traced along t, around $(0, 0, 0) \in \mathbb{R}^2 \times \mathbb{R}$.

Therefore we can (and do) apply also to $f = \Gamma$, Nishimura-Fukuda-Aoki formula for the study on bifurcation of *phase criticality.*

Example 5.1 Let us consider the bifurcation of hyperbolic singularity:
$H_t : \Psi(x, y, t) = x^2 - y^2 + t + iy, (t \in \mathbb{R})$,
In this case, we set $f = (x^2 - y^2 + t,\ y)$ and we have
$Jf = \begin{vmatrix} 2x & 2y \\ 0 & 1 \end{vmatrix} = 2x$. Then $(f, tJf) = (x^2 - y^2 + t,\ y,\ 2xt)$ and we have $Q(f, tJf) = \langle 1, x, x^2 \rangle_{\mathbb{R}}$. Moreover $J(f, tJf) = 4x^2 - 2t$ which is equal to $4x^2$ in $Q(f, tJf)$. We choose α by $\alpha(1) = 0, \alpha(x) = 0$ and $\alpha(x^2) = 1$.
Then Φ_α is represented as $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$, and $\mathrm{sign}(\Phi_\alpha) = 1$. Thus we have $\deg(f, tJf) = 1$.

For $(f, Jf) = (x^2 - y^2 + t,\ y,\ 2x)$, we have that $Q(f, Jf) = \langle 1 \rangle_{\mathbb{R}}$, $J(f, Jf) < 0$ in $Q(f, Jf)$, and that $\deg(f, Jf) = -1$. Therefore we have

$$b_+(f) + b_-(f) = 2\deg(f, tJf) = 2,$$
$$b_+(f) - b_-(f) = 2\deg(f, Jf) = -2.$$

Thus we have $b_+(f) = 0, b_-(f) = 2$ for the bifurcation of hyperbolic phase singularities of H_t.

Moreover applying to $\Gamma : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$ Nishimura-Fukuda-Aoki formula, we have $b_+(\Gamma) + b_-(\Gamma) = 4, b_+(\Gamma) - b_-(\Gamma) = -4$, and therefore we have $b_-(\Gamma) = 4, b_+(\Gamma) = 0$.

We denote by p_- (resp. p_+) the number of branches of negative (resp. positive) *phase singularities*, and by c_- (resp. c_+) the number of branches of negative (resp. positive) *phase criticalities*. Then

$$p_- = b_-(f), p_+ = b_+(f),$$
$$c_- = b_-(\Gamma) - b_-(f), c_+ = b_-(\Gamma) - b_-(f).$$

Thus we have

$$p_- = 2,\ p_+ = 0,\ c_- = 2,\ c_+ = 0$$

for the hyperbolic case.

Example 5.2 (Bifurcation of elliptic singularity). Let us consider

$$E_t : \Psi(x, y, t) = x^2 + y^2 + t + iy, (t \in \mathbb{R}).$$

Similarly to the hyperbolic case, we can calculate that

$$b_-(f) = 2,\ b_+(f) = 0,\ b_+(\Gamma) = 2,\ b_-(\Gamma) = 2,$$

and therefore we have

$$p_- = 2,\ p_+ = 0,\ c_- = 0,\ c_+ = 2,$$

for the elliptic case.

6 Phase singularity of Whitney's cusp

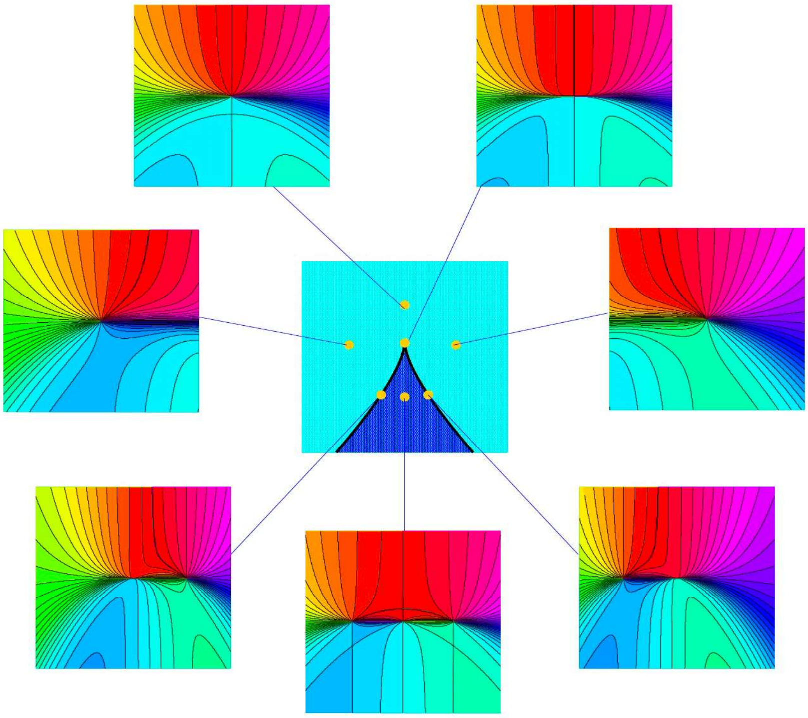
A map-germ $\psi : (\mathbb{R}^2, 0) \to (\mathbb{C}, 0) = (\mathbb{R}^2, 0)$ is called a *Whitney's cusp* if ψ is right-left equivalent to $(x, y) \mapsto (x^3 + xy, y)$.

Theorem 6.1 (The radial classification of Whitney's cusps [1]) *Any Whitney's cusp is equivalent to*

$$\psi(x, y) = x^3 + xy + iy,$$

under radial transformations. The generic bifurcation of the phase singularities of the Whitney's cusp is given by

$$\psi_{a,b}(x, y) = x^3 + xy + b + i(y + a), \ \ (a, b \in \mathbb{R}).$$



Note that our classification is closely related to the web-geometry. See [8].

7 The case of three variables

Theorem 7.1 ([1]) *For a generic complex valued function $\Psi : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{C}$, the germ of Ψ at any point $(x_0, y_0, z_0) \in \mathbb{R}^3$ and any moment t_0 with $\Psi(x_0, y_0, z_0, t_0) = 0$*

is radially equivalent to

$$
\begin{aligned}
&R : \Psi(x,y,z) = x + iy, \text{ (regular singularity)},\\
&DH : \Psi(x,y,z) = x^2 + y^2 - z^2 + iz, \text{ (definite hyperbolic)},\\
&DE : \Psi(x,y,z) = x^2 + y^2 + z^2 + iz, \text{ (definite elliptic), or}\\
&I : \Psi(x,y,z) = x^2 - y^2 - z^2 + iz, \text{ (indefinite)},
\end{aligned}
$$

at the origin $(x,y,z) = (0,0,0)$.

Similarly to the case of planar complex scalar waves, the generic bifurcations on t of the definite hyperbolic singularities, the definite elliptic singularities, and the indefinite singularities are given by

$$
\begin{aligned}
&DH_t : \Psi(x,y,z,t) = x^2 + y^2 - z^2 + t + iz,\\
&DEt : \Psi(x,y,z,t) = x^2 + y^2 + z^2 + t + iz,\\
&I_t : \Psi(x,y,z,t) = x^2 - y^2 - z^2 + t + iz.
\end{aligned}
$$

8 Open questions

1. *The question on degenerate phase singularities.* Give the classification results for degenerate phase singularities and criticalities under the radial equivalence of complex valued functions.

2. *The question on topological classification.* Two complex valued function-germs $\psi : (\mathbb{R}^2, (x_0, y_0)) \to (\mathbb{C}, 0)$ and $\psi' : (\mathbb{R}^2, (x_0', y_0'')) \to (\mathbb{C}, 0)$ are called *topologically radially equivalent* if the diagram

$$
\begin{array}{ccc}
(\mathbb{R}^2, (x_0, y_0)) & \xrightarrow{\ \psi\ } & (\mathbb{C}, 0)\\
\sigma \big\downarrow \cong & & \cong \big\downarrow \tau\\
(\mathbb{R}^2, (x_0', y_0')) & \xrightarrow[\ \psi'\ ]{} & (\mathbb{C}, 0)
\end{array}
$$

commutes, for homeomorphism-germs σ and τ such that τ maps any radial line to a radial line. Then study on phase singularities and phase criticalities under the topological radial equivalence.

3. *The question on radial codimension of complex valued function-germs.* Formulate the notion of radial codimension of complex valued function-germs.

4. *The question on phase-amplitude singularities.* Classify complex valued function-germs under the diffeomorphisms of $\mathbb{C}$ preserving both phase and amplitude.

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