Bertrand framed surfaces in the Euclidean 3-space and its applications

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1 Introduction

Bertrand and Mannheim curves are classical objects in differential geometry. A Bertrand (respectively, Mannheim) curve in the Euclidean 3-space is a space curve whose principal normal line is the same as the principal normal (respectively, bi-normal) line of another curve. By definition, another curve is a parallel curve with respect to the direction of the principal normal vector. In [5], they investigated the condition of the Bertrand and Mannheim curves of non-degenerate curves and framed curves. Moreover, we investigated the other cases, that is, a space curve whose tangent (or, principal normal, bi-normal) line is the same as the tangent (or, principal normal, bi-normal) line of another curve, respectively. We say that a Bertrand type curve if there exists such another curve. We investigated the existence conditions of Bertrand type curves in all cases in [8]. Moreover, we also investigated curves with singular points. As smooth curves with singular points, it is useful to use the framed curves in the Euclidean space (cf. [4]). We investigated the existence conditions of the Bertrand framed curves (Bertrand types of framed curves) in all cases in [8]. As a consequence, the involutes and circular evolutes of framed curves (cf. [6]) appear as the Bertrand framed curves.

A framed surface is a surface in Euclidean 3-space with a moving frame (cf. [3]). Framed surfaces may have singular points. By using a moving frame, we define Bertrand framed surfaces as the same idea as Bertrand framed curves. Then the caustic and involute appear as a Bertrand framed surface. In this paper, we give existence conditions of Bertrand framed surfaces in all cases in §3. As a consequence, then the caustics and involutes appear as Bertrand framed surfaces.

As applications, we can directly define the caustics and involutes of framed surfaces, and give conditions that the caustics and involutes are inverse operations of framed surfaces like as those of Legendre curves in §4. Furthermore, we find a new such operation, so-called, tangential direction framed surfaces in §5. Finally, we give concrete examples of caustics, involutes and tangential direction framed surfaces in §6.

We shall assume throughout the whole paper that all maps and manifolds are C^∞ unless the contrary is explicitly stated.

The content of this paper is based on joint research with Masatomo Takahashi (cf. [9]).

2 Preliminaries

Let $\mathbb{R}^3$ be the 3-dimensional Euclidean space equipped with the inner product $\boldsymbol{a} \cdot \boldsymbol{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$, where $\boldsymbol{a} = (a_1, a_2, a_3)$ and $\boldsymbol{b} = (b_1, b_2, b_3) \in \mathbb{R}^3$. The norm of $\boldsymbol{a}$ is given by $|\boldsymbol{a}| = \sqrt{\boldsymbol{a} \cdot \boldsymbol{a}}$ and the vector product is given by

$$\boldsymbol{a} \times \boldsymbol{b} = \det \begin{pmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix},$$

where $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$ are the canonical basis on $\mathbb{R}^3$. Let U be a simply connected domain of $\mathbb{R}^2$ and S^2 be the unit sphere in $\mathbb{R}^3$, that is, $S^2 = \{\boldsymbol{a} \in \mathbb{R}^3 | |\boldsymbol{a}| = 1\}$. We denote a 3-dimensional smooth manifold $\{(\boldsymbol{a}, \boldsymbol{b}) \in S^2 \times S^2 | \boldsymbol{a} \cdot \boldsymbol{b} = 0\}$ by Δ.

Definition 2.1. We say that $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s}) : U \to \mathbb{R}^3 \times \Delta$ is a *framed surface* if $\boldsymbol{x}_u(u,v) \cdot \boldsymbol{n}(u,v) = \boldsymbol{x}_v(u,v) \cdot \boldsymbol{n}(u,v) = 0$ for all $(u,v) \in U$, where $\boldsymbol{x}_u(u,v) = (\partial \boldsymbol{x}/\partial u)(u,v)$ and $\boldsymbol{x}_v(u,v) = (\partial \boldsymbol{x}/\partial v)(u,v)$. We say that $\boldsymbol{x} : U \to \mathbb{R}^3$ is a *framed base surface* if there exists $(\boldsymbol{n}, \boldsymbol{s}) : U \to \Delta$ such that $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s})$ is a framed surface.

By definition, the framed base surface is a frontal. The definition and properties of frontals see [1, 2]. On the other hand, the frontal is a framed base surface at least locally. In this paper, we consider framed base surfaces as singular surfaces.

We denote $\boldsymbol{t}(u,v) = \boldsymbol{n}(u,v) \times \boldsymbol{s}(u,v)$. Then $\{\boldsymbol{n}(u,v), \boldsymbol{s}(u,v), \boldsymbol{t}(u,v)\}$ is a moving frame along $\boldsymbol{x}(u,v)$. Thus, we have the following systems of differential equations:

$$\begin{pmatrix} \boldsymbol{x}_u \\ \boldsymbol{x}_v \end{pmatrix} = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} \begin{pmatrix} \boldsymbol{s} \\ \boldsymbol{t} \end{pmatrix},$$

$$\begin{pmatrix} \boldsymbol{n}_u \\ \boldsymbol{s}_u \\ \boldsymbol{t}_u \end{pmatrix} = \begin{pmatrix} 0 & e_1 & f_1 \\ -e_1 & 0 & g_1 \\ -f_1 & -g_1 & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{n} \\ \boldsymbol{s} \\ \boldsymbol{t} \end{pmatrix}, \quad \begin{pmatrix} \boldsymbol{n}_v \\ \boldsymbol{s}_v \\ \boldsymbol{t}_v \end{pmatrix} = \begin{pmatrix} 0 & e_2 & f_2 \\ -e_2 & 0 & g_2 \\ -f_2 & -g_2 & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{n} \\ \boldsymbol{s} \\ \boldsymbol{t} \end{pmatrix},$$

where $a_i, b_i, e_i, f_i, g_i : U \to \mathbb{R}, i = 1, 2$ are smooth functions and we call the functions *basic invariants* of the framed surface. We denote the above matrices by $\mathcal{G}, \mathcal{F}_1, \mathcal{F}_2$, respectively. We also call the matrices $(\mathcal{G}, \mathcal{F}_1, \mathcal{F}_2)$ *basic invariants* of the framed surface $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s})$. Note that (u,v) is a singular point of $\boldsymbol{x}$ if and only if $\det \mathcal{G}(u,v) = 0$.

Since the integrability conditions $\boldsymbol{x}_{uv} = \boldsymbol{x}_{vu}$ and $\mathcal{F}_{2,u} - \mathcal{F}_{1,v} = \mathcal{F}_1 \mathcal{F}_2 - \mathcal{F}_2 \mathcal{F}_1$, the basic invariants should be satisfied the following conditions:

$$\begin{cases} a_{1v} - b_1 g_2 = a_{2u} - b_2 g_1, \\ b_{1v} - a_2 g_1 = b_{2u} - a_1 g_2, \\ a_1 e_2 + b_1 f_2 = a_2 e_1 + b_2 f_1, \end{cases} \quad \begin{cases} e_{1v} - f_1 g_2 = e_{2u} - f_2 g_1, \\ f_{1v} - e_2 g_1 = f_{2u} - e_1 g_2, \\ g_{1v} - e_1 f_2 = g_{2u} - e_2 f_1. \end{cases} \tag{1}$$

We have fundamental theorems for framed surfaces, that is, the existence and uniqueness theorem for the basic invariants of framed surfaces.

Definition 2.2. We define a smooth mapping $C^F = (J^F, K^F, H^F) : U \to \mathbb{R}^3$ by

$$J^F = \det \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}, K^F = \det \begin{pmatrix} e_1 & f_1 \\ e_2 & f_2 \end{pmatrix},$$

$$H^F = -\frac{1}{2} \left\{ \det \begin{pmatrix} a_1 & f_1 \\ a_2 & f_2 \end{pmatrix} - \det \begin{pmatrix} b_1 & e_1 \\ b_2 & e_2 \end{pmatrix} \right\}.$$

We call $C^F = (J^F, K^F, H^F)$ a *curvature of the framed surface*.

3 Bertrand framed surfaces

Let $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s})$ and $(\overline{\boldsymbol{x}}, \overline{\boldsymbol{n}}, \overline{\boldsymbol{s}}) : U \to \mathbb{R}^3 \times \Delta$ be framed surfaces.

Definition 3.1. We say that $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s})$ and $(\overline{\boldsymbol{x}}, \overline{\boldsymbol{n}}, \overline{\boldsymbol{s}})$ are $(\boldsymbol{v}, \overline{\boldsymbol{w}})$-*mates* if there exists a smooth function $\lambda : U \to \mathbb{R}$ with $\lambda \not\equiv 0$ such that $\overline{\boldsymbol{x}}(u, v) = \boldsymbol{x}(u, v) + \lambda(u, v)\boldsymbol{v}(u, v)$ and $\boldsymbol{v}(u, v) = \overline{\boldsymbol{w}}(u, v)$ for all $(u, v) \in U$, where $\boldsymbol{v}$ and $\boldsymbol{w}$ are $\boldsymbol{n}, \boldsymbol{s}$ or $\boldsymbol{t}$, respectively.

We also say that $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s})$ is a $(\boldsymbol{v}, \overline{\boldsymbol{w}})$-*Bertrand framed surface* (or, $(\boldsymbol{v}, \overline{\boldsymbol{w}})$-*Bertrand-Mannheim framed surface*) if there exists another framed surface $(\overline{\boldsymbol{x}}, \overline{\boldsymbol{n}}, \overline{\boldsymbol{s}})$ such that $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s})$ and $(\overline{\boldsymbol{x}}, \overline{\boldsymbol{n}}, \overline{\boldsymbol{s}})$ are $(\boldsymbol{v}, \overline{\boldsymbol{w}})$-mates.

We clarify the notation $\lambda \not\equiv 0$. Throughout this paper, $\lambda \not\equiv 0$ means that $\{(u, v) \in U | \lambda(u, v) \neq 0\}$ is a dense subset of U. It follows that $\boldsymbol{x}$ and $\overline{\boldsymbol{x}}$ are different surfaces. Note that if λ is constant, then $\lambda \not\equiv 0$ means that λ is a non-zero constant.

Let $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s}) : U \to \mathbb{R}^3 \times \Delta$ be a framed surface with basic invariants $(\mathcal{G}, \mathcal{F}_1, \mathcal{F}_2)$. We give all characterizations of Bertrand framed surfaces. The proof of Bertrand framed surfaces in all cases see [9].

Lemma 3.2. *If* $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s})$ *and* $(\overline{\boldsymbol{x}}, \overline{\boldsymbol{n}}, \overline{\boldsymbol{s}}) : U \to \mathbb{R}^3 \times \Delta$ *are* $(\boldsymbol{n}, \overline{\boldsymbol{n}})$*-mates, then* λ *is non-zero constant.*

Theorem 3.3. $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s}) : U \to \mathbb{R}^3 \times \Delta$ *is always an* $(\boldsymbol{n}, \overline{\boldsymbol{n}})$*-Bertrand framed surface.*

By a direct calculation, we have the following (cf. [3]).

Proposition 3.4. *Suppose that* $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s})$ *and* $(\overline{\boldsymbol{x}}, \overline{\boldsymbol{n}}, \overline{\boldsymbol{s}}) : U \to \mathbb{R}^3 \times \Delta$ *are* $(\boldsymbol{n}, \overline{\boldsymbol{n}})$*-mates, where* $(\overline{\boldsymbol{x}}, \overline{\boldsymbol{n}}, \overline{\boldsymbol{s}}) = (\boldsymbol{x} + \lambda\boldsymbol{n}, \boldsymbol{n}, \boldsymbol{s})$ *and* λ *is a non-zero constant. Then the basic invariants of* $(\overline{\boldsymbol{x}}, \overline{\boldsymbol{n}}, \overline{\boldsymbol{s}})$ *are given by*

$$\overline{\mathcal{G}} = \mathcal{G} + \lambda \begin{pmatrix} e_1 & f_1 \\ e_2 & f_2 \end{pmatrix}, \ \overline{\mathcal{F}}_1 = \mathcal{F}_1, \ \overline{\mathcal{F}}_2 = \mathcal{F}_2.$$

Remark 3.5. (1) If $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s})$ and $(\overline{\boldsymbol{x}}, \overline{\boldsymbol{n}}, \overline{\boldsymbol{s}}) : U \to \mathbb{R}^3 \times \Delta$ are $(\boldsymbol{n}, \overline{\boldsymbol{n}})$-mates, then $\overline{\boldsymbol{x}}$ is a parallel surface of $\boldsymbol{x}$ (cf. [3]).

(2) On the moving frame of $(\overline{\boldsymbol{x}}, \overline{\boldsymbol{n}}, \overline{\boldsymbol{s}})$, we can also take a rotation frame $\{\boldsymbol{n}, \boldsymbol{s}^\theta, \boldsymbol{t}^\theta\}$ instead of $\{\boldsymbol{n}, \boldsymbol{s}, \boldsymbol{t}\}$.

Theorem 3.6. $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s}) : U \to \mathbb{R}^3 \times \Delta$ *is an* $(\boldsymbol{n}, \overline{\boldsymbol{s}})$*-Bertrand framed surface if and only if there exist smooth functions* $\lambda, \theta : U \to \mathbb{R}$ *with* $\lambda \not\equiv 0$ *such that*

$$\begin{pmatrix} a_1(u,v) + \lambda(u,v)e_1(u,v) & b_1(u,v) + \lambda(u,v)f_1(u,v)) \\ a_2(u,v) + \lambda(u,v)e_2(u,v) & b_2(u,v) + \lambda(u,v)f_2(u,v)) \end{pmatrix} \begin{pmatrix} \sin\theta(u,v) \\ \cos\theta(u,v) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2}$$

for all $(u, v) \in U$.

Proposition 3.7. *Suppose that* $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s})$ *and* $(\overline{\boldsymbol{x}}, \overline{\boldsymbol{n}}, \overline{\boldsymbol{s}}) : U \to \mathbb{R}^3 \times \Delta$ *are* $(\boldsymbol{n}, \overline{\boldsymbol{s}})$*-mates, where* $(\overline{\boldsymbol{x}}, \overline{\boldsymbol{n}}, \overline{\boldsymbol{s}}) = (\boldsymbol{x} + \lambda\boldsymbol{n}, \sin\theta\boldsymbol{s} + \cos\theta\boldsymbol{t}, \boldsymbol{n})$ *and* $\lambda, \theta : U \to \mathbb{R}$ *are smooth functions satisfying* $\lambda \not\equiv 0$ *and condition* (2)*. Then the basic invariants of* $(\overline{\boldsymbol{x}}, \overline{\boldsymbol{n}}, \overline{\boldsymbol{s}})$ *are given by*

$$\begin{pmatrix} \overline{a}_1 & \overline{b}_1 \\ \overline{a}_2 & \overline{b}_2 \end{pmatrix} = \begin{pmatrix} \lambda_u & (a_1 + \lambda e_1)\cos\theta - (b_1 + \lambda f_1)\sin\theta \\ \lambda_v & (a_2 + \lambda e_2)\cos\theta - (b_2 + \lambda f_2)\sin\theta \end{pmatrix},$$

$$\begin{pmatrix} \overline{e}_1 & \overline{f}_1 & \overline{g}_1 \\ \overline{e}_2 & \overline{f}_2 & \overline{g}_2 \end{pmatrix} = \begin{pmatrix} -e_1\sin\theta - f_1\cos\theta & \theta_u - g_1 & e_1\cos\theta - f_1\sin\theta \\ -e_2\sin\theta - f_2\cos\theta & \theta_v - g_2 & e_2\cos\theta - f_2\sin\theta \end{pmatrix}.$$

Remark 3.8. If $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s})$ and $(\overline{\boldsymbol{x}}, \overline{\boldsymbol{n}}, \overline{\boldsymbol{s}}) : U \to \mathbb{R}^3 \times \Delta$ are $(\boldsymbol{n}, \overline{\boldsymbol{s}})$-mates, then $\overline{\boldsymbol{x}}$ is a caustic (an evolute or a focal surface) of $\boldsymbol{x}$. By condition (2), we have

$$\det \begin{pmatrix} a_1 + \lambda e_1 & b_1 + \lambda f_1 \\ a_2 + \lambda e_2 & b_2 + \lambda f_2 \end{pmatrix} = 0. \tag{3}$$

It follows that λ must be a solution of the equation $K^F \lambda^2 - H^F \lambda + J^F = 0$. It is easy to see that the converse does not hold in general in the case of $d\boldsymbol{x}$ has a corank 2 singular point, that is, condition (2) does not follows from (3).

Theorem 3.9. *$(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s}) : U \to \mathbb{R}^3 \times \Delta$ is an $(\boldsymbol{n}, \overline{\boldsymbol{t}})$-Bertrand framed surface if and only if there exist smooth functions $\lambda, \widetilde{\theta} : U \to \mathbb{R}$ with $\lambda \not\equiv 0$ such that*

$$\begin{pmatrix} a_1(u,v) + \lambda(u,v) e_1(u,v) & b_1(u,v) + \lambda(u,v) f_1(u,v) \\ a_2(u,v) + \lambda(u,v) e_2(u,v) & b_2(u,v) + \lambda(u,v) f_2(u,v) \end{pmatrix} \begin{pmatrix} -\cos \widetilde{\theta}(u,v) \\ \sin \widetilde{\theta}(u,v) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{4}$$

for all $(u,v) \in U$.

Proposition 3.10. *Suppose that $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s})$ and $(\overline{\boldsymbol{x}}, \overline{\boldsymbol{n}}, \overline{\boldsymbol{s}}) : U \to \mathbb{R}^3 \times \Delta$ are $(\boldsymbol{n}, \overline{\boldsymbol{t}})$-mates, where $(\overline{\boldsymbol{x}}, \overline{\boldsymbol{n}}, \overline{\boldsymbol{s}}) = (\boldsymbol{x} + \lambda \boldsymbol{n}, \cos \widetilde{\theta} \boldsymbol{s} - \sin \widetilde{\theta} \boldsymbol{t}, \sin \widetilde{\theta} \boldsymbol{s} + \cos \widetilde{\theta} \boldsymbol{t})$ and $\lambda, \theta : U \to \mathbb{R}$ are smooth functions satisfying $\lambda \not\equiv 0$ and condition (4). Then the basic invariants of $(\overline{\boldsymbol{x}}, \overline{\boldsymbol{n}}, \overline{\boldsymbol{s}})$ are given by*

$$\begin{pmatrix} \overline{a}_1 & \overline{b}_1 \\ \overline{a}_2 & \overline{b}_2 \end{pmatrix} = \begin{pmatrix} (a_1 + \lambda e_1) \sin \widetilde{\theta} + (b_1 + \lambda f_1) \cos \widetilde{\theta} & \lambda_u \\ (a_2 + \lambda e_2) \sin \widetilde{\theta} + (b_2 + \lambda f_2) \cos \widetilde{\theta} & \lambda_v \end{pmatrix},$$

$$\begin{pmatrix} \overline{e}_1 & \overline{f}_1 & \overline{g}_1 \\ \overline{e}_2 & \overline{f}_2 & \overline{g}_2 \end{pmatrix} = \begin{pmatrix} g_1 - \widetilde{\theta}_u & -e_1 \cos \widetilde{\theta} + f_1 \sin \widetilde{\theta} & -e_1 \sin \widetilde{\theta} - f_1 \cos \widetilde{\theta} \\ g_2 - \widetilde{\theta}_v & -e_2 \cos \widetilde{\theta} + f_2 \sin \widetilde{\theta} & -e_2 \sin \widetilde{\theta} - f_2 \cos \widetilde{\theta} \end{pmatrix}.$$

Theorem 3.11. *$(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s}) : U \to \mathbb{R}^3 \times \Delta$ is an $(\boldsymbol{n}, \overline{\boldsymbol{t}})$-Bertrand framed surface if and only $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s}) : U \to \mathbb{R}^3 \times \Delta$ is an $(\boldsymbol{n}, \overline{\boldsymbol{s}})$-Bertrand framed surface.*

Theorem 3.12. *$(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s}) : U \to \mathbb{R}^3 \times \Delta$ is an $(\boldsymbol{s}, \overline{\boldsymbol{n}})$-Bertrand framed surface if and only if $\det(\boldsymbol{b}(u,v), \boldsymbol{g}(u,v)) = 0$ for all $(u,v) \in U$ and $\lambda : U \to \mathbb{R}$ is given by*

$$\lambda(u,v) = -\left(\int_{u_0}^{u} a_1(u,v) du + \int_{v_0}^{v} a_2(u_0, v) dv \right) + c$$

for a point $(u_0, v_0) \in U$ and constant $c \in \mathbb{R}$ with $\lambda \not\equiv 0$.

Proposition 3.13. *Suppose that $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s})$ and $(\overline{\boldsymbol{x}}, \overline{\boldsymbol{n}}, \overline{\boldsymbol{s}}) : U \to \mathbb{R}^3 \times \Delta$ are $(\boldsymbol{s}, \overline{\boldsymbol{n}})$-mates, where $(\overline{\boldsymbol{x}}, \overline{\boldsymbol{n}}, \overline{\boldsymbol{s}}) = (\boldsymbol{x} + \lambda \boldsymbol{s}, \boldsymbol{s}, \boldsymbol{t})$ and*

$$\lambda(u,v) = -\left(\int_{u_0}^{u} a_1(u,v) du + \int_{v_0}^{v} a_2(u_0, v) dv \right) + c \not\equiv 0$$

for a point $(u_0, v_0) \in U$ and constant $c \in \mathbb{R}$. Then the basic invariants of $(\overline{\boldsymbol{x}}, \overline{\boldsymbol{n}}, \overline{\boldsymbol{s}})$ are given by

$$\begin{pmatrix} \overline{a}_1 & \overline{b}_1 \\ \overline{a}_2 & \overline{b}_2 \end{pmatrix} = \begin{pmatrix} b_1 + \lambda g_1 & -\lambda e_1 \\ b_1 + \lambda g_2 & -\lambda e_2 \end{pmatrix}, \quad \begin{pmatrix} \overline{e}_1 & \overline{f}_1 & \overline{g}_1 \\ \overline{e}_2 & \overline{f}_2 & \overline{g}_2 \end{pmatrix} = \begin{pmatrix} g_1 & -e_1 & -f_1 \\ g_2 & -e_2 & -f_2 \end{pmatrix}.$$

Remark 3.14. If $(\boldsymbol{x},\boldsymbol{n},\boldsymbol{s})$ and $(\overline{\boldsymbol{x}},\overline{\boldsymbol{n}},\overline{\boldsymbol{s}}): U\to\mathbb{R}^3\times\Delta$ are $(\boldsymbol{s},\overline{\boldsymbol{n}})$-mates, then we may consider $\overline{\boldsymbol{x}}$ is one of involutes of $\boldsymbol{x}$.

Theorem 3.15. *$(\boldsymbol{x},\boldsymbol{n},\boldsymbol{s}): U\to\mathbb{R}^3\times\Delta$ is an $(\boldsymbol{s},\overline{\boldsymbol{s}})$-Bertrand framed surface if and only if there exist smooth functions $\lambda,\theta: U\to\mathbb{R}$ with $\lambda\not\equiv 0$ such that*

$$\begin{pmatrix} b_1(u,v)+\lambda(u,v)g_1(u,v) & \lambda(u,v)e_1(u,v)\\ b_2(u,v)+\lambda(u,v)g_2(u,v) & \lambda(u,v)e_2(u,v)\end{pmatrix}\begin{pmatrix}\sin\theta(u,v)\\ -\cos\theta(u,v)\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix} \tag{5}$$

for all $(u,v)\in U$.

Proposition 3.16. *Suppose that $(\boldsymbol{x},\boldsymbol{n},\boldsymbol{s})$ and $(\overline{\boldsymbol{x}},\overline{\boldsymbol{n}},\overline{\boldsymbol{s}}): U\to\mathbb{R}^3\times\Delta$ are $(\boldsymbol{s},\overline{\boldsymbol{s}})$-mates, where $(\overline{\boldsymbol{x}},\overline{\boldsymbol{n}},\overline{\boldsymbol{s}})=(\boldsymbol{x}+\lambda\boldsymbol{s},\sin\theta\boldsymbol{t}+\cos\theta\boldsymbol{n},\boldsymbol{s})$ and $\lambda,\theta: U\to\mathbb{R}$ are smooth functions satisfying $\lambda\not\equiv 0$ and condition (5). Then the basic invariants of $(\overline{\boldsymbol{x}},\overline{\boldsymbol{n}},\overline{\boldsymbol{s}})$ are given by*

$$\begin{pmatrix}\overline{a}_1 & \overline{b}_1\\ \overline{a}_2 & \overline{b}_2\end{pmatrix}=\begin{pmatrix}a_1+\lambda_u & (b_1+\lambda g_1)\cos\theta+\lambda e_1\sin\theta\\ a_2+\lambda_v & (b_2+\lambda g_2)\cos\theta+\lambda e_2\sin\theta\end{pmatrix},$$
$$\begin{pmatrix}\overline{e}_1 & \overline{f}_1 & \overline{g}_1\\ \overline{e}_2 & \overline{f}_2 & \overline{g}_2\end{pmatrix}=\begin{pmatrix}-g_1\sin\theta+e_1\cos\theta & \theta_u+f_1 & g_1\cos\theta+e_1\sin\theta\\ -g_2\sin\theta+e_2\cos\theta & \theta_v+f_2 & g_2\cos\theta+e_2\sin\theta\end{pmatrix}.$$

Remark 3.17. If $(\boldsymbol{x},\boldsymbol{n},\boldsymbol{s})$ and $(\overline{\boldsymbol{x}},\overline{\boldsymbol{n}},\overline{\boldsymbol{s}}): U\to\mathbb{R}^3\times\Delta$ are $(\boldsymbol{s},\overline{\boldsymbol{s}})$-mates, then

$$\det\begin{pmatrix} b_1+\lambda g_1 & e_1\\ b_2+\lambda g_2 & e_2\end{pmatrix}=0$$

by condition (5). It follows that $\det(\boldsymbol{b}(u,v),\boldsymbol{e}(u,v))+\lambda(u,v)\det(\boldsymbol{g}(u,v),\boldsymbol{e}(u,v))=0$. If $\det(\boldsymbol{e}(u,v),\boldsymbol{g}(u,v))\neq 0$, then $\lambda(u,v)=\det(\boldsymbol{b}(u,v),\boldsymbol{e}(u,v))/\det(\boldsymbol{e}(u,v),\boldsymbol{g}(u,v))$. Hence, we have

$$\overline{\boldsymbol{x}}(u,v)=\boldsymbol{x}(u,v)+\frac{\det(\boldsymbol{b}(u,v),\boldsymbol{e}(u,v))}{\det(\boldsymbol{e}(u,v),\boldsymbol{g}(u,v))}\boldsymbol{s}(u,v).$$

Theorem 3.18. *$(\boldsymbol{x},\boldsymbol{n},\boldsymbol{s}): U\to\mathbb{R}^3\times\Delta$ is an $(\boldsymbol{s},\overline{\boldsymbol{t}})$-Bertrand framed surface if and only if there exist smooth functions $\lambda,\widetilde{\theta}: U\to\mathbb{R}$ with $\lambda\not\equiv 0$ such that*

$$\begin{pmatrix}\lambda(u,v)e_1(u,v) & b_1(u,v)+\lambda(u,v)g_1(u,v)\\ \lambda(u,v)e_2(u,v) & b_2(u,v)+\lambda(u,v)g_2(u,v)\end{pmatrix}\begin{pmatrix}\sin\widetilde{\theta}(u,v)\\ \cos\widetilde{\theta}(u,v)\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix} \tag{6}$$

for all $(u,v)\in U$.

Proposition 3.19. *Suppose that $(\boldsymbol{x},\boldsymbol{n},\boldsymbol{s})$ and $(\overline{\boldsymbol{x}},\overline{\boldsymbol{n}},\overline{\boldsymbol{s}}): U\to\mathbb{R}^3\times\Delta$ are $(\boldsymbol{s},\overline{\boldsymbol{t}})$-mates, where $(\overline{\boldsymbol{x}},\overline{\boldsymbol{n}},\overline{\boldsymbol{s}})=(\boldsymbol{x}+\lambda\boldsymbol{s},\cos\widetilde{\theta}\boldsymbol{t}-\sin\widetilde{\theta}\boldsymbol{n},\sin\widetilde{\theta}\boldsymbol{t}+\cos\widetilde{\theta}\boldsymbol{n})$ and $\lambda,\widetilde{\theta}: U\to\mathbb{R}$ are smooth functions satisfying $\lambda\not\equiv 0$ and condition (6). Then the basic invariants of $(\overline{\boldsymbol{x}},\overline{\boldsymbol{n}},\overline{\boldsymbol{s}})$ are given by*

$$\begin{pmatrix}\overline{a}_1 & \overline{b}_1\\ \overline{a}_2 & \overline{b}_2\end{pmatrix}=\begin{pmatrix}-\lambda e_1\cos\widetilde{\theta}+(b_1+\lambda g_1)\sin\widetilde{\theta} & a_1+\lambda_u\\ -\lambda e_2\cos\widetilde{\theta}+(b_2+\lambda g_2)\sin\widetilde{\theta} & a_2+\lambda_v\end{pmatrix},$$
$$\begin{pmatrix}\overline{e}_1 & \overline{f}_1 & \overline{g}_1\\ \overline{e}_2 & \overline{f}_2 & \overline{g}_2\end{pmatrix}=\begin{pmatrix}-\widetilde{\theta}_u-f_1 & -g_1\cos\widetilde{\theta}-e_1\sin\widetilde{\theta} & -g_1\sin\widetilde{\theta}+e_1\cos\widetilde{\theta}\\ -\widetilde{\theta}_v-f_2 & -g_2\cos\widetilde{\theta}-e_2\sin\widetilde{\theta} & -g_2\sin\widetilde{\theta}+e_2\cos\widetilde{\theta}\end{pmatrix}.$$

Theorem 3.20. $(\boldsymbol{x},\boldsymbol{n},\boldsymbol{s}) : U \to \mathbb{R}^3 \times \Delta$ *is an* $(\boldsymbol{s},\overline{\boldsymbol{t}})$*-Bertrand framed surface if and only* $(\boldsymbol{x},\boldsymbol{n},\boldsymbol{s}) : U \to \mathbb{R}^3 \times \Delta$ *is an* $(\boldsymbol{s},\overline{\boldsymbol{s}})$*-Bertrand framed surface.*

We can prove from Theorem 3.21 to Proposition 3.27 by the similar calculations of proving of from Theorem 3.12 to Proposition 3.20. Therefore, we omit the proof here.

Theorem 3.21. $(\boldsymbol{x},\boldsymbol{n},\boldsymbol{s}) : U \to \mathbb{R}^3 \times \Delta$ *is a* $(\boldsymbol{t},\overline{\boldsymbol{n}})$*-Bertrand framed surface if and only if* $\det(\boldsymbol{g}(u,v),\boldsymbol{a}(u,v)) = 0$ *for all* $(u,v) \in U$ *and and* $\lambda : U \to \mathbb{R}$ *is given by*

$$\lambda(u,v) = -\left(\int_{u_0}^{u} b_1(u,v)du + \int_{v_0}^{v} b_2(u_0,v)dv\right) + c$$

for a point $(u_0,v_0) \in U$ *and constant* $c \in \mathbb{R}$ *with* $\lambda \not\equiv 0$.

Proposition 3.22. *Suppose that* $(\boldsymbol{x},\boldsymbol{n},\boldsymbol{s})$ *and* $(\overline{\boldsymbol{x}},\overline{\boldsymbol{n}},\overline{\boldsymbol{s}}) : U \to \mathbb{R}^3 \times \Delta$ *are* $(\boldsymbol{t},\overline{\boldsymbol{n}})$*-mates, where* $(\overline{\boldsymbol{x}},\overline{\boldsymbol{n}},\overline{\boldsymbol{s}}) = (\boldsymbol{x}+\lambda\boldsymbol{t},\boldsymbol{t},\boldsymbol{n})$ *and*

$$\lambda(u,v) = -\left(\int_{u_0}^{u} b_1(u,v)du + \int_{v_0}^{v} b_2(u_0,v)dv\right) + c \not\equiv 0$$

for a point $(u_0,v_0) \in U$ *and constant* $c \in \mathbb{R}$. *Then the basic invariants of* $(\overline{\boldsymbol{x}},\overline{\boldsymbol{n}},\overline{\boldsymbol{s}})$ *are given by*

$$\begin{pmatrix}\overline{a}_1 & \overline{b}_1\\ \overline{a}_2 & \overline{b}_2\end{pmatrix} = \begin{pmatrix}-\lambda f_1 & a_1-\lambda g_1\\ -\lambda f_2 & a_2-\lambda g_2\end{pmatrix}, \ \begin{pmatrix}\overline{e}_1 & \overline{f}_1 & \overline{g}_1\\ \overline{e}_2 & \overline{f}_2 & \overline{g}_2\end{pmatrix} = \begin{pmatrix}-f_1 & -g_1 & e_1\\ -f_2 & -g_2 & e_2\end{pmatrix}.$$

Theorem 3.23. $(\boldsymbol{x},\boldsymbol{n},\boldsymbol{s}) : U \to \mathbb{R}^3 \times \Delta$ *is a* $(\boldsymbol{t},\overline{\boldsymbol{s}})$*-Bertrand framed surface if and only if there exist smooth functions* $\lambda,\theta : U \to \mathbb{R}$ *with* $\lambda \not\equiv 0$ *such that*

$$\begin{pmatrix}\lambda(u,v)f_1(u,v) & a_1(u,v)-\lambda(u,v)g_1(u,v)\\ \lambda(u,v)f_2(u,v) & a_2(u,v)-\lambda(u,v)g_2(u,v)\end{pmatrix}\begin{pmatrix}-\sin\theta(u,v)\\ \cos\theta(u,v)\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} \tag{7}$$

for all $(u,v) \in U$.

Proposition 3.24. *Suppose that* $(\boldsymbol{x},\boldsymbol{n},\boldsymbol{s})$ *and* $(\overline{\boldsymbol{x}},\overline{\boldsymbol{n}},\overline{\boldsymbol{s}}) : U \to \mathbb{R}^3 \times \Delta$ *are* $(\boldsymbol{t},\overline{\boldsymbol{s}})$*-mates, where* $(\overline{\boldsymbol{x}},\overline{\boldsymbol{n}},\overline{\boldsymbol{s}}) = (\boldsymbol{x}+\lambda\boldsymbol{t},\sin\theta\boldsymbol{n}+\cos\theta\boldsymbol{s},\boldsymbol{t})$ *and* $\lambda,\theta : U \to \mathbb{R}$ *are smooth functions satisfying* $\lambda \not\equiv 0$ *and condition* (7). *Then the basic invariants of* $(\overline{\boldsymbol{x}},\overline{\boldsymbol{n}},\overline{\boldsymbol{s}})$ *are given by*

$$\begin{pmatrix}\overline{a}_1 & \overline{b}_1\\ \overline{a}_2 & \overline{b}_2\end{pmatrix} = \begin{pmatrix}b_1+\lambda_u & -\lambda f_1\cos\theta-(a_1-\lambda g_1)\sin\theta\\ b_2+\lambda_v & -\lambda f_2\cos\theta-(a_2-\lambda g_2)\sin\theta\end{pmatrix},$$

$$\begin{pmatrix}\overline{e}_1 & \overline{f}_1 & \overline{g}_1\\ \overline{e}_2 & \overline{f}_2 & \overline{g}_2\end{pmatrix} = \begin{pmatrix}f_1\sin\theta+g_1\cos\theta & \theta_u-e_1 & -f_1\cos\theta+g_1\sin\theta\\ f_2\sin\theta+g_2\cos\theta & \theta_v-e_2 & -f_2\cos\theta+g_2\sin\theta\end{pmatrix}.$$

Theorem 3.25. $(\boldsymbol{x},\boldsymbol{n},\boldsymbol{s}) : U \to \mathbb{R}^3 \times \Delta$ *is a* $(\boldsymbol{t},\overline{\boldsymbol{t}})$*-Bertrand framed surface if and only if there exist smooth functions* $\lambda,\widetilde{\theta} : U \to \mathbb{R}$ *with* $\lambda \not\equiv 0$ *such that*

$$\begin{pmatrix}\lambda(u,v)f_1(u,v) & a_1(u,v)-\lambda(u,v)g_1(u,v)\\ \lambda(u,v)f_2(u,v) & a_2(u,v)-\lambda(u,v)g_2(u,v)\end{pmatrix}\begin{pmatrix}\cos\widetilde{\theta}(u,v)\\ \sin\widetilde{\theta}(u,v)\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} \tag{8}$$

for all $(u,v) \in U$.

Proposition 3.26. *Suppose that* $(\boldsymbol{x},\boldsymbol{n},\boldsymbol{s})$ *and* $(\overline{\boldsymbol{x}},\overline{\boldsymbol{n}},\overline{\boldsymbol{s}}) : U \to \mathbb{R}^3 \times \Delta$ *are* $(\boldsymbol{t},\overline{\boldsymbol{t}})$*-mates, where* $(\overline{\boldsymbol{x}},\overline{\boldsymbol{n}},\overline{\boldsymbol{s}}) = (\boldsymbol{x}+\lambda\boldsymbol{t}, \cos\widetilde{\theta}\boldsymbol{n} - \sin\widetilde{\theta}\boldsymbol{s}, \sin\widetilde{\theta}\boldsymbol{n} + \cos\widetilde{\theta}\boldsymbol{s})$ *and* $\lambda, \widetilde{\theta} : U \to \mathbb{R}$ *are smooth functions satisfying* $\lambda \not\equiv 0$ *and condition* (8)*. Then the basic invariants of* $(\overline{\boldsymbol{x}},\overline{\boldsymbol{n}},\overline{\boldsymbol{s}})$ *are given by*

$$\begin{pmatrix} \overline{a}_1 & \overline{b}_1 \\ \overline{a}_2 & \overline{b}_2 \end{pmatrix} = \begin{pmatrix} -\lambda f_1 \sin\widetilde{\theta} + (a_1 - \lambda g_1)\cos\widetilde{\theta} & b_1 + \lambda_u \\ -\lambda f_2 \sin\widetilde{\theta} + (a_2 - \lambda g_2)\cos\widetilde{\theta} & b_2 + \lambda_v \end{pmatrix},$$
$$\begin{pmatrix} \overline{e}_1 & \overline{f}_1 & \overline{g}_1 \\ \overline{e}_2 & \overline{f}_2 & \overline{g}_2 \end{pmatrix} = \begin{pmatrix} -\widetilde{\theta}_u - e_1 & f_1\cos\widetilde{\theta} - g_1\sin\widetilde{\theta} & f_1\sin\widetilde{\theta} + g_1\cos\widetilde{\theta} \\ -\widetilde{\theta}_v - e_2 & f_2\cos\widetilde{\theta} - g_2\sin\widetilde{\theta} & f_2\sin\widetilde{\theta} + g_2\cos\widetilde{\theta} \end{pmatrix}.$$

Theorem 3.27. $(\boldsymbol{x},\boldsymbol{n},\boldsymbol{s}) : U \to \mathbb{R}^3 \times \Delta$ *is a* $(\boldsymbol{t},\overline{\boldsymbol{t}})$*-Bertrand framed surface if and only* $(\boldsymbol{x},\boldsymbol{n},\boldsymbol{s}) : U \to \mathbb{R}^3 \times \Delta$ *is a* $(\boldsymbol{t},\overline{\boldsymbol{s}})$*-Bertrand framed surface.*

Table 1 : Bertrand framed surfaces

$\boldsymbol{x}$ \ $\overline{\boldsymbol{x}}$	$\boldsymbol{n}$	$\boldsymbol{s}$	$\boldsymbol{t}$
$\overline{\boldsymbol{n}}$	parallel surface	involute	involute
$\overline{\boldsymbol{s}}$	caustic	exist	exist
$\overline{\boldsymbol{t}}$	caustic	exist	exist

4 Caustics and involutes of framed surfaces

The caustics (evolutes or focal surfaces) are classical object and it is well-known properties of caustics of regular surfaces, for instance [1, 2, 7]. By using Bertrand framed surfaces, we define caustics and involutes of framed surfaces directly. We denote

$$\mathcal{F}(U, \mathbb{R}^3 \times \Delta) := \{(\boldsymbol{x},\boldsymbol{n},\boldsymbol{s}) \in C^\infty(U, \mathbb{R}^3 \times \Delta) | (\boldsymbol{x},\boldsymbol{n},\boldsymbol{s}) \text{ is a framed surface}\}.$$

Let $(\boldsymbol{x},\boldsymbol{n},\boldsymbol{s}) : U \to \mathbb{R}^3 \times \Delta$ be a framed surface with basic invariants $(\mathcal{G}, \mathcal{F}_1, \mathcal{F}_2)$.

Definition 4.1. (1) The map $\mathcal{C}^s : \mathcal{F}(U, \mathbb{R}^3 \times \Delta) \to \mathcal{F}(U, \mathbb{R}^3 \times \Delta)$ is given by

$$\begin{aligned} &\mathcal{C}^s(\boldsymbol{x},\boldsymbol{n},\boldsymbol{s}) = (\boldsymbol{x}^{c,s}, \boldsymbol{n}^{c,s}, \boldsymbol{s}^{c,s}), \\ &\boldsymbol{x}^{c,s}(u,v) = \boldsymbol{x}(u,v) + \lambda^{c,s}(u,v)\boldsymbol{n}(u,v), \\ &\boldsymbol{n}^{c,s}(u,v) = \sin\theta^{c,s}(u,v)\boldsymbol{s}(u,v) + \cos\theta^{c,s}(u,v)\boldsymbol{t}(u,v), \\ &\boldsymbol{s}^{c,s}(u,v) = \boldsymbol{n}(u,v), \end{aligned}$$

where there exist smooth functions $\lambda^{c,s}, \theta^{c,s} : U \to \mathbb{R}$ such that

$$\begin{pmatrix} a_1(u,v) + \lambda^{c,s}(u,v)e_1(u,v) & b_1(u,v) + \lambda^{c,s}(u,v)f_1(u,v) \\ a_2(u,v) + \lambda^{c,s}(u,v)e_2(u,v) & b_2(u,v) + \lambda^{c,s}(u,v)f_2(u,v) \end{pmatrix} \begin{pmatrix} \sin\theta^{c,s}(u,v) \\ \cos\theta^{c,s}(u,v) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (9)$$

for all $(u,v) \in U$. Then we say that $\boldsymbol{x}^{c,s} : U \to \mathbb{R}^3$ is a *caustic* of the framed surface $(\boldsymbol{x},\boldsymbol{n},\boldsymbol{s})$.

(2) The map $\mathcal{C}^t : \mathcal{F}(U, \mathbb{R}^3 \times \Delta) \to \mathcal{F}(U, \mathbb{R}^3 \times \Delta)$ is given by

$$\begin{aligned}
&\mathcal{C}^t(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s}) = (\boldsymbol{x}^{c,t}, \boldsymbol{n}^{c,t}, \boldsymbol{s}^{c,t}),\\
&\boldsymbol{x}^{c,t}(u,v) = \boldsymbol{x}(u,v) + \lambda^{c,t}(u,v)\boldsymbol{n}(u,v),\\
&\boldsymbol{n}^{c,t}(u,v) = \cos\theta^{c,t}(u,v)\boldsymbol{s}(u,v) - \sin\theta^{c,t}(u,v)\boldsymbol{t}(u,v),\\
&\boldsymbol{s}^{c,t}(u,v) = \sin\theta^{c,t}(u,v)\boldsymbol{s}(u,v) + \cos\theta^{c,t}(u,v)\boldsymbol{t}(u,v),
\end{aligned}$$

where there exist smooth functions $\lambda^{c,t}, \theta^{c,t} : U \to \mathbb{R}$ such that

$$\begin{pmatrix} a_1(u,v) + \lambda^{c,t}(u,v)e_1(u,v) & b_1(u,v) + \lambda^{c,t}(u,v)f_1(u,v) \\ a_2(u,v) + \lambda^{c,t}(u,v)e_2(u,v) & b_2(u,v) + \lambda^{c,t}(u,v)f_2(u,v) \end{pmatrix} \begin{pmatrix} -\cos\theta^{c,t}(u,v) \\ \sin\theta^{c,t}(u,v) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (10)$$

for all $(u,v) \in U$. Then we say that $\boldsymbol{x}^{c,t} : U \to \mathbb{R}^3$ is a *caustic* of the framed surface $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s})$.

Remark 4.2. (1) The caustic $\boldsymbol{x}^{c,s}$ (respectively, $\boldsymbol{x}^{c,t}$) is corresponding to the $(\boldsymbol{n}, \overline{\boldsymbol{s}})$ (respectively, $(\boldsymbol{n}, \overline{\boldsymbol{t}})$)-Bertrand framed surface.

(2) By a direct calculation, we have $\boldsymbol{t}^{c,s}(u,v) = \cos\theta^{c,s}(u,v)\boldsymbol{s}(u,v) - \sin\theta^{c,s}(u,v)\boldsymbol{t}(u,v)$ and $\boldsymbol{t}^{c,t}(u,v) = \boldsymbol{n}(u,v)$.

(3) Suppose that there exist smooth functions $\lambda^{c,s}, \theta^{c,s} : U \to \mathbb{R}$ such that the condition (9) satisfies. If we take smooth functions $\lambda^{c,t}, \theta^{c,t} : U \to \mathbb{R}$ by $\lambda^{c,t} = \lambda^{c,s}$ and $\theta^{c,t} = \theta^{c,s} + \pi/2$, then the condition (10) is satisfied (cf. Theorem 3.11). The reflection frame of $\mathcal{C}^s(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s})$ is corresponding to the moving frame of $\mathcal{C}^t(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s})$. It follows that the map $\mathcal{C}^t$ is given by $\mathcal{C}^t(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s}) = \mathcal{C}^s(\boldsymbol{x}, -\boldsymbol{n}, \boldsymbol{t})$.

Definition 4.3. (1) Suppose that $\det(\boldsymbol{b}(u,v), \boldsymbol{g}(u,v)) = 0$ for all $(u,v) \in U$ and $(u_0, v_0) \in U$. The map $\mathcal{I}^s : \mathcal{F}(U, \mathbb{R}^3 \times \Delta) \to \mathcal{F}(U, \mathbb{R}^3 \times \Delta)$ is given by

$$\begin{aligned}
&\mathcal{I}^s(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s}) = (\boldsymbol{x}^{\mathcal{I},s}, \boldsymbol{n}^{\mathcal{I},s}, \boldsymbol{s}^{\mathcal{I},s}),\\
&\boldsymbol{x}^{\mathcal{I},s}(u,v) = \boldsymbol{x}(u,v) + \lambda^{\mathcal{I},s}(u,v)\boldsymbol{s}(u,v),\\
&\boldsymbol{n}^{\mathcal{I},s}(u,v) = \boldsymbol{s}(u,v),\\
&\boldsymbol{s}^{\mathcal{I},s}(u,v) = \cos\theta^{\mathcal{I},s}(u,v)\boldsymbol{t}(u,v) - \sin\theta^{\mathcal{I},s}(u,v)\boldsymbol{n}(u,v),
\end{aligned}$$

where $\theta^{\mathcal{I},s} : U \to \mathbb{R}$ is a smooth function and $\lambda^{\mathcal{I},s} : U \to \mathbb{R}$ is given by

$$\lambda^{\mathcal{I},s}(u,v) = -\left(\int_{u_0}^{u} a_1(u,v)du + \int_{v_0}^{v} a_2(u_0,v)dv\right).$$

Then we say that $\boldsymbol{x}^{\mathcal{I},s} : U \to \mathbb{R}^3$ is an *involute with respect to* $\boldsymbol{s}$ at $(u_0, v_0) \in U$ of the framed surface $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s})$.

(2) Suppose that $\det(\boldsymbol{a}(u,v), \boldsymbol{g}(u,v)) = 0$ for all $(u,v) \in U$ and $(u_0, v_0) \in U$. The map $\mathcal{I}^t : \mathcal{F}(U, \mathbb{R}^3 \times \Delta) \to \mathcal{F}(U, \mathbb{R}^3 \times \Delta)$ is given by

$$\begin{aligned}
&\mathcal{I}^t(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s}) = (\boldsymbol{x}^{\mathcal{I},t}, \boldsymbol{n}^{\mathcal{I},t}, \boldsymbol{s}^{\mathcal{I},t}),\\
&\boldsymbol{x}^{\mathcal{I},t}(u,v) = \boldsymbol{x}(u,v) + \lambda^{\mathcal{I},t}(u,v)\boldsymbol{t}(u,v),\\
&\boldsymbol{n}^{\mathcal{I},t}(u,v) = \boldsymbol{t}(u,v),\\
&\boldsymbol{s}^{\mathcal{I},t}(u,v) = \cos\theta^{\mathcal{I},t}(u,v)\boldsymbol{n}(u,v) - \sin\theta^{\mathcal{I},t}(u,v)\boldsymbol{s}(u,v),
\end{aligned}$$

where $\theta^{\mathcal{I},t}: U \to \mathbb{R}$ is a smooth function and $\lambda^{\mathcal{I},t}: U \to \mathbb{R}$ is given by

$$\lambda^{\mathcal{I},t}(u,v) = -\left(\int_{u_0}^{u} b_1(u,v)du + \int_{v_0}^{v} b_2(u_0,v)dv\right).$$

Then we say that $\boldsymbol{x}^{\mathcal{I},t}: U \to \mathbb{R}^3$ is an *involute with respect to* $\boldsymbol{t}$ at $(u_0,v_0) \in U$ of the framed surface $(\boldsymbol{x},\boldsymbol{n},\boldsymbol{s})$.

Remark 4.4. (1) The involute $\boldsymbol{x}^{\mathcal{I},s}$ (respectively, $\boldsymbol{x}^{\mathcal{I},t}$) is corresponding to the $(\boldsymbol{s},\overline{\boldsymbol{n}})$ (respectively, $(\boldsymbol{t},\overline{\boldsymbol{n}})$)-Bertrand framed surface under the condition $\theta^{\mathcal{I},s}=0$ (respectively, $\theta^{\mathcal{I},t}=0$). However, we consider a framed rotation of the framed surface in Definition 4.3. Moreover, we consider constant $c \in \mathbb{R}$ is zero.

(2) By a direct calculation, we have $\boldsymbol{t}^{\mathcal{I},s}(u,v) = \sin\theta^{\mathcal{I},s}(u,v)\boldsymbol{t}(u,v)+\cos\theta^{\mathcal{I},s}(u,v)\boldsymbol{n}(u,v)$ and $\boldsymbol{t}^{\mathcal{I},t}(u,v) = \sin\theta^{\mathcal{I},t}(u,v)\boldsymbol{n}(u,v)+\cos\theta^{\mathcal{I},t}(u,v)\boldsymbol{s}(u,v)$.

We consider conditions that caustics and involutes are inverse operations of framed surfaces.

Theorem 4.5. *Let* $(\boldsymbol{x},\boldsymbol{n},\boldsymbol{s}): U \to \mathbb{R}^3 \times \Delta$ *be a framed surface with basic invariants* $(\mathcal{G},\mathcal{F}_1,\mathcal{F}_2)$.

(1) (i) *Suppose that* $\det(\boldsymbol{b}(u,v),\boldsymbol{g}(u,v)) = 0$ *for all* $(u,v) \in U$, $\theta^{\mathcal{I},s}: U \to \mathbb{R}$ *is a smooth function and a smooth function* $\lambda^{\mathcal{I},s}: U \to \mathbb{R}$ *is given by*

$$\lambda^{\mathcal{I},s}(u,v) = -\left(\int_{u_0}^{u} a_1(u,v)du + \int_{v_0}^{v} a_2(u_0,v)dv\right),$$

for a point $(u_0,v_0) \in U$. *If we take* $\lambda^{c,s},\theta^{c,s}: U \to \mathbb{R}$ *by* $\lambda^{c,s} = -\lambda^{\mathcal{I},s}$ *and* $\theta^{c,s} = -\theta^{\mathcal{I},s}$, *then* $\mathcal{C}^s(\mathcal{I}^s(\boldsymbol{x},\boldsymbol{n},\boldsymbol{s})) = (\boldsymbol{x},\boldsymbol{n},\boldsymbol{s})$.

(ii) *Suppose that* $\det(\boldsymbol{a}(u,v),\boldsymbol{g}(u,v)) = 0$ *for all* $(u,v) \in U$, $\theta^{\mathcal{I},t}: U \to \mathbb{R}$ *is a smooth function and a smooth function* $\lambda^{\mathcal{I},t}: U \to \mathbb{R}$ *is given by*

$$\lambda^{\mathcal{I},t}(u,v) = -\left(\int_{u_0}^{u} b_1(u,v)du + \int_{v_0}^{v} b_2(u_0,v)dv\right),$$

for a point $(u_0,v_0) \in U$. *If we take* $\lambda^{c,t},\theta^{c,t}: U \to \mathbb{R}$ *by* $\lambda^{c,t} = -\lambda^{\mathcal{I},t}$ *and* $\theta^{c,t} = -\theta^{\mathcal{I},t}$, *then* $\mathcal{C}^t(\mathcal{I}^t(\boldsymbol{x},\boldsymbol{n},\boldsymbol{s})) = (\boldsymbol{x},\boldsymbol{n},\boldsymbol{s})$.

(2) (i) *Suppose that there exist smooth functions* $\lambda^{c,s},\theta^{c,s}: U \to \mathbb{R}$ *such that the condition* (9) *satisfies. If we take* $\theta^{\mathcal{I},s}: U \to \mathbb{R}$ *by* $\theta^{\mathcal{I},s} = -\theta^{c,s}$, *then* $\mathcal{I}^s(\mathcal{C}^s(\boldsymbol{x},\boldsymbol{n},\boldsymbol{s})) = (\boldsymbol{x}+\lambda^{c,s}(u_0,v_0)\boldsymbol{n},\boldsymbol{n},\boldsymbol{s})$ *for a point* $(u_0,v_0) \in U$.

(ii) *Suppose that there exist smooth functions* $\lambda^{c,t},\theta^{c,t}: U \to \mathbb{R}$ *such that the condition* (10) *satisfies. If we take* $\theta^{\mathcal{I},t}: U \to \mathbb{R}$ *by* $\theta^{\mathcal{I},t} = -\theta^{c,t}$, *then* $\mathcal{I}^t(\mathcal{C}^t(\boldsymbol{x},\boldsymbol{n},\boldsymbol{s})) = (\boldsymbol{x}+\lambda^{c,t}(u_0,v_0)\boldsymbol{n},\boldsymbol{n},\boldsymbol{s})$ *for a point* $(u_0,v_0) \in U$.

Proof. (1) (i) By Definition 4.3 (1), the map $\mathcal{I}^s: \mathcal{F}(U,\mathbb{R}^3\times\Delta) \to \mathcal{F}(U,\mathbb{R}^3\times\Delta)$ is given by $\mathcal{I}^s(\boldsymbol{x},\boldsymbol{n},\boldsymbol{s}) = (\boldsymbol{x}^{\mathcal{I},s},\boldsymbol{n}^{\mathcal{I},s},\boldsymbol{s}^{\mathcal{I},s}) = (\boldsymbol{x}+\lambda^{\mathcal{I},s}\boldsymbol{s},\boldsymbol{s},\cos\theta^{\mathcal{I},s}\boldsymbol{t}+\sin\theta^{\mathcal{I},s}\boldsymbol{n})$. The basic invariants of $\mathcal{I}^s(\boldsymbol{x},\boldsymbol{n},\boldsymbol{s})$ is given by $(\mathcal{G}^{\mathcal{I},s},\mathcal{F}_1^{\mathcal{I},s},\mathcal{F}_2^{\mathcal{I},s})$. The condition (9) for $\mathcal{I}^s(\boldsymbol{x},\boldsymbol{n},\boldsymbol{s})$ is given by

$$\begin{pmatrix} a_1^{\mathcal{I},s}+\lambda^{c,s}e_1^{\mathcal{I},s} & b_1^{\mathcal{I},s}+\lambda^{c,s}f_1^{\mathcal{I},s} \\ a_2^{\mathcal{I},s}+\lambda^{c,s}e_2^{\mathcal{I},s} & b_2^{\mathcal{I},s}+\lambda^{c,s}f_2^{\mathcal{I},s} \end{pmatrix}\begin{pmatrix}\sin\theta^{c,s} \\ \cos\theta^{c,s}\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}.$$

By a direct calculation, we have

$$\begin{pmatrix} b_1+(\lambda^{c,s}+\lambda^{\mathcal{I},s})g_1 & (\lambda^{c,s}+\lambda^{\mathcal{I},s})e_1 \\ b_2+(\lambda^{c,s}+\lambda^{\mathcal{I},s})g_2 & (\lambda^{c,s}+\lambda^{\mathcal{I},s})e_2 \end{pmatrix} \begin{pmatrix} \sin(\theta^{c,s}+\theta^{\mathcal{I},s}) \\ -\cos(\theta^{c,s}+\theta^{\mathcal{I},s}) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{11}$$

If we take $\lambda^{c,s}, \theta^{c,s} : U \to \mathbb{R}$ by $\lambda^{c,s} = -\lambda^{\mathcal{I},s}$ and $\theta^{c,s} = -\theta^{\mathcal{I},s}$, then the condition (11) is satisfied. Thus, the map $\mathcal{C}^s$ of the map $\mathcal{I}^s$ exists. By Definition 4.1 (1), the map $\mathcal{C}^s$ of the map $\mathcal{I}^s$, $\mathcal{C}^s(\mathcal{I}^s(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s}))$ is given by

$$\begin{aligned}
\mathcal{C}^s(\mathcal{I}^s(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s})) &= (\boldsymbol{x}^{c,s}(\boldsymbol{x}^{\mathcal{I},s}, \boldsymbol{n}^{\mathcal{I},s}, \boldsymbol{s}^{\mathcal{I},s}), \boldsymbol{n}^{c,s}(\boldsymbol{x}^{\mathcal{I},s}, \boldsymbol{n}^{\mathcal{I},s}, \boldsymbol{s}^{\mathcal{I},s}), \boldsymbol{s}^{c,s}(\boldsymbol{x}^{\mathcal{I},s}, \boldsymbol{n}^{\mathcal{I},s}, \boldsymbol{s}^{\mathcal{I},s})),\\
\boldsymbol{x}^{c,s}(\boldsymbol{x}^{\mathcal{I},s}, \boldsymbol{n}^{\mathcal{I},s}, \boldsymbol{s}^{\mathcal{I},s}) &= \boldsymbol{x}^{\mathcal{I},s} + \lambda^{c,s}\boldsymbol{n}^{\mathcal{I},s} = \boldsymbol{x} + (\lambda^{c,s}+\lambda^{\mathcal{I},s})\boldsymbol{s} = \boldsymbol{x},\\
\boldsymbol{n}^{c,s}(\boldsymbol{x}^{\mathcal{I},s}, \boldsymbol{n}^{\mathcal{I},s}, \boldsymbol{s}^{\mathcal{I},s}) &= \sin\theta^{c,s}\boldsymbol{s}^{\mathcal{I},s} + \cos\theta^{c,s}\boldsymbol{t}^{\mathcal{I},s}\\
&= (-\sin\theta^{c,s}\sin\theta^{\mathcal{I},s} + \cos\theta^{c,s}\cos\theta^{\mathcal{I},s})\boldsymbol{n}\\
&\quad + (\sin\theta^{c,s}\cos\theta^{\mathcal{I},s} + \cos\theta^{c,s}\sin\theta^{\mathcal{I},s})\boldsymbol{t}\\
&= \cos(\theta^{c,s}+\theta^{\mathcal{I},s})\boldsymbol{n} + \sin(\theta^{c,s}+\theta^{\mathcal{I},s})\boldsymbol{t} = \boldsymbol{n},\\
\boldsymbol{s}^{c,s}(\boldsymbol{x}^{\mathcal{I},s}, \boldsymbol{n}^{\mathcal{I},s}, \boldsymbol{s}^{\mathcal{I},s}) &= \boldsymbol{n}^{\mathcal{I},s} = \boldsymbol{s}.
\end{aligned}$$

(ii) We can also prove by the same method of (i).

(2) (i) By Definition 4.1 (1), the map $\mathcal{C}^s : \mathcal{F}(U, \mathbb{R}^3 \times \Delta) \to \mathcal{F}(U, \mathbb{R}^3 \times \Delta)$ is given by $\mathcal{C}^s(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s}) = (\boldsymbol{x}^{c,s}, \boldsymbol{n}^{c,s}, \boldsymbol{s}^{c,s}) = (\boldsymbol{x}+\lambda^{c,s}\boldsymbol{n}, \sin\theta^{c,s}\boldsymbol{s}+\cos\theta^{c,s}\boldsymbol{t}, \boldsymbol{n})$, where there exist smooth functions $\lambda^{c,s}, \theta^{c,s} : U \to \mathbb{R}$ such that the condition (9) satisfies. The basic invariants of $\mathcal{C}^s(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s})$ is given by $(\mathcal{G}^{c,s}, \mathcal{F}_1^{c,s}, \mathcal{F}_2^{c,s})$. By the integrability conditions (1), we have

$$\det(\boldsymbol{b}^{c,s}(u,v), \boldsymbol{g}^{c,s}(u,v)) = a_{1v}^{c,s}(u,v) - a_{2u}^{c,s}(u,v) = \lambda_{uv}^{c,s}(u,v) - \lambda_{vu}^{c,s}(u,v) = 0,$$

for all $(u,v) \in U$. Thus, the map $\mathcal{I}^s$ of the map $\mathcal{C}^s$ always exists. By Definition 4.3 (1), $\lambda^{\mathcal{I},s} : U \to \mathbb{R}$ is given by

$$\begin{aligned}
\lambda^{\mathcal{I},s}(u,v) &= -\left(\int_{u_0}^{u} a_1^{c,s}(u,v)du + \int_{v_0}^{v} a_2^{c,s}(u_0,v)dv\right)\\
&= -\left(\int_{u_0}^{u} \lambda_u^{c,s}(u,v)du + \int_{v_0}^{v} \lambda_v^{c,s}(u_0,v)dv\right)\\
&= -\left(\lambda^{c,s}(u,v) - \lambda^{c,s}(u_0,v_0)\right),
\end{aligned}$$

for a point $(u_0, v_0) \in U$. if we take $\theta^{\mathcal{I},s} : U \to \mathbb{R}$ by $\theta^{\mathcal{I},s} = -\theta^{c,s}$, the map $\mathcal{I}^s$ of the map $\mathcal{C}^s$, $\mathcal{I}^s(\mathcal{C}^s(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s}))$ is given by

$$\begin{aligned}
\mathcal{I}^s(\mathcal{C}^s(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s})) &= (\boldsymbol{x}^{\mathcal{I},s}(\boldsymbol{x}^{c,s}, \boldsymbol{n}^{c,s}, \boldsymbol{s}^{c,s}), \boldsymbol{n}^{\mathcal{I},s}(\boldsymbol{x}^{c,s}, \boldsymbol{n}^{c,s}, \boldsymbol{s}^{c,s}), \boldsymbol{s}^{\mathcal{I},s}(\boldsymbol{x}^{c,s}, \boldsymbol{n}^{c,s}, \boldsymbol{s}^{c,s})),\\
\boldsymbol{x}^{\mathcal{I},s}(\boldsymbol{x}^{c,s}, \boldsymbol{n}^{c,s}, \boldsymbol{s}^{c,s}) &= \boldsymbol{x}^{c,s} + \lambda^{\mathcal{I},s}\boldsymbol{s}^{c,s} = \boldsymbol{x} + (\lambda^{c,s}+\lambda^{\mathcal{I},s})\boldsymbol{n} = \boldsymbol{x} + \lambda^{c,s}(u_0,v_0)\boldsymbol{n},\\
\boldsymbol{n}^{\mathcal{I},s}(\boldsymbol{x}^{c,s}, \boldsymbol{n}^{c,s}, \boldsymbol{s}^{c,s}) &= \boldsymbol{s}^{c,s} = \boldsymbol{n},\\
\boldsymbol{s}^{\mathcal{I},s}(\boldsymbol{x}^{c,s}, \boldsymbol{n}^{c,s}, \boldsymbol{s}^{c,s}) &= \cos\theta^{\mathcal{I},s}\boldsymbol{t}^{c,s} - \sin\theta^{\mathcal{I},s}\boldsymbol{n}^{c,s}\\
&= (-\sin\theta^{c,s}\sin\theta^{\mathcal{I},s} + \cos\theta^{c,s}\cos\theta^{\mathcal{I},s})\boldsymbol{s}\\
&\quad - (\sin\theta^{c,s}\cos\theta^{\mathcal{I},s} + \cos\theta^{c,s}\sin\theta^{\mathcal{I},s})\boldsymbol{t}\\
&= \cos(\theta^{c,s}+\theta^{\mathcal{I},s})\boldsymbol{s} - \sin(\theta^{c,s}+\theta^{\mathcal{I},s})\boldsymbol{t} = \boldsymbol{s}.
\end{aligned}$$

(ii) We can also prove by the same method of (i). □

5 Tangential direction framed surfaces

Let $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s}) : U \to \mathbb{R}^3 \times \Delta$ be a framed surface with basic invariants $(\mathcal{G}, \mathcal{F}_1, \mathcal{F}_2)$.

Definition 5.1. (1) The map $\mathcal{S}^t : \mathcal{F}(U, \mathbb{R}^3 \times \Delta) \to \mathcal{F}(U, \mathbb{R}^3 \times \Delta)$ is given by

$$\begin{aligned}
\mathcal{S}^t(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s}) &= (\boldsymbol{x}^{S,t}, \boldsymbol{n}^{S,t}, \boldsymbol{s}^{S,t}),\\
\boldsymbol{x}^{S,t}(u,v) &= \boldsymbol{x}(u,v) + \lambda^{S,t}(u,v)\boldsymbol{s}(u,v),\\
\boldsymbol{n}^{S,t}(u,v) &= \cos\theta^{S,t}(u,v)\boldsymbol{t}(u,v) - \sin\theta^{S,t}(u,v)\boldsymbol{n}(u,v),\\
\boldsymbol{s}^{S,t}(u,v) &= \sin\theta^{S,t}(u,v)\boldsymbol{t}(u,v) + \cos\theta^{S,t}(u,v)\boldsymbol{n}(u,v),
\end{aligned}$$

where there exist smooth functions $\lambda^{S,t}, \theta^{S,t} : U \to \mathbb{R}$ such that

$$\begin{pmatrix} \lambda^{S,t}(u,v)e_1(u,v) & b_1(u,v) + \lambda^{S,t}(u,v)g_1(u,v) \\ \lambda^{S,t}(u,v)e_2(u,v) & b_2(u,v) + \lambda^{S,t}(u,v)g_2(u,v) \end{pmatrix} \begin{pmatrix} \sin\theta^{S,t}(u,v) \\ \cos\theta^{S,t}(u,v) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{12}$$

for all $(u,v) \in U$. We say that $(\boldsymbol{x}^{S,t}, \boldsymbol{n}^{S,t}, \boldsymbol{s}^{S,t})$ is a *tangential direction framed surface with respect to* $\boldsymbol{s}$ of the framed surface $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s})$.

(2) The map $\mathcal{T}^s : \mathcal{F}(U, \mathbb{R}^3 \times \Delta) \to \mathcal{F}(U, \mathbb{R}^3 \times \Delta)$ is given by

$$\begin{aligned}
\mathcal{T}^s(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s}) &= (\boldsymbol{x}^{T,s}, \boldsymbol{n}^{T,s}, \boldsymbol{s}^{T,s}),\\
\boldsymbol{x}^{T,s}(u,v) &= \boldsymbol{x}(u,v) + \lambda^{T,s}(u,v)\boldsymbol{t}(u,v),\\
\boldsymbol{n}^{T,s}(u,v) &= \sin\theta^{T,s}(u,v)\boldsymbol{n}(u,v) + \cos\theta^{T,s}(u,v)\boldsymbol{s}(u,v),\\
\boldsymbol{s}^{T,s}(u,v) &= \boldsymbol{t}(u,v),
\end{aligned}$$

where there exist smooth functions $\lambda^{T,s}, \theta^{T,s} : U \to \mathbb{R}$ such that

$$\begin{pmatrix} \lambda^{T,s}(u,v)f_1(u,v) & a_1(u,v) - \lambda^{T,s}(u,v)g_1(u,v) \\ \lambda^{T,s}(u,v)f_2(u,v) & a_2(u,v) - \lambda^{T,s}(u,v)g_2(u,v) \end{pmatrix} \begin{pmatrix} -\sin\theta^{T,s}(u,v) \\ \cos\theta^{T,s}(u,v) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{13}$$

for all $(u,v) \in U$. Then we say that $(\boldsymbol{x}^{T,s}, \boldsymbol{n}^{T,s}, \boldsymbol{s}^{T,s})$ is a *tangential direction framed surface with respect to* $\boldsymbol{t}$ of the framed surface $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s})$.

Remark 5.2. (1) The map $\mathcal{S}^t$ (respectively, $\mathcal{T}^s$) is corresponding to the $(\boldsymbol{s}, \overline{\boldsymbol{t}})$ (respectively, $(\boldsymbol{t}, \overline{\boldsymbol{s}})$)-Bertrand framed surface.

(2) By a direct calculation, we have $\boldsymbol{t}^{S,t}(u,v) = \boldsymbol{s}(u,v)$ and $\boldsymbol{t}^{S,t}(u,v) = \cos\theta^{T,s}(u,v)$ $\boldsymbol{n}(u,v) - \sin\theta^{T,s}(u,v)\boldsymbol{s}(u,v)$.

We give conditions that tangential direction framed surfaces are inverse operations of framed surfaces.

Theorem 5.3. *Let* $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s}) : U \to \mathbb{R}^3 \times \Delta$ *be a framed surface.*

(1) *Suppose that there exist smooth functions* $\lambda^{S,t}, \theta^{S,t} : U \to \mathbb{R}$ *such that the condition* (12) *satisfies. If we take* $\lambda^{T,s}, \theta^{T,s} : U \to \mathbb{R}$ *by* $\lambda^{T,s} = -\lambda^{S,t}$ *and* $\theta^{T,s} = -\theta^{S,t}$, *then* $\mathcal{T}^s(\mathcal{S}^t(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s})) = (\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s})$.

(2) *Suppose that there exist smooth functions* $\lambda^{T,s}, \theta^{T,s} : U \to \mathbb{R}$ *such that the condition* (13) *satisfies. If we take* $\lambda^{S,t}, \theta^{S,t} : U \to \mathbb{R}$ *by* $\lambda^{S,t} = -\lambda^{T,s}$ *and* $\theta^{S,t} = -\theta^{T,s}$, *then* $\mathcal{S}^t(\mathcal{T}^s(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s})) = (\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s})$.

Proof. (1) By Definition 5.1 (1), the map $\mathcal{S}^t : \mathcal{F}(U, \mathbb{R}^3 \times \Delta) \to \mathcal{F}(U, \mathbb{R}^3 \times \Delta)$ is given by $\mathcal{S}^t(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s}) = (\boldsymbol{x}^{S,t}, \boldsymbol{n}^{S,t}, \boldsymbol{s}^{S,t}) = (\boldsymbol{x} + \lambda^{S,t}\boldsymbol{s}, \cos\theta^{S,t}\boldsymbol{t} - \sin\theta^{S,t}\boldsymbol{n}, \sin\theta^{S,t}\boldsymbol{t} + \cos\theta^{S,t}\boldsymbol{n})$ where there exist smooth functions $\lambda^{S,t}, \theta^{S,t} : U \to \mathbb{R}$ such that the condition (12) satisfies. The basic invariants of $\mathcal{S}^t(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s})$ is given by $(\mathcal{G}^{S,t}, \mathcal{F}_1^{S,t}, \mathcal{F}_2^{S,t})$. The condition (13) for $\mathcal{S}^t(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s})$ is

$$\begin{pmatrix} \lambda^{T,s} f_1^{S,t} & a_1^{S,t} - \lambda^{T,s} g_1^{S,t} \\ \lambda^{T,s} f_2^{S,t} & a_2^{S,t} - \lambda^{T,s} g_2^{S,t} \end{pmatrix} \begin{pmatrix} -\sin\theta^{T,s} \\ \cos\theta^{T,s} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{14}$$

If we take $\lambda^{T,s}, \theta^{T,s} : U \to \mathbb{R}$ by $\lambda^{T,s} = -\lambda^{S,t}$ and $\theta^{T,s} = -\theta^{S,t}$, we have

$$\begin{aligned}
&-\lambda^{T,s} f_i^{S,t} \sin\theta^{T,s} + (a_i^{S,t} - \lambda^{T,s} g_i^{S,t}) \cos\theta^{T,s} \\
&= -\lambda^{T,s}(-g_i \cos\theta^{S,t} - e_i \sin\theta^{S,t}) \\
&\quad + \left(-\lambda^{S,t} e_i \cos\theta^{S,t} + (b_i + \lambda^{S,t} g_i) \sin\theta^{S,t} - \lambda^{T,s}(-g_i \sin\theta^{S,t} + e_i \cos\theta^{S,t})\right) \cos\theta^{T,s} \\
&= \lambda^{T,s} g_i \sin(\theta^{S,t} + \theta^{T,s}) - \lambda^{T,s} e_i \cos(\theta^{S,t} + \theta^{T,s}) \\
&\quad + \left(-\lambda^{S,t} e_i \cos\theta^{S,t} + (b_i + \lambda^{S,t} g_i) \sin\theta^{S,t}\right) \cos\theta^{T,s} \\
&= \lambda^{S,t} e_i - \lambda^{S,t} e_i \cos^2\theta^{S,t} + (b_i + \lambda^{S,t} g_i) \sin\theta^{S,t} \cos\theta^{S,t} \\
&= \lambda^{S,t} e_i \sin^2\theta^{S,t} + (b_i + \lambda^{S,t} g_i) \sin\theta^{S,t} \cos\theta^{S,t} \\
&= \sin\theta^{S,t} \left(\lambda^{S,t} e_i \sin\theta^{S,t} + (b_i + \lambda^{S,t} g_i) \cos\theta^{S,t}\right) \\
&= 0,
\end{aligned}$$

for $i = 1, 2$. It follows that the condition (14) is satisfied. Thus, the map $\mathcal{T}^s$ of the map $\mathcal{S}^t$ exists. By Definition 5.1 (2), the map $\mathcal{T}^s$ of the map $\mathcal{S}^t$, $\mathcal{T}^s(\mathcal{S}^t(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s}))$ is given by

$$\begin{aligned}
\mathcal{T}^s(\mathcal{S}^t(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s})) &= (\boldsymbol{x}^{T,s}(\boldsymbol{x}^{S,t}, \boldsymbol{n}^{S,t}, \boldsymbol{s}^{S,t}), \boldsymbol{n}^{T,s}(\boldsymbol{x}^{S,t}, \boldsymbol{n}^{S,t}, \boldsymbol{s}^{S,t}), \boldsymbol{s}^{T,s}(\boldsymbol{x}^{S,t}, \boldsymbol{n}^{S,t}, \boldsymbol{s}^{S,t})) \\
\boldsymbol{x}^{T,s}(\boldsymbol{x}^{S,t}, \boldsymbol{n}^{S,t}, \boldsymbol{s}^{S,t}) &= \boldsymbol{x}^{S,t} + \lambda^{T,s}\boldsymbol{s}^{T,s} = \boldsymbol{x} + (\lambda^{S,t} + \lambda^{T,s})\boldsymbol{s} = \boldsymbol{x}, \\
\boldsymbol{n}^{T,s}(\boldsymbol{x}^{S,t}, \boldsymbol{n}^{S,t}, \boldsymbol{s}^{S,t}) &= \sin\theta^{T,s}\boldsymbol{n}^{S,t} + \cos\theta^{T,s}\boldsymbol{s}^{S,t} \\
&= (-\sin\theta^{S,t}\sin\theta^{T,s} + \cos\theta^{S,t}\cos\theta^{T,s})\boldsymbol{n} \\
&\quad - (\sin\theta^{S,t}\cos\theta^{T,s} + \cos\theta^{S,t}\sin\theta^{T,s})\boldsymbol{t} \\
&= \cos(\theta^{S,t} + \theta^{T,s})\boldsymbol{n} - \sin(\theta^{S,t} + \theta^{T,s})\boldsymbol{t} = \boldsymbol{n}, \\
\boldsymbol{s}^{T,s}(\boldsymbol{x}^{S,t}, \boldsymbol{n}^{S,t}, \boldsymbol{s}^{S,t}) &= \boldsymbol{t}^{S,t} = \boldsymbol{s}.
\end{aligned}$$

(2) We can also prove by the same method of (1). □

Remark 5.4. (1) If $e_1(u, v) = 0$ and $e_2(u, v) = 0$ for all $(u, v) \in U$, then $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s})$ is always an $(\boldsymbol{s}, \overline{\boldsymbol{t}})$-Bertrand framed surface for any $\lambda^{S,t} : U \to \mathbb{R}$ and for any constant $\theta^{S,t}$ with $\cos\theta^{S,t} = 0$.

(2) If $f_1(u, v) = 0$ and $f_2(u, v) = 0$ for all $(u, v) \in U$, then $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s})$ is always a $(\boldsymbol{t}, \overline{\boldsymbol{s}})$-Bertrand framed surface for any $\lambda^{T,s} : U \to \mathbb{R}$ and for any constant $\theta^{T,s}$ with $\cos\theta^{T,s} = 0$.

6 Examples

We give concrete examples of caustics, involutes and tangential direction framed surfaces.

Example 6.1. (A cuspidal edge) Let $(\boldsymbol{x},\boldsymbol{n},\boldsymbol{s}):U\to\mathbb{R}^3\times\Delta$ be

$$\boldsymbol{x}(u,v)=\left(u,\frac{v^2}{2},\frac{v^3}{3}\right),\ \boldsymbol{n}(u,v)=\frac{1}{\sqrt{v^2+1}}(0,-v,1),\ \boldsymbol{s}(u,v)=(1,0,0).$$

Then $\boldsymbol{t}(u,v)=(0,1,v)/\sqrt{1+v^2}$ and $(\boldsymbol{x},\boldsymbol{n},\boldsymbol{s})$ is a framed surface with the basic invariants

$$\begin{pmatrix}a_1 & b_1\\ a_2 & b_2\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & v\sqrt{v^2+1}\end{pmatrix},\ \begin{pmatrix}e_1 & f_1 & g_1\\ e_2 & f_2 & g_2\end{pmatrix}=\begin{pmatrix}0 & 0 & 0\\ 0 & -1/(v^2+1) & 0\end{pmatrix}.$$

It follows that the curvature C^F of $(\boldsymbol{x},\boldsymbol{n},\boldsymbol{s})$ is given by

$$J^F(u,v)=v\sqrt{v^2+1},\ K^F(u,v)=0,\ H^F(u,v)=\frac{1}{2(v^2+1)}.$$

If we take $\lambda^{c,s}(u,v)=v(v^2+1)^{3/2}$ and $\theta^{c,s}(u,v)=0$, then condition (9) is satisfied. Therefore, we have a caustic of the framed surface, $\mathcal{C}^s(\boldsymbol{x},\boldsymbol{n},\boldsymbol{s})=(\boldsymbol{x}^{c,s},\boldsymbol{n}^{c,s},\boldsymbol{s}^{c,s})$,

$$\boldsymbol{x}^{c,s}(u,v)=\left(u,-v^4-\frac{v^2}{2},\frac{4}{3}v^3+v\right),\ \boldsymbol{n}^{c,s}(u,v)=\boldsymbol{t}(u,v),\ \boldsymbol{s}^{c,s}(u,v)=\boldsymbol{n}(u,v).$$

Moreover, if we take $\lambda^{c,t}(u,v)=v(v^2+1)^{3/2}$ and $\theta^{c,t}(u,v)=-\pi/2$, then condition (10) is satisfied. Therefore, we laso have a caustic of the framed surface, $\mathcal{C}^t(\boldsymbol{x},\boldsymbol{n},\boldsymbol{s})=(\boldsymbol{x}^{c,t},\boldsymbol{n}^{c,t},\boldsymbol{s}^{c,t})$,

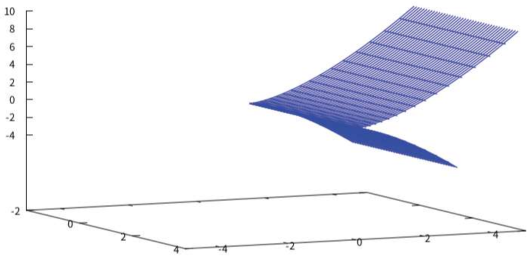
$$\boldsymbol{x}^{c,t}(u,v)=\left(u,-v^4-\frac{v^2}{2},\frac{4}{3}v^3+v\right),\ \boldsymbol{n}^{c,t}(u,v)=\boldsymbol{t}(u,v),\ \boldsymbol{s}^{c,t}(u,v)=-\boldsymbol{s}(u,v).$$

Since $\det(\boldsymbol{b}(u,v),\boldsymbol{g}(u,v))=0$ for all $(u,v)\in U$, if we take $\lambda^{\mathcal{I},s}(u,v)=-u$, $\theta^{\mathcal{I},s}(u,v)=-\pi/2$ and $(u_0,v_0)=(0,0)$, then we have an involute with respect to $\boldsymbol{s}$ at $(0,0)$,

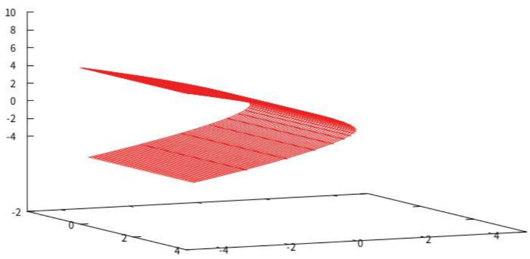
$$\begin{aligned}&\mathcal{I}^s(\boldsymbol{x},\boldsymbol{n},\boldsymbol{s})=(\boldsymbol{x}^{\mathcal{I},s},\boldsymbol{n}^{\mathcal{I},s},\boldsymbol{s}^{\mathcal{I},s}),\\ &\boldsymbol{x}^{\mathcal{I},s}(u,v)=\left(0,\frac{v^2}{2},\frac{v^3}{3}\right),\ \boldsymbol{n}^{\mathcal{I},s}(u,v)=\boldsymbol{s}(u,v),\ \boldsymbol{s}^{\mathcal{I},s}(u,v)=\boldsymbol{n}(u,v).\end{aligned}$$

Moreover, since $\det(\boldsymbol{a}(u,v),\boldsymbol{g}(u,v))=0$ for all $(u,v)\in U$, if we take $\lambda^{\mathcal{I},t}(u,v)=-\frac{1}{3}\big((v^2+1)^{\frac{3}{2}}-1\big)$, $\theta^{\mathcal{I},t}(u,v)=0$ and $(u_0,v_0)=(0,0)$, then we have an involute with respect to $\boldsymbol{t}$ at $(0,0)$, $\mathcal{I}^t(\boldsymbol{x},\boldsymbol{n},\boldsymbol{s})=(\boldsymbol{x}^{\mathcal{I},t},\boldsymbol{n}^{\mathcal{I},t},\boldsymbol{s}^{\mathcal{I},t})$,

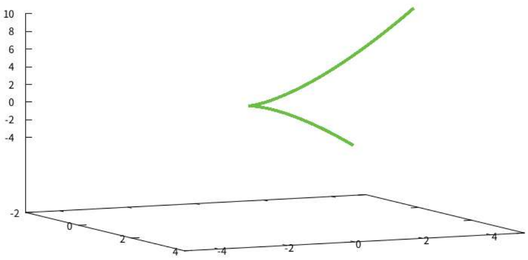
$$\begin{aligned}&\boldsymbol{x}^{\mathcal{I},t}(u,v)=\left(u,\frac{v^2-2}{6}+\frac{1}{3\sqrt{v^2+1}},-\frac{v}{3}\Big(1-\frac{1}{\sqrt{v^2+1}}\Big)\right),\\ &\boldsymbol{n}^{\mathcal{I},t}(u,v)=\boldsymbol{t}(u,v),\ \boldsymbol{s}^{\mathcal{I},t}(u,v)=\boldsymbol{n}(u,v).\end{aligned}$$



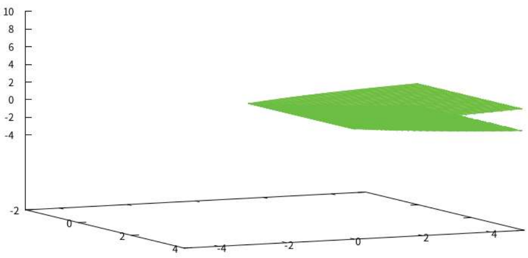
cuspidal edge $\boldsymbol{x}$



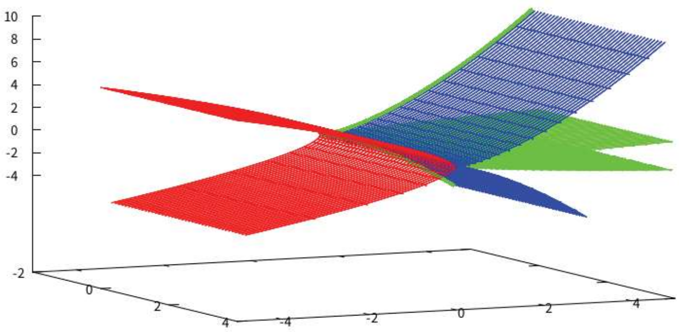
caustic $\mathcal{C}^s$ and $\mathcal{C}^t$



involute $\boldsymbol{x}^{\mathcal{I},s}$



involute $\boldsymbol{x}^{\mathcal{I},t}$



cuspidal edge $\boldsymbol{x}$, caustic $\mathcal{C}^s$ and $\mathcal{C}^t$ and involute $\boldsymbol{x}^{\mathcal{I},s}$ and $\boldsymbol{x}^{\mathcal{I},t}$

Figures 1 : $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s})$ and $(\overline{\boldsymbol{x}}, \overline{\boldsymbol{n}}, \overline{\boldsymbol{s}})$ of Example 6.1

Example 6.2. (A cuspidal cross-cap) Let $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s}) : U \to \mathbb{R}^3 \times \Delta$ be

$$\boldsymbol{x}(u,v) = \left(u, v^2, uv^3\right), \boldsymbol{n}(u,v) = \frac{1}{\sqrt{4v^6 + 9u^2v^2 + 4}}(-2v^3, -3uv, 2),$$
$$\boldsymbol{s}(u,v) = \frac{1}{\sqrt{1+v^6}}(1, 0, v^3).$$

Then $\boldsymbol{t}(u,v) = (-3uv^4, 2(v^6+1), 3uv)/\sqrt{4v^6 + 9u^2v^2 + 4}\sqrt{1+v^6}$ and $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s})$ is a framed surface with the basic invariants

$$\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} = \begin{pmatrix} \sqrt{1+v^6} & 0 \\ \frac{3uv^5}{\sqrt{1+v^6}} & \frac{v\sqrt{4v^6+9u^2v^2+4}}{\sqrt{1+v^6}} \end{pmatrix},$$
$$\begin{pmatrix} e_1 & f_1 & g_1 \\ e_2 & f_2 & g_2 \end{pmatrix} = \begin{pmatrix} 0 & -\frac{6v\sqrt{1+v^6}}{4v^6+9u^2v^2+4} & 0 \\ -\frac{6v^2\sqrt{1+v^6}}{\sqrt{1+v^6}\sqrt{4v^6+9u^2v^2+4}} & \frac{6u(2v^6-1)}{(4v^6+9u^2v^2+4)\sqrt{1+v^6}} & \frac{9uv^3}{(1+v^6)\sqrt{4v^6+9u^2v^2+4}} \end{pmatrix}.$$

It follows that the curvature C^F of $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s})$ is given by

$$J^F(u,v) = v\sqrt{4v^6 + 9u^2v^2 + 4},\ K^F(u,v) = -\frac{36v^3}{(4v^6 + 9u^2v^2 + 4)^{3/2}},$$
$$H^F(u,v) = -\frac{3u(5v^6 - 1)}{4v^6 + 9u^2v^2 + 4}.$$

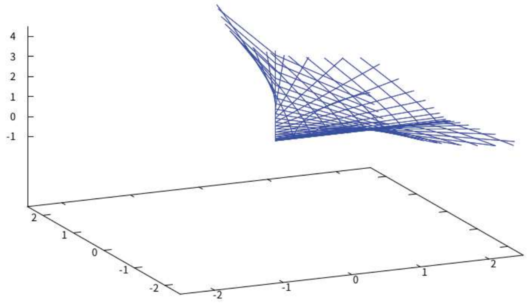
If we take

$$\lambda^{S,t}(u,v) = -\frac{(4v^6 + 9u^2v^2 + 4)\sqrt{1+v^6}}{9uv^2 + 6\sqrt{1+v^6}},\ \sin\theta^{S,t}(u,v) = -\frac{1}{\sqrt{1+v^2}},$$
$$\cos\theta^{S,t}(u,v) = \frac{v}{\sqrt{1+v^2}},$$

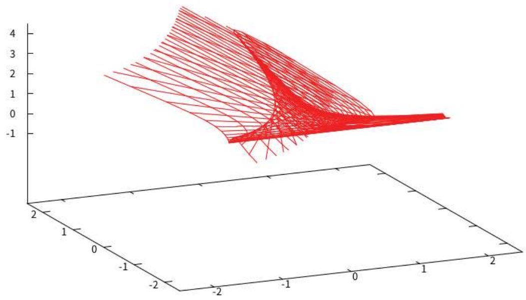
then condition (12) is satisfied. Therefore, we have tangential direction framed surface with respect to $\boldsymbol{s}$ of the framed surface $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s})$, $\mathcal{S}^t(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s}) = (\boldsymbol{x}^{S,t}, \boldsymbol{n}^{S,t}, \boldsymbol{s}^{S,t})$,

$$\begin{aligned}
\boldsymbol{x}^{S,t}(u,v) &= \boldsymbol{x}(u,v) + \lambda^{S,t}(u,v)\boldsymbol{s}(u,v) \\
&= \left(u - \frac{4v^6 + 9u^2v^2 + 4}{9uv^2 + 6\sqrt{1+v^6}}, v^2, v^3\left(u - \frac{4v^6 + 9u^2v^2 + 4}{9uv^2 + 6\sqrt{1+v^6}}\right)\right), \\
\boldsymbol{n}^{S,t}(u,v) &= \frac{v}{\sqrt{1+v^2}}\boldsymbol{t}(u,v) + \frac{1}{\sqrt{1+v^2}}\boldsymbol{n}(u,v), \\
\boldsymbol{s}^{S,t}(u,v) &= -\frac{1}{\sqrt{1+v^2}}\boldsymbol{t}(u,v) + \frac{v}{\sqrt{1+v^2}}\boldsymbol{n}(u,v).
\end{aligned}$$

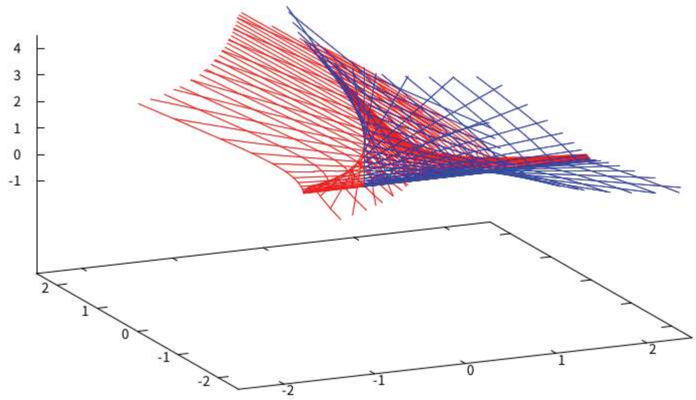
Moreover, if we take $\lambda^{T,s}(u,v) = -\lambda^{S,t}(u,v)$ and $\theta^{T,s}(u,v) = -\theta^{S,t}(u,v)$, then we have $\mathcal{T}^s \circ \mathcal{S}^t(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s}) = (\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s})$. Note that $\boldsymbol{x}^{S,t}$ at $(0,0)$ is also a cuspidal cross cap.



cuspidal cross cap $\boldsymbol{x}$



$(\boldsymbol{s}, \overline{\boldsymbol{t}})$-Bertrand framed surface $\mathcal{S}^t$



cuspidal cross cap $\boldsymbol{x}$ and $(\boldsymbol{s}, \overline{\boldsymbol{t}})$-Bertrand framed surface $\mathcal{S}^t$

Figures 2 : $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s})$ and $(\overline{\boldsymbol{x}}, \overline{\boldsymbol{n}}, \overline{\boldsymbol{s}})$ of Example 6.2

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References

[1] V. I. Arnol'd, *Singularities of Caustics and Wave Fronts*. Mathematics and Its Applications **62** Kluwer Academic Publishers, 1990.

[2] V. I. Arnol'd, S. M. Gusein-Zade, A. N. Varchenko, *Singularities of Differentiable Maps. Vol.* I, Birkhäuser, 1986.

[3] T. Fukunaga, M. Takahashi, Framed surfaces in the Euclidean space. *Bull. Braz. Math. Soc. (N.S.)* **50**, 2019, 37–65.

[4] S. Honda, M. Takahashi, Framed curves in the Euclidean space. Advances in Geometry. **16**, 2017, 265–276. doi: 10.1515/advgeom-2015-0035

[5] S. Honda, M. Takahashi, Bertrand and Mannheim curves of framed curves in the 3-dimensional Euclidean space. *Turkish J. Math.* **44**, 2020, 883–899.

[6] S. Honda, M. Takahashi, Circular evolutes and involutes of framed curves in the 3-dimensional Euclidean space. *Preprint*, 2024.

[7] S. Izumiya, M. C. Romero-Fuster, M. A. S. Ruas, F. Tari, *Differential Geometry from a Singularity Theory Viewpoint*. World Scientific Pub. Co Inc. 2015.

[8] N. Nakatsuyama, M. Takahashi, Bertrand types of regular curves and Bertrand framed curves in the Euclidean 3-space. *To appear in Hokkaido Math. J.* 2024, arXiv : `https://arxiv.org/abs/2403.19138`.

[9] N. Nakatsuyama, M. Takahashi, Bertrand framed surfaces in the Euclidean 3-space and its applications *Preprint.* 2025.

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