Some pictorial reconstructions of the Boy surface

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Abstract

We present four, not standard constructions of the Boy surface as directly and elementary as possible by pictures. The purpose of this is to give miscellaneous views of the Boy surface so that they can give a clue to the realizations of the other surfaces and higher dimensional manifolds, or to the visualizations of manifolds in the 2- or 3-spaces.

1 Introduction

The *Boy surface* in this short article means the image of a generic smooth immersion of the real projective plane $\mathbb{R}P^2$ in $\mathbb{R}^3$ which is ambient isotopic to the image of the Möbius strip described in the book of F. Apéry [Ap], pp. 51-53 when a suitable open 2-disc is removed off. In this book, it is called the *direct* Boy surface and is distinguished from its mirror called the *opposite* Boy surface. However we call both of them simply as the Boy surface.

The locus of the self-intersection points of the Boy surface is an immersed loop with one triple point. It is isotopic to a bouquet of the boundaries of the three embedded 2-discs whose interiors are mutually mutually disjoint and is called the three-bladed propeller. We denote it by P and call each 2-disc bounding a component loop of P a *blade* of the propeller. A thin neighbourhood of P in the Boy surface is made of the xy-, yz- and zw-plane sections of the unit box I^3, $I = [-1, 1]$, in $\mathbb{R}^3$ and three copies of a crossed band $X \times I$, $X = \{(s, t) \in I^2 \subset \mathbb{R}^2; st = 0\}$ connecting them in a *trivial manner*, that means, the band $\{t = 0\} \times I$ of one $X \times I$ lies on the xy-plane, that of another $X \times I$ lies on the yz-plane, and that of the third $X \times I$ lies on the zx-plane. The neighbourhood of P has four boundary loops and three of them are capped by some 2-discs contained in the blades of the propeller P (refer to Fig.20, [Ap]). The Boy surface is obtained by attaching the fourth embedded 2-disc along the fourth boundary of this neighbourhood, or in other words, along the boundary of the neighbourhood of P capped by its three blades.

We present three pictorial constructions of the Boy surface to provide its topological view as directly and elementary as possible. We do not take their explicit parametrization into account and sometimes we use combinatorial pictures to represent the Boy surface. This is because we give priority to a better understanding of its topological shape. However every combinatorial picture used in this article can be easily modified to a smooth one.

2 Putting membranes on a trefoil knot

The first model of the Boy surface we present is by putting membranes on a trefoil knot. We take a trefoil knot K passing through the 6 vertices a_1, a_4, b_1, b_4, c_1 and c_4 of the unit box in $\mathbb{R}^3$ in this order as given in Fig.1, upper. We assume that K is transverse to the three rectangles $a_1b_4a_4b_1$, $b_1c_4b_4c_1$ and $c_1a_4c_4a_1$. Let p, q, r be the intersection points of K with these rectangles, respectively. Second, for later convenience, we take a three-bladed propeller P having a triple point s at the origin of the box and passing through p, q, r so that its three blades sit in the three rectangles mentioned above. In the figure, the part of P beneath the plane passing through p, q, r is presented by broken lines, for visibility. Note that P is divided into the three regular arcs psq, qsr and rsp. We put two points a_2, a_3 on the arc a_1a_4 of K so that p, a_2, a_3 are lined in this order. Similarly, we take two points b_2, b_3 on the arc b_1b_4 and further two points c_2, c_3 on the arc c_1c_4.

Now we put three membranes Bl, Rd, Bk, each of which is topologically a closed 2-disc, on K as follows (refer to Fig.1, lower). The membrane Rd is spanned between the arcs b_1b_2 and c_2c_1 so that it passes through b_4 as a boundary point (lower left of the figure). One can make Rd smooth and contain the regular arcs psq and psr of P because these two arcs are tangent to a common plane at s by nature of P. Note that Rd is transverse to the arc a_1a_2 of K at p, by nature of P.

The membrane Bl is spanned similarly between the arcs a_1a_2 and b_2b_1 of K, passing through a_4, containing the arcs psq and qsr of P, and is transverse to c_1c_2 of K at q (lower middle of the figure). The membrane Bk is spanned similarly between the arcs c_1c_2 and a_1c_4 of K, passing through c_4, containing the arcs qsr and psr of P, and is transverse to b_1b_2 of K at r (lower right of the figure).

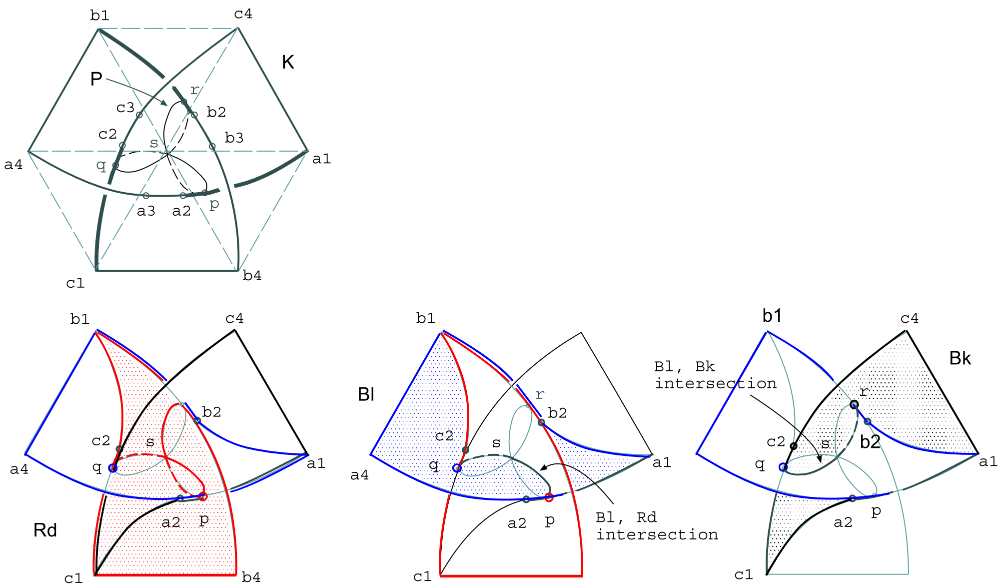

Fig. 1: Trefoil knot K (upper) and membranes Rd, Bl, Bk (lower)

The intersection of Bl and Rd is the regular arc psq of P, that of Bl and Bk is the regular arc qsr of P and that of Bk and Rd is the regular arc rsp of P. One can make these intersections *transversal*, that means, in a small neighbourhood of each intersection point in $\mathbb{R}^3$, the union of the two membranes is diffeomorphic to the union of $z = 0$ and $x = 0$ or to the union of $z = 0$ and the half plane $x = 0,\ y \geq 0$ restricted to the unit open 3-disc.

Note that each of the arcs a_1a_2, b_1b_2 and c_1c_2 of K is shared by two of the three membranes Bl, Rd, Bk and one can make the two membranes pasted smoothly along these arcs. We denote by Mo the union of Bl, Rd, Bk, which is illustrated in Fig.2, left. It is a Möbius strip when each of its self-intersection point is regarded to be two distinct points, as seen directly from the construction, and is actually an immersed Möbius strip having a triple point at s, by the transversality mentioned above. The boundary of Mo is the simple loop made by a_1b_2, $b_2b_3b_4$, b_4c_1, c_1a_2, $a_2a_3a_4$, a_4b_1, b_1c_2, $c_2c_3c_4$ and c_4a_1, as seen in the figure.

By taking new arcs a_3b_4, b_3c_4 and c_3a_4, we can put the fourth membrane Top, a 6-gon

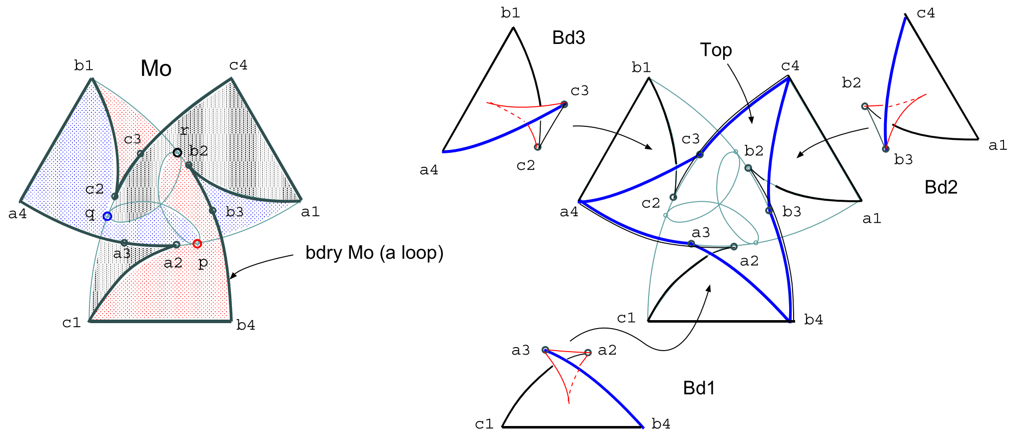

Fig. 2: Mo (left) and 1/2-twisted band attaching (right)

whose boundary is made of a_3a_4, a_4c_3, c_3c_4, c_4b_3, b_3b_4 and b_4a_3, on K so that the three sides a_3a_4, c_3c_4 and b_3b_4 are shared in common with the boundary of Mo. One can make its interior disjoint from Mo by placing it over Mo and also make the attaching along the common boundary sides smooth.

To make the union of Mo and Top closed, we attach three 1/2-twisted bands Bd_1, Bd_2 and Bd_3 as follows (refer to Fig.2, right); the band Bd_1 is spanned smoothly between a_2a_3 and b_4c_1 so that the two side edge of the band are a_3b_4 and a_2c_1. The bands Bd_2 and Bd_3 are spanned similarly between b_2b_3 and c_4a_1 and between c_2c_3 and a_4b_1.

Theorem 2.1. *Let Bo be the union of the four membranes* Bl, Rd, Bk, Top *and the three 1/2-twisted bands* Bd_1, Bd_2 *and* Bd_3*.*

1. *After smoothing, Bo is the image of an generic immersion of* $\mathbb{R}P^2$*.*
2. *Bo is the Boy surface.*

Proof. 1. The union Mo of Bl, Rd, Bk is the image of a smooth and generic immersion of the Möbius strip as mentioned. On the other hand, the union of Top and the three 1/2-wisted bands is topologically an embedded 2-disc, and one can make it also a smoothly embedded 2-disc. Hence one can make Bo the image of a smooth and generic immersion of $\mathbb{R}P^2$.

2. Let N be the neighbourhood of the three-bladed propeller in the Boy surface after the three bouquet loops of the propeller are capped by the blades of the propeller. We are enough to show that Mo, which is Bo with a suitable embedded 2-disc removed, is isotopic to N. The self-intersection locus P of Mo is the three-bladed propeller, up to isotopy, and recalling that each blade lies on one of the membranes Bl, Rd, Bk, one can see in the picture of Mo (Fig.2, left) that Mo deformation retracts to a subset of Mo which is isotopic to N. $\square$

3 Construction from an embedded Möbius strip

We make the Boy surface by starting from an embedded Möbius strip. This construction gives a reason why a trefoil knot appears in the previous construction and how the three-bladed propeller appears in the Boy surface.

Consider first the embedded Möbius strip in $\mathbb{R}^3$ as illustrated in Fig.3 (A), where the three arcs a_1a_2, b_1b_2, c_1c_2 separate the Möbius strip into three rectangles. Then pull a side of each rectangle to the location indicated by the broken lines so that they make a triple point. The result illustrated in Fig.3 (B) is an immersed Möbius strip whose locus of self-intersection points is the three line segments meeting at a triple point. To make the locus of self-intersection points closed, we stretch the three rectangles further so that each makes a transverse intersection with one of the separator above, as illustrated in Fig.3 (C). These moves are followed by a sequence of immersions and the final immersed Möbius strip illustrated in Fig.3 (D) is the image of a generic immersion with one triple point. Its locus of self-intersection points is a three-bladed propeller and it agrees with the one denoted by Mo in Section 2. By attaching the three 1/2-twisted bands as before we obtain the Boy surface (Fig.3 (E)).

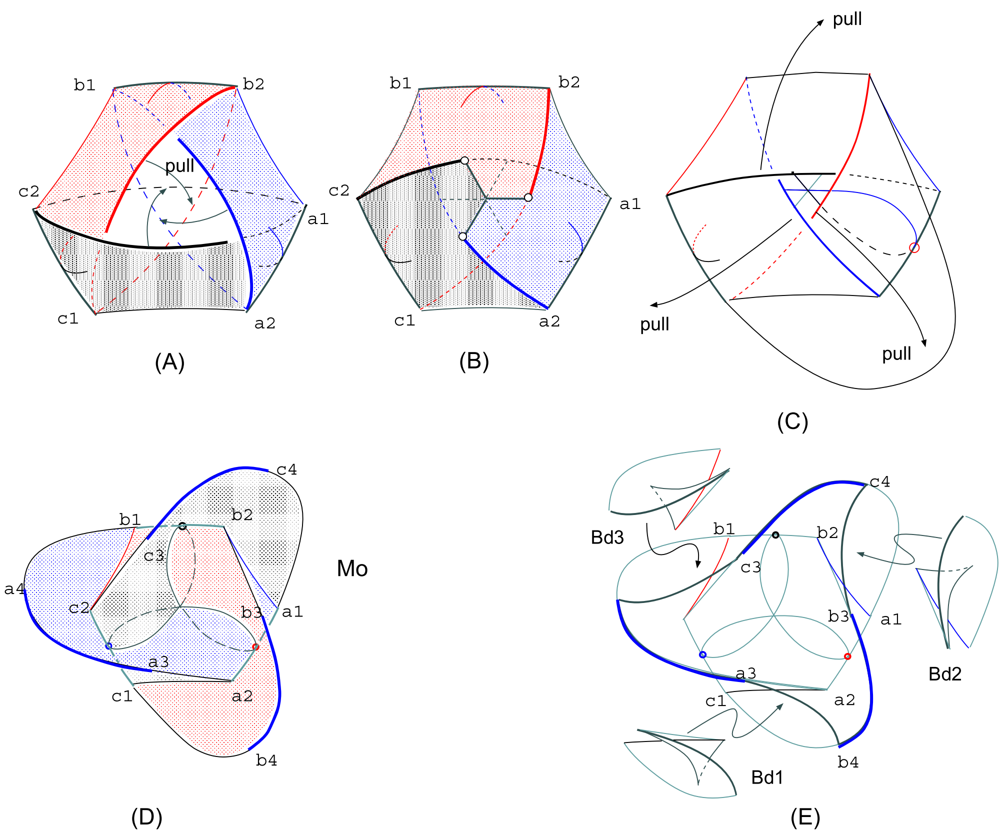

Fig. 3: Immersed Möbius strip Mo; (A) embedded Möbius strip (B) immersed one with a triple point, (C) modification to make the self-intersection locus closed, (D) Mo, (E) 1/2-twisted band attaching

Note Three arcs $a_1c_4c_3c_2$, $c_1b_4b_3b_2$ and $b_1a_4a_3a_2$ of the boundary of Mo combined with the three separators c_2c_1, b_2b_1 and a_2a_1 of Möbius strip make a trefoil knot. It appears to make the self-intersection locus of the immersed Möbius strip Fig.3 (B) closed, as mentioned, and the three-bladed propeller is obtained by this closing.

4 Realization by a combinatorial map

The Boy surface can be achieved by a combinatorial map of $\mathbb{R}P^2$ to $\mathbb{R}^3$. Let us consider the subdivision of $\mathbb{R}P^2$ into four 6-gons Bl, Rd, Bk, Top and 4-gon neighbourhoods of the three points A, B, C, as in Fig.4 (A), where the boundary of the large triangle ABC is suitably identified to represent $\mathbb{R}P^2$.

To define the map we take a frame in $\mathbb{R}^3$ by points called vertices and some line segments between them as follows. Take two unlinked triangle frames, and besides the vertices of triangles put one extra vertex on each side of the triangles, and then take six line segments between the vertices on the two triangles as in Fig.4 (B). We further attach the three-bladed propeller P with triple point s to the frame at certain point p, q, r on each side of a triangle, as in the figure. Now we map the 6-gons Top, Bl, Rd, Bk so that their boundaries are mapped onto the positions indicated in Fig.4 (B-D). Note that the map on each piece is not linear but smooth. We can adjust each piece so that the mutual intersections among Bl, Rd, Bk form the propeller P, that means, the regular arcs psq, qsr and psr of P are the loci of intersections of Bl and Rd, Bl and Bk, and Rd and Bk, respectively, and that these intersections are transversal, and further that Top is disjoint from the other three, as in the construction in Section 2.

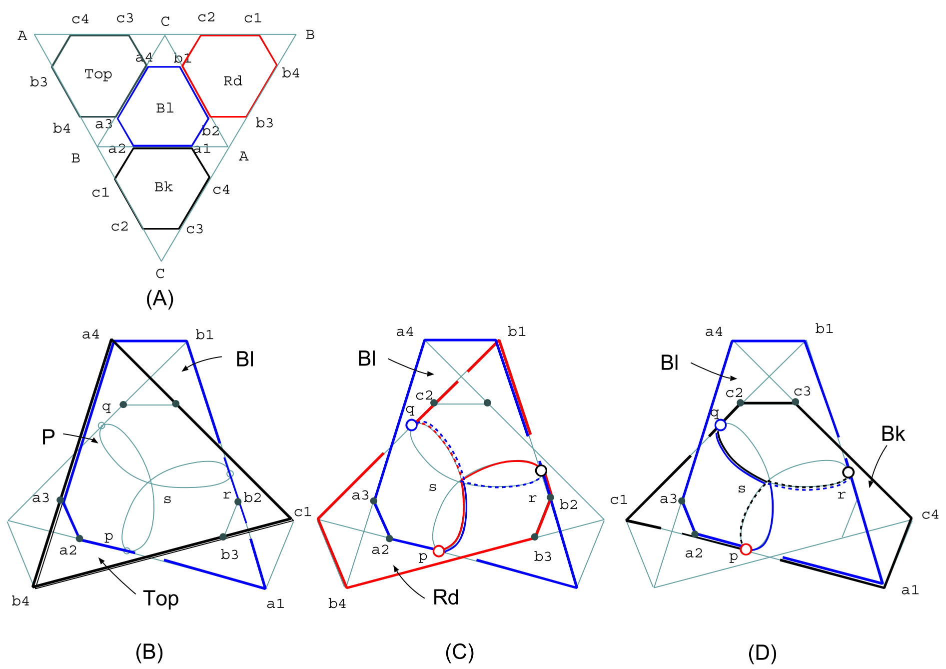

Fig. 4: Combinatorial map from $\mathbb{R}P^2$ to $\mathbb{R}^3$; (A) subdivision of $\mathbb{R}P^2$, (B) images of Bl and Top, (C) those of Bl and Rd, (D) those of Bl and Bk. In (C,D), relevant part of P is marked.

Next, the 4-gon neighbourhood of C are first made into a 1/2-twisted band Bd_3 (Fig.5, left) and then mapped so that the boundary edges are as indicated in Fig.5, right. Other two 4-gon neighbourhoods of A and B are similarly made into the 1/2-twisted bands Bd_1 and Bd_2, respectively, and then mapped as indicated in the figure.

Note that the piecewise linear loop made by the points $a_1 - a_4$, $b_1 - b_4$ and $c_1 - c_4$ in Fig.4 (B) form a trefoil knot. Then it is direct to see that this map gives the combinatorial representation of the construction in Section 2.

This construction of Boy surface implies the following observation.

Proposition 4.1. *Let $\Phi : \mathbb{R}P^2 \to \mathbb{R}^3$ be the combinatorial representation of the Boy surface defined as above. It can be deformed to the momentum map $\Phi_0 : \mathbb{R}P^2 \to \mathbb{R}^2$ onto a triangle, of a linear $O(1) \times O(1)$ action on $\mathbb{R}P^2$.*

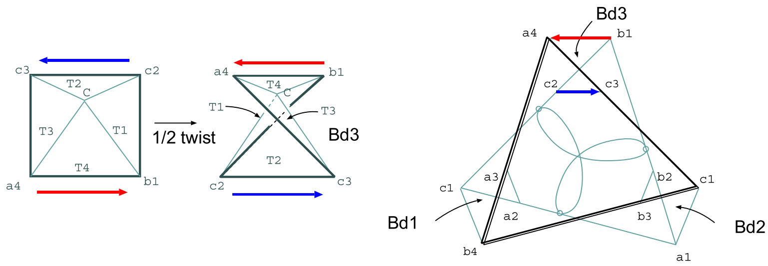

Fig. 5: Combinatorial map from $\mathbb{R}P^2$ to $\mathbb{R}^3$ on the 4-gon neighbourhoods of A, B, C; (left) make 1/2-twist, (right) positions where the twisted bands are attached

Proof. We give the deformation on the 4-gon neighbourhood of C in Fig.4 (A) as follows. In $\mathbb{R}^3$, we move the triangle T_4 in Fig.6, upper to the back of the image of T_2, so that the images of a_4 and b_1 are moved toward those of c_2 and c_3, respectively. Then the map on the 4-gon neighbourhood of C is deformed to a map that folds a square into quarter, as in Fig.6, upper. The deformations on the 4-gon neighbourhoods of A and B are similar.

On the three 6-gons Bl, Rd, Bk, we move their images in $\mathbb{R}^3$ as illustrated in Fig.6, lower, that means; the images of the edges b_4c_1 and a_2a_3, c_4a_1 and b_2b_3, a_4b_1 and c_2c_3 are made close to each other, respectively. As a result, the images of c_1c_2 and a_3a_4 and other two similar pairs of edges are made close, and hence the images of Bl, Rd and Bk are moved to the same truncated triangle as indicated in the figure.

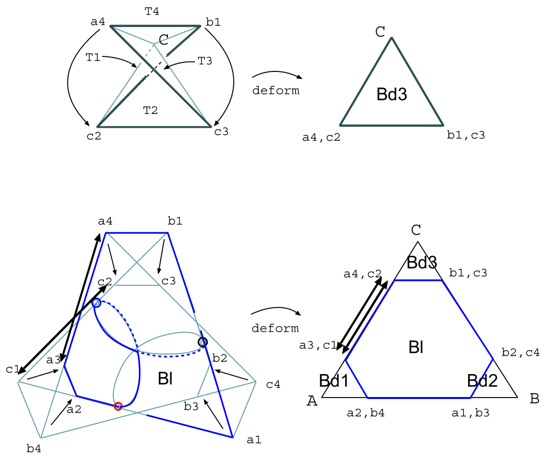

Fig. 6: Deformation of the combinatorial map; (upper) on the 4-gon neighbourhood of C, (lower) on the 6-gon Bl

The deformations on the above pieces are well pasted and extend to the 6-gon Top, so that the final map of this deformation is the map of $\mathbb{R}P^2$ onto a triangle indicated by ABC in Fig.6, lower right. It is the momentum map of the $O(1) \times O(1)$ action on $\mathbb{R}P^2$ that folds the large triangle ABC in Fig.4 (A) into a quarter triangle ABC along the three lines AB, BC and CA □

5 Putting membranes on a chain of three twisted loops

We construct the Boy surface from a cyclic chain of three loops in $\mathbb{R}^3$. Each pair of the loops meet at a point so that the tangent lines are different, and each loop is a trivial knot, that means, bounds an embedded 2-disc, but is not on any plane and makes a figure eight shape when it is orthogonally projected to the plane passing through the three intersection points of the chain, as illustrated in Fig.7. We put points a_1, a_2, b_4, b_3 on a loop, b_1, b_2, c_4, c_3 on another loop, and c_1, c_2, a_4, a_3 on the third loop, and further attach the three-bladed propeller P to this chain at certain three points p, q, r as in the figure. We may adjust P and the chain so that at the intersection points, each of the arcs a_1a_2, b_1b_2, c_1c_2 is transverse to the plane on which the blade of P lies. We denote by s the triple point of P.

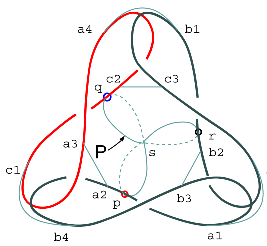

Fig. 7: Cyclic chain of three loops (bold part)

We put membranes Bl, Rd, Bk on the chain of loops as in Fig.8 so that the regular arcs qsp and rsq of P mentioned in Section 2 are on Bl, qsp and psr are on Rd and psr and rsq are on Bk. This implies that the locus of intersection among Bl, Rd, Bk is P and one can make the membranes so that the intersection of any two is transversal, as mentioned in Section 2. The intersection of Rd, Bl, Bk with the chain at p, q, r, respectively, are then transverse. We put the final membrane Top on the chain so that it is disjoint from any of Bl, Rd, Bk except the chain, as in the figure.

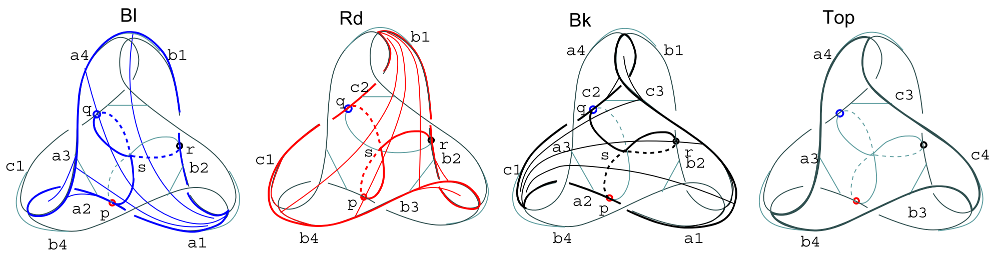

Fig. 8: The membranes Bl, Rd, Bk and Top attached to the cyclic chain

The obtained object is the same one as in Section 4 as seen in Fig.9; each component loop of the chain can be identified with the loop obtained from certain two edges of the fame used in Section 4 (for the loop $c_1c_2a_4a_3$ in Fig.9 right, for example, they are c_1c_2 and a_3a_4 in Fig.9 left) by jointing them using certain two arcs in Bd_1, Bd_2 or Bd_3 (one in Bd_1 connecting c_2 and a_4 and another in Bd_3 connecting a_3 and c_1, for the above example). The membrane Bl in this section, for example, is the union of a quarter piece of Bd_1 divided by the chain, another quarter piece of Bd_3 divided by the chain, and the 6-gon Bl in Section 4, as illustrated in Fig.9 by the regions with dotted boundary. Hence the object we have obtained by putting membranes on the chain is, after suitable smoothing, the Boy surface. We note that this chain is a cannonical divisor, which can be seen in Fig.4 (A) without so much difficulty.

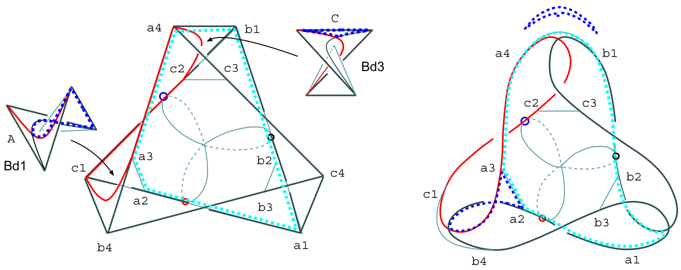

Fig. 9: Membrane Bl in the combinatorial construction and its corresponding membrane attach to the cyclic chain

Note: apparent contour of the Boy surface through an orthogonal projection We note that each Boy surface in the previous constructions, after suitably smoothed, admits an apparent contour through an orthogonal projection from $\mathbb{R}^3$ to $\mathbb{R}^2$ depicted in Fig.10.

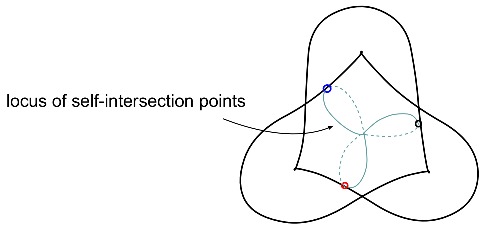

Fig. 10: Apparent contour of the constructed Boy surface in $\mathbb{R}^2$

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