A quantitative general Nullstellensatz for Jacobson rings

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Abstract

The general Nullstellensatz states that if A is a Jacobson ring, $A[X]$ is Jacobson. We introduce the notion of an α-Jacobson ring for an ordinal α and prove a quantitative version of the general Nullstellensatz: if A is an α-Jacobson ring, $A[X]$ is $(\alpha + 1)$-Jacobson. The quantitative general Nullstellensatz implies that $K[X_1, \ldots, X_n]$ is not only Jacobson but also $(1 + n)$-Jacobson for any field K. It also implies that $\mathbb{Z}[X_1, \ldots, X_n]$ is $(2 + n)$-Jacobson.

1 Introduction

In this paper, all rings are considered to be commutative with identity. This paper contains non-constructive arguments.

In classical mathematics, a ring A is called Jacobson if every prime ideal of A is an intersection of maximal ideals. One of the most important theorems about Jacobson rings is the general Nullstellensatz: a univariate polynomial ring over a Jacobson ring is Jacobson. This theorem has been independently proved by Goldman [8, Theorem 3] and Krull [11, Satz 1].

In [12, Theorem 3.9], we have extracted a constructive proof of the general Nullstellensatz from a classical proof [7, Theorem 8] and revealed the computational content of the theorem. In this paper, based on the computational content that we have obtained, we introduce the notion of an α-Jacobson ring for an ordinal α and prove a quantitative version of the general Nullstellensatz: if A is an α-Jacobson ring, $A[X]$ is $(\alpha + 1)$-Jacobson (Corollary 25).

Informally speaking, a ring A is called α-Jacobson if we can prove that A is Jacobson by an *elementary proof of complexity at most* α. If a ring A is α-Jacobson for some α, then A is Jacobson (Corollary 5). The converse does not hold (Remark 4). Every 0-dimensional ring is 1-Jacobson (Proposition 6), and $\mathbb{Z}$ is 2-Jacobson (Proposition 7).

Although the constructive general Nullstellensatz [12, Theorem 3.9] does not follow from the quantitative general Nullstellensatz (Corollary 25), the theorem provides a stronger result for typical examples. For example, we can prove that $\mathbb{Z}[X_1, \ldots, X_n]$ is not only Jacobson but also $(2 + n)$-Jacobson.

In section 4, we introduce the notion of a strongly Jacobson ring. In classical mathematics, a ring A is strongly Jacobson if and only if A is α-Jacobson for some ordinal α. In a foundation of predicative mathematics with generalized inductive definitions, the definition of a strongly Jacobson ring (Definition 26) is more acceptable than that of a Jacobson ring (Definition 2). The general Nullstellensatz for strongly Jacobson rings (Theorem 30) also holds.

2 α-Jacobson rings

We first recall the definition of a Jacobson ring.

Definition 1
Let $\langle U \rangle_A$ denote the ideal of a ring A generated by a subset $U \subseteq A$. We define two ideals $\operatorname{Nil}_A U$, $\operatorname{Jac}_A U$ of A as follows:

$$\operatorname{Nil}_A U := \{x \in A : \exists n \geq 0.\ x^n \in \langle U \rangle\},$$
$$\operatorname{Jac}_A U := \{x \in A : \forall a \in A.\ 1 \in \langle U, 1 - ax \rangle\}.$$

Here $\langle U, 1 - ax \rangle$ means $\langle U \cup \{1 - ax\} \rangle$. When the context is clear, we write $\langle U \rangle$, $\operatorname{Nil} U$, and $\operatorname{Jac} U$ for $\langle U \rangle_A$, $\operatorname{Nil}_A U$, and $\operatorname{Jac}_A U$, respectively. Note that $\operatorname{Nil} U = \operatorname{Nil} \langle U \rangle$ and $\operatorname{Jac} U = \operatorname{Jac} \langle U \rangle$ hold for all $U \subseteq A$.

Definition 2 ([17, Section 2.4.1])
We call a ring A Jacobson *if every subset U of A satisfies $\operatorname{Jac} U \subseteq \operatorname{Nil} U$.*

Example 1
Every 0-dimensional ring is Jacobson [14, Lemma IX-1.2]. The ring $\mathbb{Z}$ is Jacobson [12, Proposition 2.2]. Every algebra that is integral over a Jacobson ring is Jacobson [12, Corollary 3.8]. Every finitely generated algebra over a Jacobson ring is Jacobson [12, Corollary 3.10].

Definition 3
Let A be a ring, $x, x' \in A$, and α be an ordinal.

1. *The triple (A, x, x') is called* 0-Jacobson *if $x' \in \operatorname{Nil}_A 0$.*
2. *Let $\alpha > 0$. The triple (A, x, x') is called α-Jacobson if there exist $n \geq 0$ and $a_1, \ldots, a_n \in A$ such that for any $b_1, \ldots, b_n \in A$, there exists $\beta < \alpha$ such that $(A/\langle 1 - b_i(1 - a_i x) : i \in \{1, \ldots, n\}\rangle, x, x')$ is β-Jacobson.*

A ring A is called α-Jacobson if (A, x, x) is α-Jacobson for all $x \in A$.

Remark 1 (ZF)
We have introduced the notion of an α-Jacobson ring to measure the complexity of an elementary proof that a ring is Jacobson. In classical mathematics, a triple (A, x, x') is α-Jacobson if and only if Prover has a winning strategy for the game $J_\alpha(A, x, x')$ defined as follows:

1. *Let α be an ordinal, A be a ring, and $x \in A$. The game $J_\alpha(A, x, x')$ is played by two players called Prover and Delayer.*
2. *A possible position of the game is a pair (τ, U) of an ordinal $\tau \leq \alpha$ and a finite subset U of A.*
3. *The initial position of the game is $(\alpha, \emptyset)$.*
4. *Let (τ, U) be the current position.*
   - *If $\tau > 0$, then Prover declares a natural number $n \in \mathbb{N}$ and n elements $a_1, \ldots, a_n \in A$. Then Delayer declares n elements $b_1, \ldots, b_n \in A$. Then Prover declares an ordinal $\tau' < \tau$. The next position is (τ', U'), where $U' := U \cup \{1 - b_i(1 - a_i x) : i \in \{1, \ldots, n\}\}$.*
   - *If $\tau = 0$, then the game ends. Prover wins if $x' \in \operatorname{Nil} U$. Delayer wins if $x' \notin \operatorname{Nil} U$.*

In the game $J_\alpha(A, x, x')$, Prover is trying to give an elementary proof that $(x \in \operatorname{Jac} U) \to (x' \in \operatorname{Nil} U)$ holds for all subsets $U \subseteq A$. The idea of using the Prover-Delayer game comes from the theory of proof complexity, which the author has learned from Ken [9]. Another game related to the Noetherianity of rings is considered in [2, Section 5].

Remark 2
In constructive mathematics, we have to be careful what the term ordinal *means. See [5, 10] for the constructive theory of ordinals. We only need finite ordinals to treat the examples in this paper.*

Remark 3
Let α, α' be ordinals, A be a ring, and $x, x', y, z \in A$.

1. *If $\alpha \leq \alpha'$ and (A, x, x') is α-Jacobson, then (A, x, x') is α'-Jacobson.*
2. *If the triple (A, x, x') is α-Jacobson and I is an ideal of A, then $(A/I, x, x')$ is α-Jacobson.*
3. *If the triple (A, xy, x') is α-Jacobson, then $(A, x, x'z)$ is α-Jacobson.*
4. *The ring A is 0-Jacobson if and only if $1 =_A 0$.*

We next prove that if a ring A is α-Jacobson, then A is Jacobson.

Theorem 4
Let α be an ordinal, A be a ring, $x, x' \in A$, and $U \subseteq A$. If (A, x, x') is α-Jacobson and $x \in \operatorname{Jac} U$, then $x' \in \operatorname{Nil} U$.

Proof We prove this by induction on α.

1. If $\alpha = 0$, then $x' \in \operatorname{Nil} 0 \subseteq \operatorname{Nil} U$ holds.
2. Let $\alpha > 0$. There exist $n \geq 0$ and $a_1, \ldots, a_n \in A$ such that for all $b_1, \ldots, b_n \in A$, there exists $\beta < \alpha$ such that $(A/\langle 1 - b_i(1 - a_i x) : i \in \{1, \ldots, n\}\rangle, x, x')$ is β-Jacobson. Since $x \in \operatorname{Jac} U$, there exist $b_1, \ldots, b_n \in A$ such that $1 - b_i(1 - a_i x) \in \langle U \rangle$ for any $i \in \{1, \ldots, n\}$. Hence there exists $\beta < \alpha$ such that $(A/\langle U \rangle, x, x')$ is β-Jacobson. Hence $x' \in \operatorname{Nil}_{A/\langle U \rangle} 0$ by the inductive hypothesis. Hence $x' \in \operatorname{Nil}_A U$. ∎

Corollary 5
Let α be an ordinal. Then every α-Jacobson ring is Jacobson.

We present some examples of α-Jacobson rings.

Proposition 6 (a quantitative version of [12, Example 2.6])
Let A be a ring, $x, a \in A$, and $e \geq 0$. If $x^e(1 - ax) = 0$, then (A, x, x) is 1-Jacobson. In particular, every 0-dimensional ring A is 1-Jacobson.

Proof Let $b \in A$. Then $x^e = x^e(1 - b(1 - ax)) \in \langle 1 - b(1 - ax) \rangle$. ∎

Let $\operatorname{Reg} A$ denote the set of regular elements of a ring A.

Proposition 7 (a quantitative version of [12, Corollary 2.8])
Every ring A satisfying the following conditions is 2-Jacobson:

1. *The ring A is integral (i.e., every element is null or regular).*
2. *The Krull dimension of A is less than 2.*
3. *For any $x \in \operatorname{Reg} A$, there exists $a \in A$ such that $(1 - ax \in A^\times) \to (1 =_A 0)$.*

In particular, $\mathbb{Z}$ is 2-Jacobson, and $K[X]$ is 2-Jacobson for every discrete field K. Note that a ring K is called a discrete field if every element of K is null or invertible.

Proof It suffices to prove that (A, x, x) is 2-Jacobson for all $x \in \operatorname{Reg} A$. Let $x \in \operatorname{Reg} A$. Then there exists $a \in A$ such that $(1 - ax \in A^\times) \to (1 =_A 0)$. Let $b \in A$.

1. If $1 - b(1 - ax) = 0$, then $1 - ax \in A^\times$. Hence $1 =_A 0$, and $(A/\langle 1 - b(1 - ax) \rangle, x, x)$ is 1-Jacobson.
2. If $1 - b(1 - ax) \in \operatorname{Reg} A$, then $A/\langle 1 - b(1 - ax) \rangle$ is 0-dimensional. Hence $(A/\langle 1 - b(1 - ax) \rangle, x, x)$ is 1-Jacobson by Proposition 6.

Hence (A, x, x) is 2-Jacobson. ∎

We use the following lemmas to prove that $\mathbb{Z}$ and $A[X]$ are not 1-Jacobson, where A is a non-trivial ring:

Lemma 8 ([12, Lemma 3.4])
Let A be a ring, U be a subset of A, and $x, y \in A$. If $xy \in \mathrm{Nil}\, U$ and $x \in \mathrm{Nil}\,(U \cup \{y\})$, then $x \in \mathrm{Nil}\, U$.

Lemma 9 ([14, Lemma II-2.6])
For all rings A, we have $A[X]^\times \subseteq A^\times + (\mathrm{Nil}_A\, 0)[X]$.

Proposition 10
If N is an integer such that $|N| \geq 2$, the triple $(\mathbb{Z}, N, N)$ is not 1-Jacobson. In particular, $\mathbb{Z}$ is not 1-Jacobson.

Proof Assume that $(\mathbb{Z}, N, N)$ is 1-Jacobson. Then there exist $a_1, \ldots, a_n \in \mathbb{Z}$ such that for all $b_1, \ldots, b_n \in \mathbb{Z}$, we have $N \in \mathrm{Nil}\langle 1 - b_1(1 - a_1 N), \ldots, 1 - b_n(1 - a_n N)\rangle$. Let $c := 1 + |N(1 - a_1 N) \cdots (1 - a_n N)|$. Then there exist $b_1, \ldots, b_n \in \mathbb{Z}$ such that $1 - b_i(1 - a_i N) = c$ for all $i \in \{1, \ldots, n\}$. Hence $N \in \mathrm{Nil}\, c$. Hence $1 \in \mathrm{Nil}\, c$ by $1 \in \langle c, N\rangle$ and Lemma 8. This contradicts the fact that $c \geq 2$. ∎

Proposition 11
Let A be a ring. If $(A[X], X, X)$ is 1-Jacobson, then $1 =_A 0$.

Proof Since $(A[X], X, X)$ is 1-Jacobson, there exist $f_1, \ldots, f_n \in A[X]$ such that for all $g_1, \ldots, g_n \in A[X]$, we have $X \in \mathrm{Nil}\langle 1 - g_1(1 - f_1 X), \ldots, 1 - g_n(1 - f_n X)\rangle$. Let $h := 1 - X(1 - f_1 X) \cdots (1 - f_n X)$. Then there exist $g_1, \ldots, g_n \in \mathbb{Z}$ such that $1 - g_i(1 - f_i X) = h$ for all $i \in \{1, \ldots, n\}$. Hence $X \in \mathrm{Nil}\, h$. Hence $1 \in \mathrm{Nil}\, h$ by $1 \in \langle X, h\rangle$ and Lemma 8. Hence $h \in A[X]^\times \subseteq A^\times + (\mathrm{Nil}_A\, 0)[X]$ by Lemma 9. Hence $1 =_A 0$. ∎

We next provide a Jacobson ring that is not α-Jacobson for any α. For a ring A, let $A[X_k : k \in \mathbb{N}]$ denote the polynomial ring in countably infinitely many variables over A.

Lemma 12
Let K be a ring, $A := K[X_k : k \in \mathbb{N}]$, α be an ordinal, $n \geq 0$, and $f_1, \ldots, f_n \in A$. Let $l : \{1, \ldots, n\} \to \mathbb{N}_{\geq 1}$ be a function. Assume $f_i \in K[X_0, \ldots, X_{l(i)-1}]$ and $\forall j < i.\ l(j) < l(i)$ for all $i \in \{1, \ldots, n\}$. Let $U := \{1 - X_{l(i)}(1 - f_i X_0) : i \in \{1, \ldots, n\}\}$. If $(A/\langle U\rangle, X_0, X_0)$ is α-Jacobson, then $1 =_K 0$.

Proof We prove this by induction on α.

1. If $(A/\langle U\rangle, X_0, X_0)$ is 0-Jacobson, then $X_0 \in \mathrm{Nil}_{A/\langle U\rangle}\, 0$. Let $S := 1 + XK[X] \subseteq K[X]$ and $B := S^{-1}(K[X])$. We define a K-homomorphism $\varphi : A/\langle U\rangle \to B$ by the following equation:

$$\varphi(X_k) := \begin{cases} X & \text{if } \forall i \in \{1, \ldots, n\}.\ k \neq l(i), \\ (1 - \varphi(f_i)X)^{-1} & \text{if } k = l(i). \end{cases}$$

   Then $X = \varphi(X_0) \in \mathrm{Nil}_B\, 0$. Since $K[X] \subseteq B$, we have $1 =_K 0$.

2. Let $\alpha > 0$. Since $(A/\langle U\rangle, X_0, X_0)$ is α-Jacobson, there exist $m \geq n$ and $f_{n+1}, \ldots, f_m \in A$ such that for all $g_{n+1}, \ldots, g_m \in A$, there exists $\beta < \alpha$ such that

$$(A/\langle U \cup \{1 - g_i(1 - f_i X_0) : i \in \{n+1, \ldots, m\}\}\rangle, X_0, X_0)$$

   is β-Jacobson. There exist $l(n+1), \ldots, l(m) \in \mathbb{N}$ such that $f_i \in K[X_0, \ldots, X_{l(i)-1}]$ and $\forall j < i.\ l(j) < l(i)$ hold for every $i \in \{1, \ldots, m\}$. Hence there exists $\beta < \alpha$ such that

$$(A/\langle 1 - X_{l(i)}(1 - f_i X_0) : i \in \{1, \ldots, m\}\rangle, X_0, X_0)$$

   is β-Jacobson. Hence $1 =_K 0$ by the inductive hypothesis. ∎

Corollary 13
Let K be a ring. If $K[X_k : k \in \mathbb{N}]$ is α-Jacobson for some α, then $1 =_K 0$.

Proof Let $n := 0$ in Lemma 12. ∎

Remark 4 (ZFC)
Krull [11, Satz 4] has proved that if K is an uncountable field, then $K[X_k : k \in \mathbb{N}]$ is Jacobson. This also follows from Amitsur's theorem [1, Corollary 3]. Hence $\mathbb{R}[X_k : k \in \mathbb{N}]$ is Jacobson, but it is not α-Jacobson for any α by Corollary 13.

3 The quantitative general Nullstellensatz

In this section, we prove the quantitative general Nullstellensatz: if A is an α-Jacobson ring, $A[X]$ is $(\alpha+1)$-Jacobson. The proof is similar to the constructive proof of the general Nullstellensatz in [12], but we need more exact lemmas.

Lemma 14
Let A be a ring and $x, y, z \in A$. If (A, y, xz) and $(A/\langle x \rangle, y, z)$ are α-Jacobson, then (A, y, z) is α-Jacobson.

Proof We prove this by induction on α.

1. If $\alpha = 0$, then $xz \in \operatorname{Nil}_A 0$ and $z \in \operatorname{Nil}_A x$. Hence $z \in \operatorname{Nil}_A 0$ by Lemma 8.
2. Let $\alpha > 0$. Then there exist $m, n \geq 0$, $a_1, \ldots, a_m \in A$, and $a'_1, \ldots, a'_n \in A$ such that for all $b_1, \ldots, b_m \in A$ and $b'_1, \ldots, b'_n \in A$, there exist ordinals β, β' such that
   - $(A/\langle 1 - b_i(1 - a_i y) : i \in \{1, \ldots, m\}\rangle, y, xz)$ is β-Jacobson, and
   - $(A/(\langle x \rangle + \langle 1 - b'_i(1 - a'_i y) : i \in \{1, \ldots, n\}\rangle), y, z)$ is β'-Jacobson.

   Note that the above two items together imply that

   - $(A/(\langle 1 - b_i(1 - a_i y) : i \in \{1, \ldots, m\}\rangle + \langle 1 - b'_i(1 - a'_i y) : i \in \{1, \ldots, n\}\rangle), y, z)$ is γ-Jacobson, where $\gamma := \max(\beta, \beta')$.

   by the inductive hypothesis. Hence (A, y, z) is α-Jacobson. ∎

Lemma 15
Let A be a ring, B be an A-algebra which is integral over A, and $x \in A$. Then

$$\forall b \in B.\ \exists a \in A.\ 1 - ax \in \langle 1 - bx \rangle_B.$$

Proof Let $b \in B$. Since $C := B/\langle 1 - bx \rangle$ is integral over A, and $x \in C^\times$, there exists $a \in A$ such that $ax =_C 1$ by [14, Theorem IX-1.7]. Hence we have $1 - ax \in \langle 1 - bx \rangle_B$. ∎

Lemma 16
Let A be a ring, $a, a_1, a'_2 \in A$, and $e \geq 0$. Then there exists $a_2 \in A$ such that

$$1 - a_2(1 - a_1 a) \in \langle a^e - a'_2(1 - a_1 a) \rangle_A.$$

Proof Let $a_2 := (1 + \cdots + (a_1 a)^{e-1}) + a_1^e a'_2$. Then,

$$\begin{aligned} 1 - a_2(1 - a_1 a) &= 1 - ((1 + \cdots + (a_1 a)^{e-1}) + a_1^e a'_2)(1 - a_1 a) \\ &= (a_1 a)^e - a_1^e a'_2(1 - a_1 a) \\ &= a_1^e(a^e - a'_2(1 - a_1 a)). \end{aligned}$$

Hence $1 - a_2(1 - a_1 a) \in \langle a^e - a'_2(1 - a_1 a) \rangle_A$. ∎

For a ring A and an element $a \in A$, let A_a denote the ring $A[1/a]$.

Lemma 17
Let A be a ring, $a, a_0 \in A$, B be an A-algebra such that B_a is integral over A_a. Then

$$\forall a_1 \in A.\ \forall b_2 \in B.\ \exists a_2 \in A.\ 1 - a_2(1 - a_1 a a_0) \in \langle 1 - b_2(1 - a_1 a a_0)\rangle_B.$$

Proof Let $a_1 \in A$ and $b_2 \in B$. By Lemma 15, there exists $a_2' \in A_a$ such that

$$1 - a_2'(1 - a_1 a a_0) \in \langle 1 - b_2(1 - a_1 a a_0)\rangle_{B_a}.$$

There exists $a_2'' \in A$ and $e \geq 0$ such that $a_2' = a_2''/a^e$. Then there exists $e' \geq 0$ such that

$$a^{e'}(a^e - a_2''(1 - a_1 a a_0)) \in \langle 1 - b_2(1 - a_1 a a_0)\rangle_B.$$

By Lemma 16, there exists $a_2 \in A$ such that

$$1 - a_2(1 - a_1 a a_0) \in \langle a^{e+e'} - a^{e'} a_2''(1 - a_1 a a_0)\rangle_A.$$

Hence

$$1 - a_2(1 - a_1 a a_0) \in \langle 1 - b_2(1 - a_1 a a_0)\rangle_B$$

holds. ∎

The following lemma follows from the definition:

Lemma 18
Let (A, x, x') be an α-Jacobson triple, B be an A-algebra, and $y \in B$. If

$$\forall a_1 \in A.\ \exists b_1 \in B.\ \forall b_2 \in B.\ \exists a_2 \in A.\ 1 - a_2(1 - a_1 x) \in \langle 1 - b_2(1 - b_1 y)\rangle_B,$$

then there exist $n \geq 0$, $a_{1,1}, \ldots, a_{1,n} \in A$, and $b_{1,1}, \ldots, b_{1,n} \in B$ such that for all $b_{2,1}, \ldots, b_{2,n} \in B$, there exists $a_{2,1}, \ldots, a_{2,n} \in A$ such that

- *there exists $\beta < \alpha$ such that $(A/\langle 1 - a_{2,i}(1 - a_{1,i}x) : i \in \{1, \ldots, n\}\rangle, x, x')$ is β-Jacobson, and*
- *$1 - a_{2,i}(1 - a_{1,i}x) \in \langle 1 - b_{2,i}(1 - b_{1,i}y)\rangle_B$ holds for all $i \in \{1, \ldots, n\}$.*

Lemma 19
Let α be an ordinal, A be a ring, $a \in A$, B be an A-algebra such that B_a is integral over A_a, and $y \in B$. If there exist $d \geq 0$, $c_0, \ldots, c_{d-1} \in A$, and $l \geq 0$ such that $a^l y^d =_B c_{d-1}y^{d-1} + \cdots + c_0$ and (A, ac_k, ac_k) is α-Jacobson for all $k \in \{0, \ldots, d-1\}$, then (B, y, ay) is α-Jacobson.

Proof We prove this by induction on α.

1. If $\alpha = 0$, then $ac_0, \ldots, ac_{d-1} \in \mathrm{Nil}_A\, 0$. Hence $ay \in \mathrm{Nil}_B\, 0$, and (B, y, ay) is 0-Jacobson.
2. Let $\alpha > 0$. Let $f_k := a^l y^k - (c_{d-1}y^{k-1} + \cdots + c_{d-k})$ and $B_k := B/\langle f_k, \ldots, f_d\rangle$ for $k \in \{0, \ldots, d\}$. Note that $f_d =_B 0$. We prove that (B_k, y, ay) is α-Jacobson for all k by induction.
   (a) Since $a^l = f_0 =_{B_0} 0$, the triple (B_0, y, ay) is 0-Jacobson.
   (b) Let $k \geq 1$.
   - By Lemma 17 and Lemma 18, there exist $n \geq 0$, $a_{1,1}, \ldots, a_{1,n} \in A$, and $b_{1,1}, \ldots, b_{1,n} \in B_k$ such that for all $b_{2,1}, \ldots, b_{2,n} \in B_k$, there exist $a_{2,1}, \ldots, a_{2,n} \in A$ such that
     – there exists $\beta < \alpha$ such that

     $$(A/\langle 1 - a_{2,i}(1 - a_{1,i}ac_{d-k}) : i \in \{1, \ldots, n\}\rangle, ac_{d-k}, ac_{d-k})$$

     is β-Jacobson, and
     – $1 - a_{2,i}(1 - a_{1,i}ac_{d-k}) \in \langle 1 - b_{2,i}(1 - b_{1,i}ac_{d-k})\rangle_B$ holds for all $i \in \{1, \ldots, n\}$.

     The above two items together imply that

– there exists $\beta < \alpha$ such that

$$(B_k/\langle 1 - b_{2,i}(1 - (b_{1,i}a)c_{d-k}) : i \in \{1, \dots, n\}\rangle, c_{d-k}, ac_{d-k})$$

is β-Jacobson

by the inductive hypothesis. Hence (B_k, c_{d-k}, ac_{d-k}) is α-Jacobson. Since $c_{d-k} = f_{k-1}y - f_k =_{B_k} f_{k-1}y$, the triple $(B_k, y, af_{k-1}y)$ is α-Jacobson by Remark 3-3.

- The triple (B_{k-1}, y, ay) is α-Jacobson by the inductive hypothesis.

Hence (B_k, y, ay) is α-Jacobson by Lemma 14.

Hence (B_d, y, ay) is α-Jacobson. ∎

The above lemma implies the following theorem:

Theorem 20 (a quantitative version of [12, Lemma 3.6])
Let α be an ordinal, A be an α-Jacobson ring, $a \in A$, B be an A-algebra such that B_a is integral over A_a, and $y \in B$. Then (B, y, ay) is α-Jacobson.

Corollary 21 (a quantitative version of [12, Corollary 3.7])
Let α be an ordinal, A be an α-Jacobson ring, and $a \in A$. Then A_a is α-Jacobson.

Corollary 22 (a quantitative version of [12, Corollary 3.8])
Let α be an ordinal, A be an α-Jacobson ring. Then every A-algebra B that is integral over A is α-Jacobson.

Using Lemma 19, we prove the quantitative general Nullstellensatz, which is a quantitative version of [12, Theorem 3.9]. We first recall the following lemma:

Lemma 23 ([15, Corollary VI-1.3])
Let B be an A-algebra. If $b \in B$ is integral over A, then the algebra $A[b] \subseteq B$ is integral over A.

Theorem 24
Let α be an ordinal, A be an α-Jacobson ring, and $f, g \in A[X]$. Then

$$(A[X]/\langle 1 - g(1 - Xf)\rangle, f, f)$$

is α-Jacobson.

Proof Let $h := 1 - g(1 - Xf)$. There exist $d \geq 0$ and $a_0, \dots, a_d \in A$ such that $h = a_d X^d + \cdots + a_0$. Let $C_k := A[X]/\langle h, a_{k+1}, \dots, a_d\rangle$ for $k \in \{-1, \dots, d\}$. We prove that (C_k, f, f) is α-Jacobson by induction on k.

1. Since $h =_{C_{-1}} 0$, we have $1 - Xf \in C_{-1}^{\times}$. Since $C_{-1} = (A/\langle a_0, \dots, a_d\rangle)[X]$, we have $f \in \mathrm{Nil}_{C_{-1}} 0$. Hence (C_{-1}, f, f) is 0-Jacobson.

2. Let $k \geq 0$.

   - The A_{a_k}-algebra $(C_k)_{a_k}$ is integral over A_{a_k} by Lemma 23. Hence the triple $(C_k, f, a_k f)$ is α-Jacobson by Lemma 19.
   - The triple (C_{k-1}, f, f) is α-Jacobson by the inductive hypothesis.

   Hence (C_k, f, f) is α-Jacobson by Lemma 14.

Hence (C_d, f, f) is α-Jacobson. ∎

Corollary 25 (The quantitative general Nullstellensatz)
If A is an α-Jacobson ring, then $A[X]$ is $(\alpha + 1)$-Jacobson.

In [12], we have proved that $K[X_1, \dots, X_n]$ is Jacobson for any discrete field K, and that $\mathbb{Z}[X_1, \dots, X_n]$ is Jacobson. These results provide a solution to the first two problems of Lombardi's list [13]. By Proposition 6, Proposition 7, and Corollary 25, we have the following stronger results: $K[X_1, \dots, X_n]$ is $(1 + n)$-Jacobson for any discrete field K, and $\mathbb{Z}[X_1, \dots, X_n]$ is $(2 + n)$-Jacobson.

4 Strongly Jacobson rings

In this section, we introduce the notion of a strongly Jacobson ring.

Definition 26
Let A be a ring and $x, x' \in A$. The proposition that "The triple (A, x, x') is strongly Jacobson*" is inductively generated by the following constructors:*

1. *If $x' \in \mathrm{Nil}_A\, 0$, then (A, x, x') is strongly Jacobson.*
2. *If there exist $n \geq 0$ and $a_{1,1}, \ldots, a_{1,n} \in A$ such that for any $a_{2,1}, \ldots, a_{2,n} \in A$, the triple*

$$(A/\langle 1 - a_{2,i}(1 - a_{1,i}x) : i \in \{1, \ldots, n\}\rangle, x, x')$$

   is strongly Jacobson, then (A, x, x') is strongly Jacobson.

A ring A is called strongly Jacobson if (A, x, x) is strongly Jacobson for all $x \in A$.

Remark 5
Definition 26 is a generalized inductive definition. We have simultaneously defined the proposition that "The triple $(A/I, x, x')$ is strongly Jacobson" for all finitely generated ideals I of A. Generalized inductive definitions are accepted in some foundations of predicative mathematics. They are also used to define a well-founded relation [16, Section 10.3] and a Noetherian ring [6, 4, 3].

The following theorem easily follows from the definition.

Theorem 27
Let A be an α-Jacobson ring for some α. Then A is strongly Jacobson.

Theorem 28 (ZF)
Let (A, x, x') be a strongly Jacobson triple. Then there exists an ordinal α such that (A, x, x') is α-Jacobson.

Proof We prove this by induction on the proof of the proposition that "The triple (A, x, x') is strongly Jacobson."

1. If $x' \in \mathrm{Nil}_A\, 0$, then (A, x, x') is 0-Jacobson.
2. Let there exist $n \geq 0$ and $a_{1,1}, \ldots, a_{1,n} \in A$ such that for any $a_{2,1}, \ldots, a_{2,n} \in A$, the triple

$$(A/\langle 1 - a_{2,i}(1 - a_{1,i}x) : i \in \{1, \ldots, n\}\rangle, x, x')$$

   is strongly Jacobson. By the inductive hypothesis, for any $a_{2,1}, \ldots, a_{2,n} \in A$, there exists the minimum ordinal $\beta_{a_{2,1},\ldots,a_{2,n}}$ such that $(A/\langle 1 - a_{2,i}(1 - a_{1,i}x) : i \in \{1, \ldots, n\}\rangle, x, x')$ is $(\beta_{a_{2,1},\ldots,a_{2,n}})$-Jacobson. Then A is $(\sup_{a_{2,1},\ldots,a_{2,n}} \beta_{a_{2,1},\ldots,a_{2,n}} + 1)$-Jacobson. ∎

We can prove the following theorem by an argument similar to Corollary 5.

Theorem 29
Every strongly Jacobson ring is Jacobson.

The converse does not hold by Remark 4. We can prove the following theorem by an argument similar to Corollary 25.

Theorem 30 (The general Nullstellensatz for strongly Jacobson rings)
If A is a strongly Jacobson ring, then $A[X]$ is strongly Jacobson.

5 Future work

We present three problems on α-Jacobson rings and strongly Jacobson rings.

Problem 1
Let K be a field. What is the minimum ordinal α such that $K[X_1, \ldots, X_n]$ is α-Jacobson?

Corollary 25 implies that $K[X_1, \ldots, X_n]$ is $(1+n)$-Jacobson, and Proposition 11 implies that if $K[X]$ is 0-Jacobson, then $1 =_K 0$.

Problem 2
What is the minimum ordinal α such that $\mathbb{Z}[X_1, \ldots, X_n]$ is α-Jacobson?

Corollary 25 implies that $\mathbb{Z}[X_1, \ldots, X_n]$ is $(2+n)$-Jacobson, and Proposition 10 implies that $\mathbb{Z}$ is not 1-Jacobson.

Problem 3
Is there a Noetherian Jacobson ring that is not strongly Jacobson?

The ring $\mathbb{R}[X_k : k \in \mathbb{N}]$ is a non-Noetherian Jacobson ring that is not strongly Jacobson by Remark 4.

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