An extension of Chapple's formula by Blaschke-like maps: the case of parabolas

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Abstract

We define the Blaschke-like map on a domain with a parabola boundary using conformal deformation and study the geometric properties of the maps. In particular, we give an extension of Chapple's theorem and show each Poncelet's triangle inscribed in a parabola and circumscribed about an ellipse constructed from a Blaschke-like map.

1 Introduction

Every triangle is bicentric, i.e., it has both an inscribed circle and a circumscribed circle. The distance d between the circumcenter and incenter of a triangle is given by $d^2 = R(R-2r)$, where R and r are the circumradius and inradius, respectively. In particular, if the circumscribed circle is the unit circle, then the distance is given by $d^2 = 1-2r$. This formula is known as Chapple's formula [Cha46].

This Chapple's formula gives no information about the location of the triangle. But the following Poncelet's theorem [Pon66] guarantees that any point on the outer circle can be a vertex of an inscribed triangle. See [Fla08] for details about Poncelet's theorem.

Theorem 1 (Poncelet [Pon66])
Let E_1 and E_2 be two conics. If there exists an n-sided polygon inscribed in E_1 and simultaneously circumscribed about E_2, then for any point P_0 of E_1, there exists an n-sided polygon with P_0 as a vertex, inscribed in E_1 and circumscribed about E_2.

The n-sided polygon that satisfies the above conditions is called *Poncelet n-polygon with respect to E_1 and E_2*. The above Chapple's formula and Poncelet's theorem are the subject of algebraic geometry. Here we approach these problems analytically.

A *Blaschke product* of degree d is a rational function defined by

$$B(z) = e^{i\theta} \prod_{k=1}^{d} \frac{z-a_k}{1-\overline{a_k}z} \qquad (a_k \in \mathbb{D},\ \theta \in \mathbb{R}).$$

In the case that $\theta = 0$ and $B(0) = 0$, B is called *canonical*.

For a Blaschke product of degree d, set

$$f_1(z) = e^{-\frac{\theta}{d}i} z, \quad \text{and} \quad f_2(z) = \frac{z-(-1)^d a_1 \cdots a_d e^{i\theta}}{1-(-1)^d \overline{a_1 \cdots a_d e^{i\theta}} z}.$$

Then, the composition $f_2 \circ B \circ f_1$ is a canonical, and geometric properties with respect to the preimages of B and $f_2 \circ B \circ f_1$ are the same. So, we will only consider canonical Blaschke products in the following discussions.

Remark that there are d distinct preimages $z_1, \cdots, z_d$ of $\lambda \in \partial\mathbb{D}$ by B because the derivative of B has no zeros on $\partial\mathbb{D}$ (see, for example, [Mas13]).

Let $z_1, \cdots, z_d$ be the d distinct preimages of $\lambda \in \partial\mathbb{D}$ by B, and ℓ_λ the set of lines joining z_j and z_k ($j \neq k$). Here, we consider the envelope I_B of the family of lines $\{\ell_\lambda\}_{\lambda\in\mathbb{D}}$. We call the envelope I_B the *interior curve associated with* B.

The interior curve associated with a Blaschke product of degree 3 forms an ellipse, and corresponds to the inner ellipse of Poncelet's theorem. See also [Fuj13] for the case of degree 4.

Theorem 2 (Daepp, Gorkin, and Mortini [DGM02])
Let $B(z) = z\dfrac{z-a}{1-\overline{a}z}\dfrac{z-b}{1-\overline{b}z}$ be a canonical Blaschke product of degree 3. For $\lambda \in \partial\mathbb{D}$, let z_1, z_2, and z_3 denote the points mapped to λ under B. Then the lines joining z_j and z_k for $j \neq k$ are tangent to the ellipse E with equation

$$|z-a| + |z-b| = |1-\overline{a}b|.$$

The following result guarantees that any 3-inscribed ellipse, i.e., Poncelet's inner ellipse of a triangle, in $\partial\mathbb{D}$ can be constructed from a Blaschke product of degree 3.

Theorem 3 (Frantz [Fra04])
For the case of a triangle inscribed in $\partial\mathbb{D}$, the ellipse E is a Poncelet's inner ellipse if and only if E is the interior curve for some Blaschke product of degree three.

The two theorems above allow the result of Chapple's formula to extend the inner circle to the inner ellipse. If we could extend the outer circle to a conic in the above theorems, would we obtain a result more similar to Poncelet's theorem? In [Fuj23], we extended the outer circle to an ellipse. The main aim of this report is to extend the outer circle to a parabola.

2 Blaschke-like maps on a domain with parabola boundary

Let ψ_t be

$$z = \psi_t(w) = \Big(\frac{1-w}{1+w} + t\Big)^2 - t^2 \quad (t > 0),$$

and $\mathbb{P}_t = \{z \in \mathbb{C}\,;\, z = x+iy,\ y^2 + 4t^2x > 0\}$. Then, ψ_t conformally maps $\mathbb{D}$ onto $\mathbb{P}_t$ and continuously maps $\overline{\mathbb{D}}$ to $\overline{\mathbb{P}}_t$ (see Fig. 1).

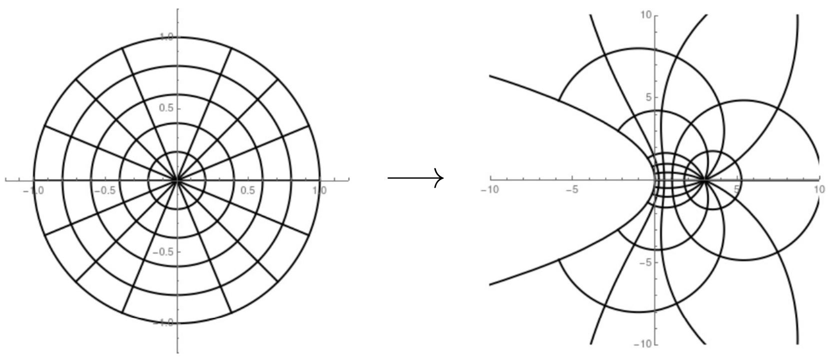

Figure 1: The map ψ_t conformally maps $\mathbb{D}$ onto $\overline{\mathbb{P}}_t$.

Any two parabolas are similar to each other. So, without loss of generality, we can assume $t = 1$. Let $\psi(w) = \psi_1(w) = \dfrac{(3+w)(1-w)}{(1+w)^2}$ and $\mathbb{P} = \mathbb{P}_1 = \{z \in \mathbb{C}\,;\, z = x+iy,\ y^2 + 4x > 0\}$.

For a canonical Blaschke product B, let $B_\psi = \psi \circ B \circ \psi^{-1}$.

$$\begin{array}{ccc} \mathbb{D} & \xrightarrow{B} & \mathbb{D} \\ {\scriptstyle\psi}\downarrow & & \downarrow{\scriptstyle\psi} \\ \mathbb{P} & \xrightarrow[B_\psi]{} & \mathbb{P} \end{array}$$

Since

$$\psi\big(-(2+w)\big) = \frac{(3-(2+w))(1+(2+w))}{(1-(2+w))^2} = \psi(w)$$

holds, the map ψ conformally maps $\{|w+2|<1\}$ onto $\mathbb{P}$ as well as conformally maps $\mathbb{D}$ onto $\mathbb{P}$. So, for each $z \in \mathbb{P}$ we can choose a unique branch ψ^{-1} that satisfies $|w|<1$. Hence, B_ψ is well-defined and maps $\mathbb{P}$ onto itself. We call B_ψ a *Blaschke-like map associated with B and ψ.*

For Blaschke-like map associated with a canonical Blaschke product of degree 3, we have the following results (see Fig.2 and [FG24]).

Theorem 4
Let B_ψ be a Blaschke-like map associated with a Blaschke product B of degree 3 and ψ. Then, the interior curve with respect to B_ψ is an ellipse.

Proof Using Risa/Asir, a symbolic computation system, a defining equation of the interior curve is given by

$$g_I^\psi(z) = \overline{U}z^2 + Pz\overline{z} + U\overline{z}^2 + \overline{V}z + V\overline{z} + Q = 0,$$

where

$$\begin{aligned}
U &= (|ab|^2-1)^2 - 2(a+b+2)(\overline{a}+\overline{b})(|ab|^2+1) + 4(a+b)(\overline{a}^2+\overline{b}^2) + 4(\overline{a}-\overline{b})^2 + |a+b|^4, \\
P &= 2\big((|ab|^2-|a+b|^2)^2 - 2(a+b+\overline{a}+\overline{b}+1)(|ab|^2-|a+b|^2+1) \\
&\quad - 4ab(\overline{a}+\overline{b}+\overline{a}\overline{b}) - 4\overline{a}\overline{b}(a+b+ab) - 5\big), \\
V &= -4\big((|ab|^2-|a+b|^2+1)^2 + (|ab|^2-1)(a+b-(\overline{a}+\overline{b})) \\
&\quad - (\overline{a}+\overline{b})((a+1)^2+(b+1)^2) + (a+b-2)(\overline{a}^2+\overline{b}^2+2) + 4(\overline{a}\overline{b}+2)\big), \\
Q &= 4\big((|ab|^2-|a+b|^2)^2 + 2(|ab|^2+1)(a+b+\overline{a}+\overline{b}-3) \\
&\quad - 2(\overline{a}+\overline{b})(a^2+b^2) - 2(a+b)(\overline{a}^2+\overline{b}^2) + (a-b)^2 + (\overline{a}-\overline{b})^2 + 3\big).
\end{aligned}$$

Here, P and Q are written as

$$\begin{aligned}
P &= 2\Big((|ab|^2-|a+b|^2)^2 - 2(2\mathrm{Re}(a+b)+1)(|ab|^2-|a+b|^2+1) \\
&\quad - 8\mathrm{Re}\big(ab(\overline{a}+\overline{b}+\overline{a}\overline{b})\big) - 5\Big), \\
Q &= 4\big((|ab|^2-|a+b|^2)^2 + 2(|ab|^2+1)(2\mathrm{Re}(a+b)-3) - 4\mathrm{Re}\big((\overline{a}+\overline{b})(a^2+b^2)\big) \\
&\quad + 2\mathrm{Re}\big((a-b)^2\big) + 3\big).
\end{aligned}$$

Hence, $P, Q \in \mathbb{R}$, and $g_I^\psi(z)=0$ gives a defining equation of a conic.

Next, we need to check $P^2 - 4U\overline{U} > 0$ in order to verify that $g_I^\psi(z)=0$ gives a defining equation of an ellipse. We have,

$$\begin{aligned}
&P^2 - 4U\overline{U} \\
&\quad = -64|1-\overline{a}\overline{b}b|^2(1-|a|^2)(1-|b|^2)\Big((1-|a|^2)(1-|b|^2) - 4\big(1-\mathrm{Re}(a)\big)\big(1-\mathrm{Re}(b)\big)\Big).
\end{aligned}$$

The last factor of the above equality is written as

$$\begin{aligned}&(1-|a|^2)(1-|b|^2)-4\big(1-\mathrm{Re}(a)\big)\big(1-\mathrm{Re}(b)\big)\\ &\quad<\big(1-\mathrm{Re}(a)^2\big)\big(1-\mathrm{Re}(b)^2\big)-4\big(1-\mathrm{Re}(a)\big)\big(1-\mathrm{Re}(b)\big)\\ &\quad=\big(1-\mathrm{Re}(a)\big)\big(1-\mathrm{Re}(b)\big)\Big(\big(1+\mathrm{Re}(a)\big)\big(1+\mathrm{Re}(b)\big)-4\Big).\end{aligned}$$

Since $-1<\mathrm{Re}(a),\ \mathrm{Re}(b)<1$, the last factor is negative and $(1-|a|^2)(1-|b|^2)-4\big(1-\mathrm{Re}(a)\big)\big(1-\mathrm{Re}(b)\big)<0$ holds. Therefore, $P^2-4U\overline{U}>0$ and the conic defined by $g_I^{\psi}(z)=0$ is an ellipse or its degenerate. ∎

Remark 1
On the unit circle $\partial\mathbb{D}$, ψ is written as

$$\Psi(w):=\psi|_{\partial\mathbb{D}}(w)=\frac{-w+3\overline{w}-2}{w+\overline{w}+2},$$

that is,

$$\begin{array}{cccc}\Psi: & \partial\mathbb{D} & \to & \partial\mathbb{P}\\ & \cup & & \cup\\ & u+iv & \mapsto & \frac{u-1}{u+1}-i\frac{2v}{u+1}.\end{array}$$

The inverse is given by

$$\Psi^{-1}(z)=\psi^{-1}|_{\partial\mathbb{P}}(z)=\frac{-2\overline{z}-2}{z+\overline{z}-2},$$

that is,

$$\begin{array}{cccc}\Psi^{-1}: & \partial\mathbb{P} & \to & \partial\mathbb{D}\\ & \cup & & \cup\\ & x+iy & \mapsto & -\frac{x+1}{x-1}+i\frac{2y}{x-1}.\end{array}$$

Note that Ψ maps the unit circle to the parabola $\partial\mathbb{P}$, and Ψ^{-1} maps $\partial\mathbb{P}$ to $\partial\mathbb{D}$.

Let $g_I(w)=0$ be a defining equation of the ellipse $|w-a|+|w-b|=|1-a\overline{b}|$, i.e., the interior curve of $B(z)=z\dfrac{z-a}{1-\overline{a}z}\dfrac{z-b}{1-\overline{b}z}$. Then, from the calculation of substitutions, we can verify the following

$$(z+\overline{z}-2)^2 g_I\circ\Psi^{-1}(z)=g_I^{\psi}(z),$$

and

$$\frac{1}{64}(w+\overline{w}+2)^2 g_I^{\psi}\circ\Psi(w)=g_I(w).$$

The map Ψ has the following properties.

Lemma 5
Ψ is a map which corresponds a line on the z-plane to a line on the w-plane.

Proof Consider a line on the z-plane $L(z)=\overline{\alpha}z+\alpha\overline{z}+\beta=0\quad(\beta\in\mathbb{R})$.
Then,

$$L(\Psi(w))=\frac{1}{w+\overline{w}+2}\Big((3\alpha-\overline{\alpha}+\beta)w+(3\overline{\alpha}-\alpha+\beta)\overline{w}-4\mathrm{Re}(\alpha)+2\beta\Big)=0\qquad(w\neq-1).$$

The factor

$$(3\alpha-\overline{\alpha}+\beta)w+(3\overline{\alpha}-\alpha+\beta)\overline{w}-4\mathrm{Re}(\alpha)+2\beta=0$$

represents a line on the w-plane. ∎

Lemma 6
Ψ is a map which corresponds an ellipse on the z-plane to an ellipse on the w-plane.

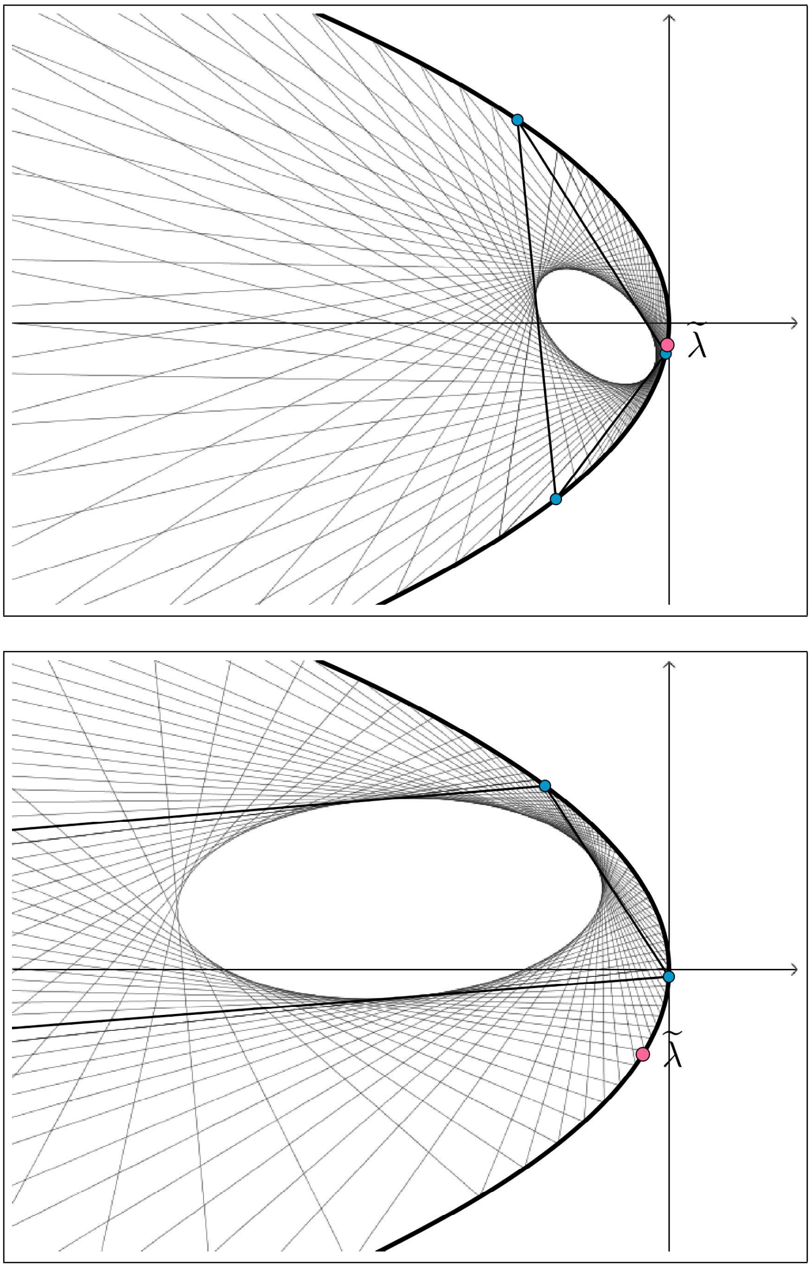

Figure 2: The envelopes indicate the interior curves of the Blaschke-like maps associated with the canonical Blaschke products with zeros $0,\ \frac{1}{2}+\frac{1}{2}i,\ -0.3-0.2i$ and ψ (upper) and $0,\ -0.7,\ -0.2-0.7i$ and ψ (lower), where $\psi(w)=\dfrac{(3+w)(1-w)}{(1+w)^2}$.

Proof Consider an ellipse $\widetilde{E}$ in the z-plane that is contained in the $\mathbb{C} \setminus \overline{\mathbb{P}}$ and is given by the equation $|z-c|+|z-d|=r$. Note that $c, d \in \mathbb{C} \setminus \overline{\mathbb{P}}$. This ellipse has following general form,

$$\begin{aligned}\widetilde{E}(z) &= ((z-c)(\overline{z}-\overline{c})+(z-d)(\overline{z}-\overline{d})-r^2)^2-4(z-c)(\overline{z}-\overline{c})(z-d)(\overline{z}-\overline{d})\\ &= (\overline{c}-\overline{d})^2z^2+2(|c-d|^2-2r^2)z\overline{z}+(c-d)^2\overline{z}^2\\ &\quad -2\big((\overline{c}-\overline{d})(|c|^2-|d|^2)-(\overline{c}+\overline{d})r^2\big)z-2\big((c-d)(|c|^2-|d|^2)-(c+d)r^2\big)\overline{z}\\ &\quad +(|c|^2-|d|^2)^2-2(|c|^2+|d|^2)r^2+r^4=0.\end{aligned}$$

Then, we have $\widetilde{E}(\Psi(w)) = \dfrac{1}{(w+\overline{w}+2)^2}(\overline{u}w^2+pw\overline{w}+u\overline{w}^2+\overline{v}w+v\overline{w}+q)=0$, where

$$\begin{aligned} u &= r^4-2\big((d-3)\overline{d}+(\overline{c}+1)c+d-3\overline{c}-6\big)r^2+(d-3)\overline{d}+(-\overline{c}-1)c+d+3\overline{c})^2,\\ p &= 2\Big(r^4-2\big((d-1)\overline{d}+(\overline{c}-1)c-d-\overline{c}+10\big)r^2\\ &\quad +\big((d-3)\overline{d}+(-\overline{c}-1)c+d+3\overline{c}\big)\big((d+1)\overline{d}+(-\overline{c}+3)c-3d-\overline{c}\big)\Big),\\ v &= 4\Big(r^4-2\big((d-1)\overline{d}+(\overline{c}+1)c+d-\overline{c}-2\big)r^2\\ &\quad +\big((d-3)\overline{d}+(-\overline{c}-1)c+d+3\overline{c}\big)\big((d+1)\overline{d}+(-\overline{c}-1)c+d-\overline{c}\big)\Big),\\ q &= 4\Big(r^4-2\big((d+1)\overline{d}+(\overline{c}+1)c+d+\overline{c}+2\big)r^2+\big((d+1)\overline{d}+(-\overline{c}-1)c+d-\overline{c}\big)^2\Big).\end{aligned}$$

Moreover, the coefficients p and q are expressed as

$$p = 2\Big(r^4-2\big(|d|^2+|c|^2-2\mathrm{Re}(c+d)+10\big)r^2+\big||d|^2-|c|^2-3\overline{d}+d-c+3\overline{c}\big|^2\Big) \in \mathbb{R},$$

and

$$q = 4\Big(r^4-2\big(|d|^2+|c|^2+2\mathrm{Re}(c+d)+2\big)r^2+\big(|d|^2-|c|^2+2\mathrm{Re}(d-c)\big)^2\Big) \in \mathbb{R},$$

respectively. Therefore, the factor $\overline{u}w^2+pw\overline{w}+u\overline{w}^2+\overline{v}w+v\overline{w}+q=0$ represents a conic on the w-plane.

Next, we will check $p^2-4u\overline{u}>0$. In fact, we have

$$p^2-4u\overline{u} = -256r^2(r^2-|d+\overline{c}-2|^2)(r^2-|c-d|^2).$$

Since $\widetilde{E}$ is an ellipse, the inequality $r^2>|c-d|^2$ holds and the last factor of the right side of the above equality is positive. So, we need to check $r^2<|d+\overline{c}-2|^2$ holds.

Now we will show that the ellipse $\widetilde{E}(z)=0$ intersects the imaginary axis if $r^2=|d+\overline{c}-2|^2$. The intersection points of the ellipse $\widetilde{E}(z)=0$ and the imaginary axis are given as the solution to the equation

$$\begin{aligned}&\big(\mathrm{Re}(c+d)-2\big)^2z^2+2\Big(2\mathrm{Im}(c+d)\big(\mathrm{Re}(c+d)-1\big)-\mathrm{Re}(c+d)\mathrm{Im}(cd)\Big)iz\\ &\quad +4\big(\mathrm{Re}(c+d)-1\big)^2-2\big(\mathrm{Re}(c+d)-1\big)\mathrm{Re}(cd)-\mathrm{Im}(cd)=0, \end{aligned}\tag{1}$$

where (1) is obtained by eliminating $\overline{z}$ from $\widetilde{E}(z)=0$ and $z+\overline{z}=0$. The above equation (1) has two pure imaginary solution,

$$\begin{aligned}&\frac{1}{\big(\mathrm{Re}(c+d)-2\big)^2}\Big(\mathrm{Re}(c+d)\mathrm{Im}(cd))-2\mathrm{Im}(c+d)(\mathrm{Re}(c+d)-1)\\ &\quad \pm 2|c+\overline{d}-2|^2\sqrt{2(\mathrm{Re}(c)-1)(\mathrm{Re}(d)-1)(\mathrm{Re}(c+d)-1)}\Big)i.\end{aligned}\tag{2}$$

Therefore, the ellipse $\widetilde{E}(z)=0$ and the imaginary axis intersect at exactly two points. Such an ellipse always intersects with $\partial\mathbb{P}$. So, the ellipse defined by $\widetilde{E}(z)=0$ with $r^2 \geq |d+\overline{c}-2|^2$ always intersects $\partial\mathbb{P}$. This is contrary to the assumption that $\widetilde{E}$ contained in the $\mathbb{C}\setminus\overline{\mathbb{P}}$.

Therefore, $r^2 < |d+\overline{c}-2|^2$ holds, and we have $p^2 - 4u\overline{u} > 0$. Hence, $\widetilde{E}(\Psi(w)) = 0$ gives a defining equation of an ellipse or its degenerate. ∎

Although Ψ is not conformal, we can see that the point of tangency between a line and an ellipse corresponds to the point of tangency between a line and an ellipse.

Theorem 7
For the parabola $\partial\mathbb{P}$, C_2 is a 3-inscribed ellipses in $\mathbb{C}\setminus\overline{\mathbb{P}}$ if and only if C_2 is the interior curve with respect to a Blaschke-like map B_ψ for some Blaschke product B of degree 3 and ψ.

Proof From Theorem 4, since the interior curve of B_ψ is a 3-inscribed ellipse in $\mathbb{C}\setminus\overline{\mathbb{P}}$, it is sufficient to show that any 3-inscribed ellipse in $\mathbb{C}\setminus\overline{\mathbb{P}}$ is the interior curve of Blaschke-like map for some Blaschke product of degree 3 and ψ.

Suppose that C_2 is a 3-inscribed ellipse in $\mathbb{C}\setminus\overline{\mathbb{P}}$. That is, there exists a triangle $\triangle abc$ inscribed in $\partial\mathbb{P}$ and circumscribed about C_2.

From Lemmas 5 and 6, the map Ψ^{-1} maps the parabola $\partial\mathbb{P}$, $\triangle abc$ inscribed in $\partial\mathbb{P}$, and C_2 inscribed in $\triangle abc$ to the unit circle $\partial\mathbb{D}$, and the triangle $\widetilde{\triangle}\big(=\triangle\psi^{-1}(a)\psi^{-1}(b)\psi^{-1}(c)\big)$ inscribed in $\partial\mathbb{D}$, an ellipse $\widetilde{E}$ inscribed in $\widetilde{\triangle}$, respectively.

From Theorem 3, this ellipse $\widetilde{E}$ is the interior curve for some Blaschke product $\widetilde{B}$ of degree 3. From the construction method, it is clear that C_2 is the interior curve of the Blaschke-like map associated with $\widetilde{B}$ and ψ. ∎

Although the details are omitted here, the characterization of an ellipse inscribed in a triangle inscribed in the parabola $\partial\mathbb{P}$ is given as follows. For the parabola $\partial\mathbb{P}$ and the ellipse $E : |z-c|+|z-d| = r$ $(c, d \in \mathbb{C}\setminus\overline{\mathbb{P}})$, if there is a triangle inscribed in $\partial\mathbb{P}$ and circumscribed about E, then r satisfies the following condition,

$$r^4 + 2\big(2\mathrm{Re}\,(cd) - |c-4|^2 - |d-4|^2\big)r^2 + \big|(c-\overline{d})^2 - 8(c+\overline{d})\big|^2 = 0.$$

For a positive solution r of the above equation, if $E \subset \mathbb{P}$, then there is a Poncelet triangle with respect to $\partial\mathbb{D}$ and $\partial\mathbb{P}$.

Remark 2
For a quadrilateral inscribed in the unit circle, an ellipse E is a Poncelet's inner ellipse if and only if E is an interior curve of the composition of two Blaschke products of degree 2 ([Fuj13, Theorem 2]). So, Fusse's theorem on bicentric quadrilateral can be extended in a similar way.

The hyperbolic case remains. This is a future work in progress.

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