A new minimal element theorem and new generalizations of Ekeland's variational principle in complete lattice optimization problem

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This manuscript summarizes the results of [2] and presents a new application based on [1] and [2]. For more detailed results and previous researches, please refer to [2].

1 Preliminaries

We first recall some notations, definitions and well-known results, which will be used in this paper. Let $\mathbb{R}^n$ be n-dimensional Euclidean space,

$$\mathbb{R}^n_+ := \{x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n \mid x_1 \geq 0, x_2 \geq 0, ..., x_n \geq 0\}$$

be its nonnegative orthant and $\mathbf{0}$ be the origin of $\mathbb{R}^n$, respectively.

For a set $A \subset \mathbb{R}^n$, $\mathrm{int}(A)$, $\mathrm{cl}(A)$, $\mathrm{cor}(A)$ $\mathrm{conv}(A)$ denote the topological interior, the topological closure, the algebraic interior and the convex hull, respectively. The symbol $\mathcal{P}(\mathbb{R}^n)$ denote the family of nonempty subsets of $\mathbb{R}^n$ including the empty set $\emptyset$ and $\mathcal{V}$ denote the family of nonempty subsets of $\mathbb{R}^n$. The sum of two sets $V_1, V_2 \in \mathcal{V}$ and the product of $\alpha \in \mathbb{R}$ and $V \in \mathcal{V}$ are defined by

(OP) $V_1 + V_2 := \{v_1 + v_2 \,|v_1 \in V_1, v_2 \in V_2\}, \qquad \alpha V := \{\alpha v \,|v \in V\}.$

In this paper, we assume that $C \subset \mathbb{R}^n$ is a solid pointed closed convex cone, that is, $\mathrm{int}C \neq \emptyset$, $C \cap (-C) = \{\mathbf{0}\}$, $\mathrm{cl}C = C$, $C + C \subset C$ and $t \cdot C \subset C$ for all $t \in [0, \infty)$.

Definition 1.1. *For $a, b \in \mathbb{R}^n$ and a solid convex cone $C \subset \mathbb{R}^n$, we define*

$$a \leq_C b \quad \text{by} \quad b - a \in C \qquad\qquad a \leq_{\mathrm{int}C} b \quad \text{by} \quad b - a \in \mathrm{int}(C).$$

Proposition 1.1. *For $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$, the following statements hold:*

(i) *$x \leq_C y$ implies that $x + z \leq_C y + z$ for all $z \in \mathbb{R}^n$,*

(ii) *$x \leq_C y$ implies that $\alpha x \leq_C \alpha y$ for all $\alpha \geq 0$,*

(iii) *$\leq_C$ is reflexive and transitive. Moreover, if C is pointed, $\leq_C$ is antisymmetric and hence a partial order.*

We next introduce the concept of minimal elements in vector optimization problem, which are also known as Edgeworth-Pareto-minimal or efficient elements.

Definition 1.2. *Let Z denote a real vector space that is pre-ordered by some convex cone $C \subset Z$ and let A denote some nonempty subset of Z. We also suppose that* $\mathrm{cor}(C) \neq \emptyset$.

- *An element $\bar{z} \in A$ is called a minimal element of the set A, if*

  $$A \cap (\bar{z} - C) \subset \{\bar{z}\} + C.$$

  If C is pointed, then the above inclusions can be replaced by

  $$A \cap (\bar{z} - C) = \{\bar{z}\}.$$

- *An element $\bar{z} \in A$ is called a weakly minimal element of the set A, if*

  $$A \cap (\bar{z} - \mathrm{cor}(C)) = \emptyset.$$

Lemma 1.1 (Jahn)**.** *Let C have a nonempty algebraic interior and $C \neq Z$. Then every minimal element of the set A is also a weakly minimal element of the set A.*

2 Set optimization and complete lattice optimization problem

2.1 Preliminaries in set optimization

Definition 2.1 (Kuroiwa-Tanaka-Ha)**.** *For A, $B \in \mathcal{V}$ and a solid closed convex cone $C \subset \mathbb{R}^n$, we define*

$$A \leq^l_C B \quad \text{by} \quad B \subset A + C, \qquad A \leq^u_C B \quad \text{by} \quad A \subset B - C.$$

Proposition 2.1 ([2])**.** *For A, B, $D \in \mathcal{V}$ and $\alpha \geq 0$, the following statements hold.*

(i) $\leq^l_C$ *and* $\leq^u_C$ *are reflexive and transitive.*

(ii) $A \leq^l_C B \iff -B \leq^u_C -A \iff B \leq^l_{-C} -A$.

(iii) $A \leq^l_C B \iff B + C \subset A + C$ *and* $A \leq^u_C B \iff A - C \subset B - C$.

(iv) $A \leq^l_C B$ *implies* $A + D \leq^l_C B + D$ *and* $A \leq^u_C B$ *implies* $A + D \leq^u_C B + D$.

(v) $A \leq^l_C B$ *implies* $\alpha A \leq^l_C \alpha B$ *and* $A \leq^u_C B$ *implies* $\alpha A \leq^u_C \alpha B$.

Definition 2.2 (Luc, Hernandez)**.** *It is said that $A \in \mathcal{V}$ is*

(i) *C-proper (resp. $(-C)$-proper) if $A + C \neq \mathbb{R}^n$ (resp. $A - C \neq \mathbb{R}^n$).*

(ii) *C-closed (resp. $(-C)$-closed) if $A + C$ (resp. $A - C$) is a closed set,*

(iii) *C-bounded (resp. $(-C)$-bounded) if for each neighborhood U of zero in $\mathbb{R}^n$ there is some positive number $t > 0$ such that*

$$A \subset tU + C \quad (\textit{resp. } A \subset tU - C),$$

(iv) C-*compact (resp.* $(-C)$-*compact) if any cover of* A *the form*

$$\{U_\alpha + C \mid U_\alpha \text{ are open}\} \quad (\textit{resp. } \{U_\alpha - C \mid U_\alpha \text{ are open}\})$$

admits a finite subcover,

(v) C-*convex (resp.* $(-C)$-*convex) if* $A + C$ *(resp.* $A - C$*) is a convex set.*

It is easy to see that every C-*compact set is* C-*closed and* C-*bounded.*

Introducing the equivalence relations

$$A \simeq_l B \Longleftrightarrow A \leq_C^l B \quad \text{and} \quad B \leq_C^l A,$$

$$A \simeq_u B \Longleftrightarrow A \leq_C^u B \quad \text{and} \quad B \leq_C^u A,$$

we can generate the set of equivalence classes which are denoted by $[\cdot]^l$ and $[\cdot]^u$, respectively. The followings are easily confirmed.

$$(\diamond) \quad A \in [B]^l \Longleftrightarrow A + C = B + C, \quad A \in [B]^u \Longleftrightarrow A - C = B - C.$$

Definition 2.3 (l-minimal element, u-minimal element)**.** *Let* $\mathcal{S} \subset \mathcal{V}$. *We say that* $\bar{A} \in \mathcal{S}$ *is a* $l[u]$-*minimal element if for any* $A \in \mathcal{S}$,

$$A \leq_C^{l[u]} \bar{A} \quad \text{implies} \quad \bar{A} \leq_C^{l[u]} A.$$

The symbols $l[u]\text{-Min}(\mathcal{S}; C)$ *denote the family of* $l[u]$-*minimal elements of* $\mathcal{S}$.

2.2 Complete lattice optimization problem

The property (iii) in Proposition 2.1 and $(\diamond)$ allow to define the following set

$$\mathcal{L} := \{A \in \mathcal{P}(\mathbb{R}^n) \,|\, A = A + C\}.$$

We can easily see that $(\mathcal{L}, \supseteq)$ is a partially ordered set (that is, the above order relation satisfies the antisymmetric property).

Proposition 2.2 (Hamel et al.[6])**.** *The pair* $(\mathcal{L}, \supseteq)$ *is a complete lattice. Moreover, for a subset* $\mathcal{A} \subseteq \mathcal{L}$, *the infimum and supremum of* $\mathcal{A}$ *are given by*

$$\inf \mathcal{A} = \bigcup_{A \in \mathcal{A}} A, \qquad \sup \mathcal{A} = \bigcap_{A \in \mathcal{A}} A$$

where it is understood that $\inf \mathcal{A} = \emptyset$ *and* $\sup \mathcal{A} = \mathbb{R}^n$ *whenever* $\mathcal{A} = \emptyset$. *The greatest (top) element of* $\mathcal{L}$ *with respect to* $\supseteq$ *is* $\emptyset$, *the least (bottom) element is* $\mathbb{R}^n$.

Proposition 2.3 (Hamel et al.[6])**.** *The following statements hold.*

(i) *For* $A, B, D, E \in \mathcal{L}$, $A \supseteq B$, $D \supseteq E$ *implies* $A + D \supseteq B + E$.

(ii) *For* $A, B \in \mathcal{L}$, $A \supseteq B$, $s \geq 0$ *implies* $sA \supseteq sB$.

(iii) $\mathcal{A} \subseteq \mathcal{L}$, $B \in \mathcal{L}$ *implies* $\inf(\mathcal{A} + B) = (\inf \mathcal{A}) + B$ *and*
$\mathcal{A} \subseteq \mathcal{L}$, $B \in \mathcal{L}$ *implies* $\sup(\mathcal{A} + B) \supseteq (\sup \mathcal{A}) + B$,
where $\mathcal{A} + B = \{A + B \,|\, A \in \mathcal{L}\}$.

Inspired by Definition 2.2, we introduce the following new concepts.

Definition 2.4 ([2]). *It is said that $A \in \mathcal{L}$ is*

(i) *$\mathcal{L}$-proper if $A \neq \mathbb{R}^n$,*

(ii) *$\mathcal{L}$-closed if A is a closed set,*

(iii) *$\mathcal{L}$-bounded if for each neighborhood $U_1 = U_1 + C$ of zero in $\mathbb{R}^n$ there is some positive number $t > 0$ such that $A \subset tU_1$,*

(iv) *$\mathcal{L}$-compact (resp. $\mathcal{U}$-compact) if any cover of A the form*

$$\{U_\alpha |\ U_\alpha \text{ are open and } U_\alpha + C = U_\alpha\}$$

admits a finite subcover,

(v) *$\mathcal{L}$-convex if A is a convex set.*

Remark 2.1. *Every $\mathcal{L}$-compact set is $\mathcal{L}$-closed and $\mathcal{L}$-bounded.*

We conclude this subsection by introducing the solution concept in complete lattice-valued optimization problem. We set

$$\mathrm{cl}(\mathcal{L}) := \{A \in \mathcal{P}(\mathbb{R}^n)\,|\,A = \mathrm{cl}(A + C)\,\},$$

$$\mathrm{clconv}(\mathcal{L}) := \{A \in \mathcal{P}(\mathbb{R}^n)\,|\,A = \mathrm{clconv}(A + C)\,\}.$$

Definition 2.5 (Hamel et al.[6]). *Let $\mathcal{A} \subseteq \mathrm{cl}(\mathcal{L})$. An element $\bar{A} \in \mathcal{A}$ is called l-minimal for $\mathcal{A}$ if it satisfies*

$$A \in \mathcal{A}, \quad A \supseteq \bar{A} \quad \Longrightarrow \quad A = \bar{A}.$$

The set of all l-minimal elements of $\mathcal{A}$ is denoted by $\mathrm{Min}\mathcal{A}$.

Let M be a nonempty set and $F : M \to \mathrm{cl}(\mathcal{L})$ a set-valued mapping. Similar to Zhang-Huang[8] in 2021, we consider the following complete lattice-valued optimization problem:

(CLOP) Minimize $F(x)$ subject to $x \in M$.

Definition 2.6 (Minimal solutions [2]). *A point $x_0 \in M$ is said to be*

(i) *an $\mathcal{L}$-minimal solutions of (CLOP) if for any $x \in M$, $F(x) \subset F(x_0)$ implies $F(x) = F(x_0)$. The set of all $\mathcal{L}$-minimal solutions of (CLOP) is denoted by* $\mathrm{Min}(F(M); \subset)$.

(ii) *a weak $\mathcal{L}$-minimal solutions of (CLOP) if for any $x \in M$, $F(x) \subset \mathrm{int}(F(x_0))$ implies $F(x) = F(x_0)$. The set of all weak $\mathcal{L}$-minimal solutions of (CLOP) is denoted by* $\mathrm{wMin}(F(M); \subset)$.

Remark 2.2. *Hamel et al.[6] introduced complete lattice optimization problem (CL) using the concept of the infimum and minimal elements. Given a set $\mathcal{A} \subseteq \mathrm{cl}(\mathcal{L})$ or $\mathcal{A} \subseteq \mathrm{clconv}(\mathcal{L})$, complete lattice optimization problem look for*

(CL) *a set $\mathcal{B} \subseteq \mathcal{A}$ such that*

$$\inf \mathcal{B} = \inf \mathcal{A} \qquad \text{and} \qquad \mathcal{B} \subseteq \mathrm{Min}\mathcal{A}.$$

In this paper, for simplicity, we adopt the definition of the minimal solution of (CLOP) using the definition 2.5 and 2.6. The concept of minimal solutions using (CL) is a subject for future research.

2.3 Cancellation laws in complete lattices

The Rådström cancellation law is a well-known fundamental result. After, Prakash-Sertel generalized the above result. As a direct consequence of Durea-Florea[4] in 2024, we obtain the following cancellation laws in complete lattices.

Proposition 2.4 ([2]). *The following statements hold.*

(i) *Let $A, B, D \in \mathcal{L}$ be such that $\mathcal{L}$-closed, $\mathcal{L}$-bounded and $\mathcal{L}$-convex. Then*

$$A \supseteq B \iff A + D \supseteq B + D.$$

(ii) *Let $A, B, D \in \mathcal{U}$ be such that $\mathcal{U}$-closed, $\mathcal{U}$-bounded and $\mathcal{U}$-convex. Then*

$$A \subseteq B \iff A + D \subseteq B + D.$$

Remark 2.3. *We have found that the concept of C-closedness, C-convexity and C-boundedness play an important role to obtain cancellation laws in set optimization. Using Rådström's idea, Nuriya-Kuroiwa introduced parametrized embedding functions on compact and convex subset to observe l-type solutions. Moreover in 2023, Araya assumed C-convexity to establish algebraic operations on $\mathcal{V}$. The subject of next research is to investigate the relationships among embedding theorems, cancellation laws and algebraic operations of set order relations. See also Kuroiwa's manuscript in 2024 RIMS workshop.*

3 Existence results

The aim of this section is to present a minimal element theorem with set perturbation in complete lattice optimization problem using Brézis-Browder's principle, sublinear scalarizing functions for complete lattices proposed in [2]. In 2006, Hamel-Löhne[5] defined the following new order relations on $X \times \mathcal{V}$,

$$(x_1, V_1) \preceq^l_{k^0} (x_2, V_2) \iff V_1 + d(x_1, x_2)k^0 \leq^l_C V_2,$$

where X is a metric space. We see that $\preceq^l_{k^0}$ is reflexive and transitive on $X \times \mathcal{V}$.

Let $D_L \subset \mathbb{R}^n$ be a convex set. As a natural generalization of the above order relations, we define the following new order relation on $X \times \mathcal{L}$, where X is a metric space:

$$(x_1, V_1) \preceq_{D_L} (x_2, V_2) \iff V_2 + d(x_1, x_2)D_L \subset V_1.$$

Proposition 3.1 ([2]). *Let $D_L \subset \mathbb{R}^n$ be a convex set. Then $\preceq_{D_L}$ is reflexive and transitive on $X \times \mathcal{L}$.*

Let P_X and P_Y be projections of $X \times Y$ onto X and Y, respectively, that is, for every $(x, y) \in X \times Y$

$$P_X(x, y) = x \qquad P_Y(x, y) = y.$$

Theorem 3.1 (A minimal element theorem [2]). *Let X be a complete metric space, $C \subset \mathbb{R}^n$ a solid pointed closed convex cone, $\mathcal{L}$ a family of L-proper and L-closed subsets of $\mathbb{R}^n$, $k^0 \in \operatorname{int}C$, $D_L \in \operatorname{clconv}(\mathcal{L})$ a L-proper, L-closed and L-convex subset of $\mathbb{R}^n$ such that $\mathbf{0} \in \operatorname{int}(D_L)$ and $\mathcal{A} \subset X \times \mathcal{L}$ a nonempty set. We assume the following conditions:*

(i) *$\mathcal{A}$ is bounded below (there exists $\tilde{V} \in \mathcal{L}$ such that $\tilde{V} \supset P_{\mathcal{L}}(\mathcal{A})$);*

(ii) *For all $\preceq_{D_L}$-decreasing sequence $\{(x_n, V_n)\}_{n\in\mathbb{N}} \subset \mathcal{A}$ with $x_n \to x \in X$, there exists $(x, V) \in \mathcal{A}$ such that $(x, V) \preceq_{D_L} (x_n, V_n)$ for all $n \in \mathbb{N}$.*

Then for every $(x_0, V_0) \in \mathcal{A}$ there exists $(\bar{x}, \bar{V}) \in \mathcal{A}$ such that

(a) *$(\bar{x}, \bar{V}) \preceq_{D_L} (x_0, V_0)$, and*

(b) *If $(\hat{x}, \hat{V}) \in \mathcal{A}$ such that $(\hat{x}, \hat{V}) \preceq_{D_L} (\bar{x}, \bar{V})$ then $\hat{x} = \bar{x}$.*

Using scalarizing functions for complete lattices defined in [2] and applying Theorem 3.1, we obtain the following new weak form of Ekeland's variational principle for complete lattices with set perturbation.

Theorem 3.2 (Weak form of generalized Ekeland's variational principle [2])**.** *We suppose that X is a complete metric space, $C \subset \mathbb{R}^n$ is a solid pointed closed convex cone, $k^0 \in \mathrm{int}C$, $D_L \in \mathrm{clconv}(\mathcal{L})$ is a L-proper, L-closed and L-convex set such that $\mathbf{0} \in \mathrm{int}(D_L)$, $F : X \to \mathcal{L}$ is a L-proper and L-closed valued function. We also assume that*

(i) F is bounded below
(there exists $\tilde{V} \in \mathcal{L}$ such that $\tilde{V} \supset F(x)$ for all $x \in X$);

(ii) $\{\hat{x} \in X | (\hat{x}, F(\hat{x})) \preceq_{D_L} (x, F(x))\}$ is closed for all $x \in X$.

Then for any $x_0 \in X$, there exists $\bar{x} \in X$ such that

(a) $F(\bar{x}) \supset F(x_0)$,

(b) $F(x) \not\supset F(\bar{x}) + d(\bar{x}, x)D_L$ for all $x \in X$ with $x \neq \bar{x}$.

4 Application to Game Theory under Uncertainty

In this section, we apply Ekeland's variational principle to derive the existence of robust Nash equilibrium solutions for vector-valued games under uncertainty.

Let $\mathcal{G} := (\Lambda, \{X_i\}, \{g_i\}, \mathcal{U})_{i\in\Lambda}$ be a vector-valued game under uncertainty, where

(a) $\Lambda := \{1, 2, \cdots, n\}$ is the set of n players;

(b) The symbol X_i denotes the set of strategies of the i^{th} player for each $i \in \Lambda$, which is a nonempty subset of a complete metric space X with metric d_i;

(c) set $\tilde{X} := \Pi_{i\in\Lambda} X_i$;

(d) For each $i \in \Lambda$ and the set $\mathcal{U}$ of all uncertainties, $g_i : \tilde{X} \times \mathcal{U} \to \mathbb{R}^n$ is a vector-valued loss function corresponding to the i^{th} player.

Set $X_{-i} := \Pi_{j\in\Lambda\setminus\{i\}} X_j$. For each $i \in \Lambda$, we define

$$x_{-i} := \{x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n\} \in X_{-i}, \quad \forall\, x = (x_1, \cdots, x_n) \in \tilde{X}.$$

If for each $i \in \Lambda$, $w_i \in X_i$, then we define

$$(w_i, x_{-i}) := \{x_1, \cdots, x_{i-1}, w_i, x_{i+1}, \cdots, x_n\} \in \tilde{X}.$$

The image of the uncertainty set $\mathcal{U}$ and all $x \in \tilde{X}$ under g_i is the set $g_i(x,\mathcal{U}) := \{g_i(x,u) \mid u \in \mathcal{U}\}$. These sets will be compared by using complete lattices in order to obtain the notions of robust Nash equilibria. Based on [1, 3], we now define the notion of robust Nash equilibria for the game $\mathcal{G}$ with complete lattices.

Definition 4.1. *An element $\tilde{x} = (\tilde{x}_1, \tilde{x}_2, \cdots, \tilde{x}_n) \in \tilde{X}$ is said to be a robust Nash equilibrium for the game $\mathcal{G}$ if and only if for any $i \in \Lambda$,*

$$g_i(x_i, \tilde{x}_{-i}, \mathcal{U}) \supset g_i(\tilde{x}, \mathcal{U}),\ x_i \in X_i \ \Rightarrow\ g_i(\tilde{x}, \mathcal{U}) = g_i(x_i, \tilde{x}_{-i}, \mathcal{U});$$

The symbol $\mathrm{RMin}(\mathcal{G}, \supset)$ denotes the set of robust Nash equilibrium for the game $\mathcal{G}$.
Define a metric $\tilde{d}$ on $\tilde{X} = \Pi_{i\in\Lambda} X_i$ by

$$\tilde{d}(x, w) = \sum_{i=1}^{n} d_i(x_i, w_i),$$

where $x = (x_1, x_2, \cdots, x_n) \in \tilde{X}$ and $w = (w_1, w_2, \cdots, w_n) \in \tilde{X}$. Clearly, $(\tilde{X}, \tilde{d})$ is a complete metric space.

An element $\tilde{x} = (\tilde{x}_1, \tilde{x}_2, \cdots, \tilde{x}_n) \in \tilde{X}$ is a robust Nash equilibrium if and only if $\tilde{x}_i \in X_i$ is a l-minimal solution, respectively, of the following complete lattice optimization problem

$$\begin{aligned} &\text{Minimize } g_i(x_i, \tilde{x}_{-i}, \mathcal{U}) \\ &\text{subject to } x_i \in X_i, \end{aligned} \tag{P_i}$$

for each $i \in \Lambda$. We now establish the existence of the solution for vector-valued game $\mathcal{G}$.

Theorem 4.1. *Let $C \subset \mathbb{R}^n$ be a solid pointed closed convex cone, $k^0 \in \mathrm{int}C$, $D_L \in \mathrm{clconv}(\mathcal{L})$ be a L-proper, L-closed and L-convex set such that $\mathbf{0} \in \mathrm{int}(D_L)$. We also assume that*

(a) *$x \mapsto g_i(x,\mathcal{U})$ is L-proper and L-closed valued function on $\tilde{X}$;*

(b) *$g_i(\cdot,\mathcal{U})$ is bounded below*
(there exists $\tilde{V} \in \mathcal{L}$ such that $\tilde{V} \supset g_i(x,\mathcal{U})$ for all $x \in \tilde{X}$);

(c) *the set $\{\hat{x} \in \tilde{X} \mid (\hat{x}, g_i(\hat{x},\mathcal{U})) \preceq_{D_L} (x, g_i(x,\mathcal{U}))\}$ is closed for all $x \in \tilde{X}$;*

(d) *for each $x \in \tilde{X}$ with $x \notin \mathrm{RMin}(\mathcal{G}, \supset)$, there exists $\hat{x} \in \tilde{X} \setminus \{x\}$ such that*

$$g_i(\hat{x},\mathcal{U}) \supset g_i(x,\mathcal{U}) + \tilde{d}(\hat{x}, x) D_L.$$

Then $\mathrm{RMin}(\mathcal{G}, \supset)$ admits a robust Nash equilibrium.

Proof. It follows from Theorem 3.2 that there exists $\bar{x} \in \tilde{X}$ such that

$$g_i(x,\mathcal{U}) \not\supset g_i(\bar{x},\mathcal{U}) + \tilde{d}(x, \bar{x}) D_L, \quad \forall x \in \tilde{X}, x \neq \bar{x}. \tag{1}$$

We need to prove that $\bar{x} \in \mathrm{RMin}(\mathcal{G}, \supset)$. Assume that $\bar{x} \notin \mathrm{RMin}(\mathcal{G}, \supset)$, then there exists $\hat{x} \in \tilde{X} \setminus \{\bar{x}\}$ such that

$$g_i(\hat{x},\mathcal{U}) \supset g_i(\bar{x},\mathcal{U}) + \tilde{d}(\hat{x}, \bar{x}) D_L,$$

which contradicts to (1) and hence $\bar{x} \in \mathrm{RMin}(\mathcal{G}, \supset)$. □

5 Conclusions

In this paper, we established new cancellation laws of set order relations. Moreover, we introduced new concepts on complete lattice optimization problem. Applying nonlinear scalarizing techniques in complete lattice which is a generalization of Gerstewitz's scalarizing function [7], we presented a new type of minimal element theorem and generalized Ekeland's variational principles in complete lattice optimization problem. Moreover, as an application, we proposed an existence theorem of robust Nash equilibrium.

We have found that the family of C-closed, bounded and convex subset of $\mathbb{R}^n$ allow cancellation laws and algebraic operations on some complete lattice. This fact may bring a new insight into the complete lattice optimization problems and new existence results are expected.

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