The integral representation theorems of nonlinear functionals on an abstract space

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1 Introduction

This paper discusses the problem of expressing a monotone functional on a collection of functions in terms of the Choquet integral with respect to a continuous nonadditive measure. The Choquet [3, 23], Sugeno [21, 26], and Shilkret [24, 31] integrals are known as integrals related to a nonadditive measure, namely, a monotone set function vanishing at the empty set. Among them, the Choquet integral is important from the viewpoint of measure theory because it coincides with the Lebesgue integral when the nonadditive measure is σ-additive. It also plays an important role in game theory and expected utility theory, where it is necessary to consider set functions that do not satisfy additivity due to interactions among sets of players or states of nature [2, 5]; see also [15] for a clear educational example of a "workshop".

The Choquet integral representation theorem for a monotone functional on a collection of functions can be generally described as follows: Let X be a nonempty set. Let $\Phi \subset [0,\infty]^X$ contain the zero function (this is denoted by 0 with a slightly mixed symbol). Suppose that a functional $I\colon \Phi \to [0,\infty]$ is monotone with $I(0)=0$. Then there exists a nonadditive measure μ on X such that

$$I(f) = \int_0^\infty \mu(\{f>t\})dt$$

for every $f \in \Phi$. This type of representation is called the *Choquet integral representation*, and μ is called a *representing measure* of I. Since no additivity is assumed for the functional I, the representing measure μ is generally nonadditive.

This presentation is a summary of the contents of a recent paper [10]; see also [1,6,7,13, 16,17,22,23,25,27,29,32]. It discusses the following items based on the scheme proposed by König [12, 13].

1. For a given continuous monotone functional, find a representing measure that is simultaneously inner and outer continuous on appropriate collections of sets such as open, closed, compact, and measurable.
2. Formulate continuous Choquet representation theorems in an abstract setting so as to contain as many existing results as possible.

Concerning other types of nonlinear integrals, he Shilkret and Sugeno integral representations will be discussed elsewhere.

2 Preliminaries

Let $\mathbb{R}$ and $\mathbb{N}$ denote the set of the real numbers and the set of the natural numbers. Let $\overline{\mathbb{R}} := [-\infty,\infty]$ with the usual total order $\leq$. For a nonempty subset A of $\mathbb{R}$, let $\sup A := \infty$ (resp. $\inf A := -\infty$) if A is not bounded from above (resp. below) by a real number. With this convention, $\overline{\mathbb{R}}$ is an order complete lattice, that is, it a lattice in which any nonempty subset of $\overline{\mathbb{R}}$ has a supremum and an infimum. For the set consisting of two numbers $a, b \in \overline{\mathbb{R}}$, its supremum is denoted by $\sup(a, b)$ or by $a \vee b$, and its infimum is denoted by $\inf(a, b)$ or $a \wedge b$. We adopt the usual conventions for algebraic operations on $\overline{\mathbb{R}}$. We also adopt the convention that $(\pm\infty) \cdot 0 = 0 \cdot (\pm\infty) = 0$ and $\inf \emptyset := \infty$. Instead of the ambiguous expression $c > 0$, we use $c \in (0, \infty]$ when the value of c may take ∞. In other words, $c > 0$ always means $c \in (0, \infty)$. This notation convention will be used for similar cases.

For nonempty sets X and Y, let Y^X denote the collection of all functions $f \colon X \to Y$. For any $f, g \in \overline{\mathbb{R}}^X$, the order $f \leq g$ is defined as $f(x) \leq g(x)$ for every $x \in X$. This is a partial order for which $\overline{\mathbb{R}}^X$ is order complete, that is, for any nonempty $\Phi \subset \overline{\mathbb{R}}^X$, its supremum $\sup_{f \in \Phi} f$ and infimum $\inf_{f \in \Phi} f$ exist in such a way that

$$\Big(\sup_{f \in \Phi} f\Big)(x) = \sup_{f \in \Phi} f(x), \quad \Big(\inf_{f \in \Phi} f\Big)(x) = \inf_{f \in \Phi} f(x)$$

for every $x \in X$. For the set consisting of two functions $f, g \in \overline{\mathbb{R}}^X$, its supremum is denoted by $\sup(f, g)$ or $f \vee g$, and its infimum is denoted by $\inf(f, g)$ or $f \wedge g$. For any $c \in \overline{\mathbb{R}}$, the constant function $c_X \colon X \to \overline{\mathbb{R}}$, which is defined by $c_X(x) := c$ for every $x \in X$, is denoted by c with a slight abuse of notation. Let $f^+ := f \vee 0$, $f^- := (-f) \vee 0$, $|f| := f \vee (-f)$, and $\|f\|_\infty := \sup_{x \in X} |f(x)|$. Let χ_A denote the characteristic function of a set A, that is, $\chi_A(x) = 1$ if $x \in A$ and $\chi_A(x) = 0$ otherwise.

A *directed set* is a partially ordered set in which every pair of elements has an upper bound. By definition, a *net* is a function defined on a directed set. For a net $\{a_\gamma\}_{\gamma \in \Gamma}$ in $\overline{\mathbb{R}}$ and $a \in \overline{\mathbb{R}}$, the notation $a_\gamma \uparrow a$ (resp. $a_\gamma \downarrow a$) means that $\{a_\gamma\}_{\gamma \in \Gamma}$ is nondecreasing (resp. nonincreasing) and $a_\gamma \to a$. For a net $\{f_\gamma\}_{\gamma \in \Gamma}$ in $\overline{\mathbb{R}}^X$ and $f \in \overline{\mathbb{R}}^X$, the notation $f_\gamma \to f$ means *pointwise convergence*, that is, $f_\gamma(x) \to f(x)$ for every $x \in X$. Then, $f_\gamma \uparrow f$ (resp. $f_\gamma \downarrow f$) means that $\{f_\gamma\}_{\gamma \in \Gamma}$ is nondecreasing (resp. nonincreasing) and $f_\gamma \to f$. Finally, for a net $\{A_\gamma\}_{\gamma \in \Gamma}$ of sets and a set A, the notation $A_\gamma \uparrow A$ (resp. $A_\gamma \downarrow A$) means that $\{A_\gamma\}_{\gamma \in \Gamma}$ is nondecreasing (resp. nonincreasing) and $A = \bigcup_{\gamma \in \Gamma} A_\gamma$ (resp. $A = \bigcap_{\gamma \in \Gamma} A_\gamma$). These notation rules also apply to sequences of numbers, functions, and sets.

2.1 Nonadditive measures and measurable functions

Let X be a nonempty set and $\mathcal{D}$ a collection of subsets of X with $\emptyset \in \mathcal{D}$. Let 2^X denote the collection of all subsets of X. A set function $\mu \colon \mathcal{D} \to [0, \infty]$ is called a *nonadditive measure* on $(X, \mathcal{D})$ if $\mu(\emptyset) = 0$ and $\mu(A) \leq \mu(B)$ whenever $A, B \in \mathcal{D}$ and $A \subset B$. If $\mathcal{D} = 2^X$, then μ is referred to as a nonadditive measure on X. In the literature, this type of set function is also called a monotone measure [28, 30], a capacity [1, 3], a fuzzy measure [21, 26], and a pre-measure [25]. If $X \in \mathcal{D}$ and $\mu(X) < \infty$, then μ is said to be *finite*.

A function $f \colon X \to [0, \infty]$ is defined as *lower $\mathcal{D}$-measurable* if $\{f > t\} \in \mathcal{D}$ for every $t > 0$ and *upper $\mathcal{D}$-measurable* if $\{f \geq t\} \in \mathcal{D}$ for every $t > 0$. Let $\mathrm{Lm}(\mathcal{D})$ and $\mathrm{Um}(\mathcal{D})$ denote the collection of the lower $\mathcal{D}$-measurable and the upper $\mathcal{D}$-measurable functions $f \colon X \to [0, \infty]$, respectively. It is easy to see that $cf, f \wedge c, (f - c)^+ = f - f \wedge c \in \mathrm{Lm}(\mathcal{D})$ whenever $f \in \mathrm{Lm}(\mathcal{D})$ and $c > 0$. Moreover, $D \in \mathcal{D}$ if and only if $\chi_D \in \mathrm{Lm}(\mathcal{D})$. These are true for $\mathrm{Um}(\mathcal{D})$ as well. In particular, $0 \in \mathrm{Lm}(\mathcal{D}) \cap \mathrm{Um}(\mathcal{D})$ since $\emptyset \in \mathcal{D}$ is always assumed, while $1 \notin \mathrm{Lm}(\mathcal{D}) \cup \mathrm{Um}(\mathcal{D})$ unless $X \in \mathcal{D}$. See [4, 19, 30] for further information on nonadditive measures.

2.2 The Choquet integral

The *Choquet integral* [3] of a function $f\colon X \to [0,\infty]$ with respect to a nonadditive measure μ on $(X, \mathcal{D})$ is defined by

$$\mathrm{Ch}(\mu, f) := \begin{cases} \displaystyle\int_0^\infty \mu(\{f > t\})dt & \text{if } f \in \mathrm{Lm}(\mathcal{D}), \\ \displaystyle\int_0^\infty \mu(\{f \geq t\})dt & \text{if } f \in \mathrm{Um}(\mathcal{D}), \end{cases}$$

where the integral on the right-hand side is the Lebesgue integral of a nonincreasing $[0,\infty]$-valued function defined for all $t > 0$. This integral is also known as the *horizontal integral* [12]. If $f \in \mathrm{Lm}(\mathcal{D}) \cap \mathrm{Um}(\mathcal{D})$, then

$$\mathrm{Ch}(\mu, f) = \int_0^\infty \mu(\{f > t\})dt = \int_0^\infty \mu(\{f \geq t\})dt, \tag{1}$$

which follows from the fact that

$$\mu(\{f \geq t\}) \geq \mu(\{f > t\}) \geq \mu(\{f \geq t + \varepsilon\})$$

for any $\varepsilon > 0$ and $t > 0$ and the monotone convergence theorem for the Lebesgue integral. When $\mathcal{D}$ is a σ-field and μ is σ-additive, the Choquet integral $\mathrm{Ch}(\mu, f)$ is the Lebesgue integral of f with respect to μ. The following are basic properties of the Choquet integral that will be used later. For the proof of the other properties see [4, 12, 30].

Proposition 2.1 *Let X be a nonempty set and $\mathcal{D}$ a collection of subsets of X with $\emptyset \in \mathcal{D}$. Let μ be a nonadditive measure on $(X, \mathcal{D})$. Let $f, g \in \mathrm{Lm}(\mathcal{D}) \cup \mathrm{Um}(\mathcal{D})$. Then the following statements hold.*

(1) $\mathrm{Ch}(\mu, c) = c\mu(X)$ *if $c > 0$ and $X \in \mathcal{D}$. (generativity)*

(2) $\mathrm{Ch}(\mu, f) \leq \mathrm{Ch}(\mu, g)$ *if $f \leq g$. (monotonicity)*

(3) $\mathrm{Ch}(\mu, cf) = c\,\mathrm{Ch}(\mu, f)$ *for every $c > 0$ (positive homogeneity).*

(4) $\mathrm{Ch}(\mu, f + c) = \mathrm{Ch}(\mu, f) + \mathrm{Ch}(\mu, c)$ *if $c > 0$ and $X \in \mathcal{D}$. (positive translatability)*

(5) $\mathrm{Ch}(\mu, f) = \mathrm{Ch}(\mu, f \wedge c) + \mathrm{Ch}(\mu, (f - c)^+)$ *for any $c > 0$. (horizontal additivity)*

(6) $\mathrm{Ch}(\mu, f) = \sup\{\mathrm{Ch}(\mu, (f - a)^+ \wedge (b - a))\colon 0 < a < b < \infty\}$. *(marginal continuity)*

(7) $\mathrm{Ch}(\mu, f) \leq \mu(X)\|f\|_\infty$ *if $X \in \mathcal{D}$. (boundedness)*

See [4, 12, 19, 30] for the general theory of nonadditive measures and nonlinear integrals. See also [11] for an overview of various nonlinear integrals containing the Choquet integral.

3 The collections of nonnegative functions and sets

This section introduces some collections consisting of nonnegative functions and sets according to the König scheme [12, 13]. For a nonempty subset Φ of $[0,\infty]^X$, define the following collections of functions and sets. In what follows, *subsets* and *subcollections* are considered to be nonempty, and *countable* sets and *countable* collections are considered to be at most countable.

$$\Phi^\sigma := \left\{ \sup_{f\in\Phi_0} f \colon \Phi_0 \text{ is a countable subset of } \Phi \right\}$$

$$\Phi^\tau := \left\{ \sup_{f\in\Phi_0} f \colon \Phi_0 \text{ is a subset of } \Phi \right\}$$

$$\Phi_\sigma := \left\{ \inf_{f\in\Phi_0} f \colon \Phi_0 \text{ is a countable subset of } \Phi \right\}$$

$$\Phi_\tau := \left\{ \inf_{f\in\Phi_0} f \colon \Phi_0 \text{ is a subset of } \Phi \right\}$$

$$\mathcal{H}_\Phi := \big\{ \{f > t\} \colon f \in \Phi, t > 0 \big\}$$

$$\mathcal{H}_\Phi^\sigma := \left\{ \bigcup_{H\in\mathcal{H}_0} H \colon \mathcal{H}_0 \text{ is a countable subcollection of } \mathcal{H}_\Phi \right\}$$

$$\mathcal{H}_\Phi^\tau := \left\{ \bigcup_{H\in\mathcal{H}_0} H \colon \mathcal{H}_0 \text{ is a subcollection of } \mathcal{H}_\Phi \right\}$$

$$\mathcal{L}_\Phi := \big\{ \{f \geq t\} \colon f \in \Phi, t > 0 \big\}$$

$$\mathcal{L}_\Phi^\sigma := \left\{ \bigcap_{L\in\mathcal{L}_0} L \colon \mathcal{L}_0 \text{ is a countable subcollection of } \mathcal{L}_\Phi \right\}$$

$$\mathcal{L}_\Phi^\tau := \left\{ \bigcap_{L\in\mathcal{L}_0} L \colon \mathcal{L}_0 \text{ is a subcollection of } \mathcal{L}_\Phi \right\}$$

$$\mathcal{X}_\Phi := \big\{ A \subset X \colon \chi_A \in \Phi \big\}$$

A subset Φ of $[0,\infty]^X$ is called *positively homogeneous* if $cf \in \Phi$ for any $f \in \Phi$ and $c > 0$, a *lattice* if $f \vee g, f \wedge g \in \Phi$ for any $f, g \in \Phi$, and *Stonean* if $f \wedge c, (f-c)^+ \in \Phi$ for any $f \in \Phi$ and $c > 0$. To simplify the description, the symbol $\bullet$ is used to represent the theory that runs parallel to both $\bullet = \sigma$ (sequential or countable) and $\bullet = \tau$ (nonsequential or arbitrary). An *approximate σ-identity* in Φ is a sequence $\{p_n\}_{n\in\mathbb{N}} \subset \Phi$ such that $0 \leq p_n \leq 1$ for all $n \in \mathbb{N}$ and $p_n \uparrow 1$. Likewise, an approximate *τ-identity* in Φ is a net $\{p_\gamma\}_{\gamma\in\Gamma} \subset \Phi$ such that $0 \leq p_\gamma \leq 1$ for all $\gamma \in \Gamma$ and $p_\gamma \uparrow 1$. Obviously, Φ possesses an approximate $\bullet$-identify whenever $1 \in \Phi$.

4 Some modes of continuity of nonadditive measures and functionals

This section introduces some modes of continuity of nonadditive measures and nonlinear functionals used in this presentation.

Definition 4.1 Let $\mathcal{D}$ and $\mathcal{E}$ be collections of subsets of X containing $\emptyset$. We say that a nonadditive measure $\mu\colon 2^X \to [0,\infty]$ is

- *inner τ-continuous* on $\mathcal{D}$ if $\mu(D) = \sup_{\gamma\in\Gamma} \mu(D_\gamma)$ for any net $\{D_\gamma\}_{\gamma\in\Gamma}$ in $\mathcal{D}$ and $D \in \mathcal{D}$ with $D_\gamma \uparrow D$.
- *outer τ-continuous* on $\mathcal{D}$ if $\mu(D) = \inf_{\gamma\in\Gamma} \mu(D_\gamma)$ for any net $\{D_\gamma\}_{\gamma\in\Gamma}$ in $\mathcal{D}$ and $D \in \mathcal{D}$ with $D_\gamma \downarrow D$; and *conditionally outer τ-continuous* on $\mathcal{D}$ if this holds true whenever $\mu(D_{\gamma_0}) < \infty$ for some $\gamma_0 \in \Gamma$.
- *inner σ-continuous* on $\mathcal{D}$ if $\mu(D) = \sup_{n\in\mathbb{N}} \mu(D_n)$ for any sequence $\{D_n\}_{n\in\mathbb{N}}$ in $\mathcal{D}$ and $D \in \mathcal{D}$ with $D_n \uparrow D$.
- *outer σ-continuous* on $\mathcal{D}$ if $\mu(D) = \inf_{n\in\mathbb{N}} \mu(D_n)$ for any sequence $\{D_n\}_{n\in\mathbb{N}}$ in $\mathcal{D}$ and $D \in \mathcal{D}$ with $D_n \downarrow D$; and *conditionally outer σ-continuous* on $\mathcal{D}$ if this holds true whenever $\mu(D_{n_0}) < \infty$ for some $n_0 \in \mathbb{N}$.
- *inner $\mathcal{E}$ regular* on $\mathcal{D}$ if $\mu(D) = \sup\{\mu(E) \colon E \subset D, E \in \mathcal{E}\}$ for any $D \in \mathcal{D}$.
- *outer $\mathcal{E}$ regular* on $\mathcal{D}$ if $\mu(D) = \inf\{\mu(E) \colon E \supset D, E \in \mathcal{E}\}$ for any $D \in \mathcal{D}$.

A collection $\mathcal{K}$ of sets is called *σ-compact* if every countable subcollection of $\mathcal{K}$ whose intersection is empty has a further finite subcollection whose intersection is empty [18, Definition I.6.2] or [20, Definition 1.1]; it is called *τ-compact* if every subcollection of $\mathcal{K}$ whose intersection is empty has a further finite subcollection whose intersection is empty [14, page 111]. It is easy to see that every collection of compact subsets of a Hausdorff space is τ-compact, and hence, σ-compact.

Next we turn to the continuity of a functional on Φ.

Definition 4.2 Let Φ be a subset of $[0,\infty]^X$ with $0 \in \Phi$. We say that a functional $I \colon \Phi \to [0,\infty]$ is

- *inner τ-continuous* if $I(f_\gamma) \to I(f)$ for any net $\{f_\gamma\}_{\gamma\in\Gamma}$ in Φ and $f \in \Phi$ with $f_\gamma \uparrow f$.
- *outer τ-continuous* if $I(f_\gamma) \to I(f)$ for any net $\{f_\gamma\}_{\gamma\in\Gamma}$ in Φ and $f \in \Phi$ with $f_\gamma \downarrow f$; and *conditionally outer τ-continuous* if it holds under the additional condition that $I(f_{\gamma_0}) < \infty$ for some $\gamma_0 \in \Gamma$.
- *inner σ-continuous* if $I(f_n) \to I(f)$ for any sequence $\{f_n\}_{n\in\mathbb{N}}$ in Φ and $f \in \Phi$ with $f_n \uparrow f$.
- *outer σ-continuous* if $I(f_n) \to I(f)$ for any sequence $\{f_n\}_{n\in\mathbb{N}}$ in Φ and $f \in \Phi$ with $f_n \downarrow f$; and *conditionally outer σ-continuous* if it holds under the additional condition that $I(f_{n_0}) < \infty$ for some $n_0 \in \mathbb{N}$.

5 Continuous Choquet integral representation theorems

This section presents continuous Choquet integral representation theorems for a monotone functional on a collection of nonnegative functions on X. It should be mentioned here that a prototype of the Choquet integral representation theorem, due to Greco [7], is essentially used in their proofs. Let Φ be a subset of $[0,\infty]^X$ with $0 \in \Phi$. Throughout this section, let $I \colon \Phi \to [0,\infty]$ be a functional with $I(0) = 0$, which is always assumed to be monotone, that is, $I(f) \le I(g)$ whenever $f, g \in \Phi$ and $f \le g$.

Definition 5.1 We say that I is

- *positively homogeneous* if $I(cf) = cI(f)$ for any $f \in \Phi$ and $c > 0$, provided that Φ is positively homogeneous.

- *horizontally additive* if $I(f) = I(f \wedge c) + I((f-c)^+)$ for any $f \in \Phi$ and $c > 0$, provided that Φ is Stonean.
- *marginal continuous* if $I(f) = \sup\{I((f-a)^+ \wedge (b-a)) \colon 0 < a < b < \infty\}$ for any $f \in \Phi$, provided that Φ is Stonean.
- *bounded* if there is a constant $M > 0$ such that $I(f) \leq M\|f\|_\infty$ for any $f \in \Phi$.

In what follows, the uniqueness of the representing measure will not be mentioned to simplify the description of results since it follows from the following proposition.

Proposition 5.2 *Let Φ be a positively homogeneous Stonean lattice. Let μ and ν be nonadditive measures on X that represent I, that is, $I(f) = \mathrm{Ch}(\mu, f) = \mathrm{Ch}(\nu, f)$ for every $f \in \Phi$. Then $\mu(A) = \nu(A)$ for every $A \in 2^X$ if μ and ν are inner $\bullet$-continuous on $\mathcal{H}_\Phi^\bullet$ and outer $\mathcal{H}_\Phi^\bullet$ regular on 2^X.*

Our continuous Choquet integral representation theorems are classified into three cases. The first case requires the continuity of the functional I. For two nonempty collections $\mathcal{D}$ and $\mathcal{E}$ of subsets of X, we say that Φ *separates sets in $\mathcal{D}$ and $\mathcal{E}$* if for any $D \in \mathcal{D}$ and $E \in \mathcal{E}$ with $D \subset E$, there exist an $f \in \Phi$ such that $\chi_D \leq f \leq \chi_E$.

Theorem 5.3 *Let Φ be a positively homogeneous Stonean lattice. Let I be horizontally additive and marginal continuous. Assume that Φ separates sets in $\mathcal{L}_\Phi^\bullet$ and $\mathcal{H}_\Phi^\bullet$. Then, there exists a nonadditive measure μ on X such that*

(i) *$I(f) = \mathrm{Ch}(\mu, f)$ for every $f \in \Phi$,*

(ii) *μ is inner $\bullet$-continuous on $\mathcal{H}_\Phi^\bullet$ whenever I is inner $\bullet$-continuous on Φ,*

(iii) *μ is outer (resp. conditionally outer) $\bullet$-continuous on $\mathcal{L}_\Phi^\bullet$ whenever I is outer (resp. conditionally outer) $\bullet$-continuous on Φ,*

(iv) *μ is inner $\mathcal{L}_\Phi^\bullet$ regular on $\mathcal{H}_\Phi^\bullet$ and outer $\mathcal{H}_\Phi^\bullet$ regular on 2^X,*

(v) *μ is finite if I is bounded and Φ possesses an approximate $\bullet$-identity.*

The second case requires the compactness of $\mathcal{L}_\Phi$ instead of the continuity of the functional I.

Theorem 5.4 *Let Φ be positively homogeneous and Stonean. Assume that $(1-f) \wedge g \in \Phi$ whenever $f, g \in \Phi$ and $0 \leq f \leq 1$. Let I be horizontally additive and marginal continuous. Assume that Φ separates sets in $\mathcal{L}_\Phi^\bullet$ and $\mathcal{H}_\Phi^\bullet$ and that $\mathcal{L}_\Phi$ is $\bullet$-compact. Then, there exists a nonadditive measure μ on X such that*

(i) *$I(f) = \mathrm{Ch}(\mu, f)$ for every $f \in \Phi$,*

(ii) *μ is inner $\bullet$-continuous on $\mathcal{H}_\Phi^\bullet$ and outer $\bullet$-continuous on $\mathcal{L}_\Phi^\bullet$,*

(iii) *μ is inner $\mathcal{L}_\Phi^\bullet$ regular on $\mathcal{H}_\Phi^\bullet$ and outer $\mathcal{H}_\Phi^\bullet$ regular on 2^X,*

(iv) *μ is finite if I is bounded and Φ possesses an approximate $\bullet$-identity.*

The last is the case where Φ contains the characteristic functions of sets in a given collection $\mathcal{D}$ of subsets of X. Unlike Theorems 5.3 and 5.4, in this case, the representing measure μ is inner and outer $\bullet$-continuous on the same collection $\mathcal{D}$.

Theorem 5.5 *Let $\mathcal{D}$ be a collection of subsets of X with $\emptyset \in \mathcal{D}$. Let Φ be positively homogeneous and Stonean. Let I be horizontally additive and marginal continuous. Assume that $\mathcal{D} \subset \mathcal{X}_\Phi$. Then there exists a nonadditive measure μ on X such that*

(i) *$I(f) = \mathrm{Ch}(\mu, f)$ for every $f \in \Phi$,*

(ii) *μ is inner $\bullet$-continuous on $\mathcal{D}$ whenever I is inner $\bullet$-continuous on Φ,*

(iii) *μ is outer (resp. conditionally outer) $\bullet$-continuous on $\mathcal{D}$ whenever I is outer (resp. conditionally outer) $\bullet$-continuous on Φ,*

(iv) *μ is finite if and only if I is bounded.*

Example 5.6 (1) Let Φ be the collection of all bounded continuous nonnegative real-valued functions on a normal space X. Then, it is positively homogeneous Stonean lattice with the identity function 1. Moreover, $\mathcal{H}_\Phi^\tau$ is the collection of the open sets and $\mathcal{L}_\Phi^\tau$ is the collection of the closed sets, hence Φ separates sets in $\mathcal{L}_\Phi^\bullet$ and $\mathcal{H}_\Phi^\bullet$.

(2) Let Φ be the collection of all continuous nonnegative real-valued functions on a locally compact space X with compact support. Then, it is positively homogeneous Stonean lattice possessing an approximate τ-identity. It also satisfies the condition that $(1-f)\wedge g \in \Phi$ for any $f, g \in \Phi$ with $0 \leq f \leq 1$. Moreover, $\mathcal{H}_\Phi^\tau$ is the collection of the open sets and $\mathcal{L}_\Phi^\tau$ is the collection of the compact sets, hence $\mathcal{L}_\Phi$ is τ-compact and Φ separates sets in $\mathcal{L}_\Phi^\bullet$ and $\mathcal{H}_\Phi^\bullet$.

(3) Let Φ be the collection of all bounded nonnegative real-valued functions in $\mathrm{Lm}(\mathcal{D}) \cap \mathrm{Um}(\mathcal{D})$, where X is a nonempty set and $\mathcal{D}$ is a collection of subsets of X with $\emptyset \in \mathcal{D}$. Then, it is positively homogeneous Stonean and satisfies $\mathcal{D} \subset \mathcal{X}_\Phi$. Moreover, $\mathcal{H}_\Phi = \mathcal{L}_\Phi = \mathcal{D}$.

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