On scalarization methods for sets and generalized cone-continuity for set-valued maps*

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Dedicate to Professor Lai-Jiu Lin for his 77th birthday celebration.

1 Introduction

It is a key point in this paper that a composite function is a function which is the nesting of two or more functions to form a single new function. Such operation frequently preserves several mathematical properties of each nested function. For instance, a composition of continuous maps is continuous on topological spaces. This kind of mathematical inheritation is very useful and productive to get new results in Mathematics. On the other hand, in the theory of global convergency for algorithm in optimization, it is well-known that a sequence defined by $x^{k+1} \in A(x^k)$ for some algorithm map $A : X \mapsto 2^X$ has a global convergence property if (i) the set-valued map A is upper semicontinuous, (ii) there is a real-valued continuous function $Z : X \mapsto \mathbb{R}$ such that $Z(y) < Z(x)$ for any $y \in A(x)$, and (iii) the sequence is contained in a compact set in X. This property is one of the properties for a composition of a set-valued map and its scalarization function.

From the view point of vector optimization and set optimization, this kind of inheritance by composite operations is important and useful to prove extended results and to get characterizations of optimal solutions through scalarization. This is a typical approach by which optimization problems with vector-valued or set-valued maps can be easily handled by converting vectors or sets into real numbers; see [4] and [5, 6].

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Therefore, it is interesting to investigate the inheritance of cone continuity for set-valued maps via general scalarization functions for sets shown in [1]. Besides, by transforming set relations to twofold scalar optimization problems of some (real-valued) characterizing functions, the set-relations in [10] can be established by solving real-valued twofold optimization problems; see [7]. For each of the six set-relations, some necessary conditions and sufficient conditions are established on the negativity or the nonpositivity of the associated twofold scalar problems.

The aim of the paper based on [1, 7] is to intrduce the mechanism by which composite functions of a set-valued map and a scalarization function transmit semicontinuity of parent set-valued maps through several scalarization functions for sets, and to show the method how to characterize the set-relation through some characterizing functions for the convex cone.

2 Scalarizing functions for vectors

A linear functional on a real vector space is a bilinear form as a function of two variables of the original space and its dual space; it is an inner product of two vectors in the case of a finite-dimensional space. Also, it is one of the most useful tools for evaluation with respect to some index of the adequacy of efficiency in multiobjective programming or vector optimization. This is a typical approach to comparing vectors or sets, and it is referred to as "scalarization"; see [5]. Each object is mapped to an (extended) real number so that any two objects become comparable as elements of the totally ordered space $\mathbb{R}$ (or $\overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$). Naturally, converting vectors or sets into scalars involves a decrease of the amount of information; however, a well-designed conversion makes the problem less significant and can be a powerful tool. A norm $\|\cdot\|$ and an inner product $\langle w, \cdot \rangle$ with a fixed vector w are familiar examples of a scalarization for vectors. In addition, Gerstewitz's (Tammer's) sublinear scalarizing functional has exerted a prominent presence in vector optimization (e.g., see [5, Sections 2.3 and 3.1]):

$$h_C(v; d) := \inf \{t \in \mathbb{R} \mid v \in td - C\} \tag{1}$$

where C is a convex cone and $d \in C$. This scalarizing functional $h_C(\cdot; d)$ is sublinear (i.e., $h_C(v_1 + v_2; d) \leq h_C(v_1) + h_C(v_2)$ and $h_C(tv; d) = th_C(v; d)$ for $t > 0$) and hence this conversion is called "sublinear scalarization," found early in [3]. If convex cone C is a half space, that is, $C = \{v \mid \langle w, v \rangle \geq 0\}$ with w satisfying $\langle w, w \rangle = 1$, then $h_C(v; w) = \langle w, v \rangle$. Therefore this special functional is a certain generalization of linear scalarization including the notions of weighted sum and inner product. Accordingly, this idea has inspired some researchers to develop particular scalarization methods for sets, leading to several applicative results.

For example, we consider a scalarizing function $\psi \colon V \to \overline{\mathbb{R}}$ for vectors in a real vector space V. Let θ_V and C be the zero vector and a (nonempty) convex cone in V with $\theta_V \in C$. We define a preorder $\leq_C$ in V induced by C as follows: for v_1,

$v_2 \in V$, $v_1 \leq_C v_2 \overset{\text{def}}{\iff} v_2 - v_1 \in C$. This preorder is compatible with the linear structure:

$$v_1 \leq_C v_2 \implies v_1 + v_3 \leq_C v_2 + v_3 \text{ for all } v_1, v_2, v_3 \in V; \tag{2}$$

$$v_1 \leq_C v_2 \implies tv_1 \leq_C tv_2 \text{ for all } v_1, v_2 \in V \text{ and } t > 0. \tag{3}$$

For well-designed conversions in scalarization, we need the following "order-monotone" (that is, order preserving) property:

$$v_1 \leq_C v_2 \implies \psi(v_1) \leq \psi(v_2).$$

If the scalarizing function is an inner product, that is, $\psi(v) = \langle w, v \rangle$ with the weight vector w chosen as an element of the dual cone C^* in V^*, where

$$C^* = \{w \in V^* \mid \langle w, v \rangle \geq 0 \quad \forall v \in C\},$$

ψ has the order-monotone property. Owing to this property, any minimal or maximal element of a convex set in a vector optimization problem is characterized by optimal solutions of its scalarized problem with a certain nonzero weight vector in C^*, which is guaranteed by separation theorems for two convex sets.

3 Inheritance of cone continuity by scalarization

Throughout the paper, let X be a topological space and Y a real topological vector space. Let θ_Y be the zero vector in Y and $\mathcal{P}(Y)$ denote the set of all nonempty subsets of Y. The topological interior, topological closure, convex hull, and complement of a set $A \in \mathcal{P}(Y)$ are denoted by $\operatorname{int} A$, $\operatorname{cl} A$, $\operatorname{co} A$, and A^{c}, respectively. Let $\mathcal{N}(x)$ and $\preccurlyeq$ be a neighborhood system of a point $x \in X$ and a binary relation on $\mathcal{P}(Y)$, respectively. Furthermore, we assume that C is a convex cone in Y with $\operatorname{int} C \neq \emptyset$ and $\theta_Y \in C$. Then, $C + C = C$ holds, and $\operatorname{int} C$ and $\operatorname{cl} C$ are also convex cones. Accordingly, we can define a preoder $\leq_C$ on Y induced by C as follows:

$$\text{for } y_1, y_2 \in Y, y_1 \leq_C y_2 \overset{\text{def}}{\iff} y_2 - y_1 \in C.$$

When C is pointed (i.e., $C \cap (-C) = \{\theta_Y\}$), $\leq_C$ is antisymmetric and then a partial order. In this section, we consider the transmiting mechanism for the semicontinuity of parent set-valued maps through several scalarization functions for sets.

Definition 1 (Definition 2.6 in [1])**.** Let $F : X \to \mathcal{P}(Y)$, $x_0 \in X$, $\preccurlyeq$ a binary relation on $\mathcal{P}(Y)$ and $C \subset Y$ a convex cone. We say that F is $(\preccurlyeq, C)$-continuous at x_0 if

$$\forall W \subset Y, W \text{ open }, W \preccurlyeq F(x_0), \exists V \in \mathcal{N}_X(x_0) \text{ s.t. } W + C \preccurlyeq F(x), \forall x \in V.$$

Remark 2. If $C = \{0\}$ then $(\preccurlyeq, C)$-continuity for set-valued maps becomes $\preccurlyeq$-continuity in Definition 3.2 in [8].

Definition 3 (Definition 2.9 in [1])**.** Let $\varphi : \mathcal{P}(Y) \to \mathbb{R} \cup \{\pm\infty\}$, $A_0 \in \mathcal{P}(Y)$, $\preccurlyeq$ a binary relation on $\mathcal{P}(Y)$, and C a convex cone in Y with $C \neq Y$. Then, we say that φ is

(i) $(\preccurlyeq, C)$-lower semicontinuous at A_0 if $\forall r < \varphi(A_0), \exists W \in \mathcal{P}(Y)$, W open, s.t. $W \preccurlyeq A_0$ and $r < \varphi(A), \forall A \in U(W + C, \preccurlyeq)$;

(ii) $(\preccurlyeq, C)$-upper semicontinuous at A_0 if $\forall r > \varphi(A_0), \exists W \in \mathcal{P}(Y)$, W open, s.t. $W \preccurlyeq A_0$ and $r > \varphi(A), \forall A \in U(W + C, \preccurlyeq)$,

where $U(V, \preccurlyeq) := \{A \in \mathcal{P}(Y) \,| V \preccurlyeq A\}$.

Remark 4. When $C = \{0\}$, $(\preccurlyeq, C)$-lower and $(\preccurlyeq, C)$-upper semicontinuities are coincident with $\preccurlyeq$-lower and $\preccurlyeq$-upper semicontinuities, respectively, which are introduced in Definition 3.3 of [8]. In Definition 3, we adopt that if $\varphi(A_0) = -\infty$ (resp. $+\infty$) then φ is $(\preccurlyeq, C)$-lower (resp. upper) semicontinuous at A_0.

Therefore, we can easily show the following results as generalizations of Theorems 3.1 and 3.2 in [8].

Theorem 5 (Theorem 3.2 in [1])**.** *Let $F : X \to \mathcal{P}(Y)$, $\varphi : \mathcal{P}(Y) \to \mathbb{R} \cup \{\pm\infty\}$, $x_0 \in X$, $\preccurlyeq$ a binary relation on $\mathcal{P}(Y)$, and $C \subset Y$ a convex cone. If F is $(\preccurlyeq, C)$-continuous at x_0 and φ is $(\preccurlyeq, C)$-lower semicontinuous at $F(x_0)$, then $\varphi \circ F$ is lower semicontinuous at x_0.*

Theorem 6 (Theorem 3.3 in [1])**.** *Let $F : X \to \mathcal{P}(Y)$, $\varphi : \mathcal{P}(Y) \to \mathbb{R} \cup \{\pm\infty\}$, $x_0 \in X$, $\preccurlyeq$ a binary relation on $\mathcal{P}(Y)$, and $C \subset Y$ a convex cone. If F is $(\preccurlyeq, C)$-continuous at x_0 and φ is $(\preccurlyeq, C)$-upper semicontinuous at $F(x_0)$, then $\varphi \circ F$ is upper semicontinuous at x_0.*

Remark 7. Theorems 5 and 6 are generalizations of and reduced to Theorems 3.1 and 3.2 in [8] whenever $C = \{0\}$; see Remarks 2 and 4.

Besides, in case of singleton function, both theorems generalize a usual inheritance property for a composite function of a set-valued map and a continuous linear functional. If f is C-continuous (vector-valued) function from X to Y then the composite function $\varphi \circ f$ is lower semicontinuous for each $\varphi \in C^+$, where

$$C^+ := \{y^* \in Y^* \,| y^*(y) \geq 0 \;\forall y \in C\}$$

and Y^* denotes the set of all continuous linear functionals on Y. Similarly, if f is $(-C)$-continuous function from X to Y then the composite function $\varphi \circ f$ is upper semicontinuous for each $\varphi \in C^+$. For instance, refer to Proposition 2.3 in [11].

4 Characterization of set-relations by scalar problems

As generalizations of partial orderings for vectors, we give a definition of certain binary relations between sets in Y, called set relations.

Definition 8 (set relations, [10]). For $A, B \in \mathcal{P}(Y)$, we define the following eight types of binary relations on $\mathcal{P}(Y)$.

(i) $A \preccurlyeq_C^{(1)} B \overset{\text{def}}{\iff} \forall a \in A,\ \forall b \in B,\ a \leq_C b \iff A \subset \bigcap_{b\in B}(b - C) \iff B \subset \bigcap_{a\in A}(a + C)$;

(ii) $A \preccurlyeq_C^{(2L)} B \overset{\text{def}}{\iff} \exists a \in A \text{ s.t. } \forall b \in B,\ a \leq_C b \iff A \cap \left(\bigcap_{b\in B}(b - C)\right) \neq \emptyset$;

(iii) $A \preccurlyeq_C^{(2U)} B \overset{\text{def}}{\iff} \exists b \in B \text{ s.t. } \forall a \in A,\ a \leq_C b \iff \left(\bigcap_{a\in A}(a + C)\right) \cap B \neq \emptyset$;

(iv) $A \preccurlyeq_C^{(2)} B \overset{\text{def}}{\iff} A \preccurlyeq_C^{(2L)} B \text{ and } A \preccurlyeq_C^{(2U)} B \iff A \cap \left(\bigcap_{b\in B}(b - C)\right) \neq \emptyset \text{ and } \left(\bigcap_{a\in A}(a + C)\right) \cap B \neq \emptyset$;

(v) $A \preccurlyeq_C^{(3L)} B \overset{\text{def}}{\iff} \forall b \in B,\ \exists a \in A \text{ s.t. } a \leq_C b \iff B \subset A + C$;

(vi) $A \preccurlyeq_C^{(3U)} B \overset{\text{def}}{\iff} \forall a \in A,\ \exists b \in B \text{ s.t. } a \leq_C b \iff A \subset B - C$;

(vii) $A \preccurlyeq_C^{(3)} B \overset{\text{def}}{\iff} A \preccurlyeq_C^{(3L)} B \text{ and } A \preccurlyeq_C^{(3U)} B \iff B \subset A + C \text{ and } A \subset B - C$;

(viii) $A \preccurlyeq_C^{(4)} B \overset{\text{def}}{\iff} \exists a \in A,\ \exists b \in B \text{ s.t. } a \leq_C b \iff A \cap (B - C) \neq \emptyset \iff (A + C) \cap B \neq \emptyset$.

For each of the six set-relations, we shall give necessary conditions and sufficient conditions for set-relations on two subsets in a vector space under several types of binary relations via the corresponding twofold scalar optimization problems such as inf-inf, inf-sup, sup-inf, sup-sup problems on the two sets for a characterizing function of an ordering convex cone.

Definition 9. Let $C \subset Y$ be a convex cone and $f : Y \to \mathbb{R} \cup \{+\infty\}$. We say that f is a characterizing function of $-C$ if

$$f(y) \leq 0 \iff y \in -C. \tag{4}$$

Remark 10. Condition (4) means that the 0-lower level set of f coincides with $-C$. This idea is introduced by Eichfelder and Jahn [2] and applied to the characterization of set-relation by Jahn [9].

Theorem 11 (Type 1; Theorem 3.5 in [7]). *Let $f : Y \to \mathbb{R} \cup \{+\infty\}$ be a characterizing function of $-C$. For $A, B \in \mathcal{P}(Y)$, the following statements hold:*

$$A \preccurlyeq_C^{(1)} B \iff \sup_{a\in A}\sup_{b\in B} f(a - b) \leq 0 \quad \left(\textit{equivalently,} \quad \sup_{b\in B}\sup_{a\in A} f(a - b) \leq 0\right).$$

Theorem 12 (Type 2L; Theorem 3.6 in [7]). *Let $f : Y \to \mathbb{R} \cup \{+\infty\}$ be a characterizing function of $-C$. For $A, B \in \mathcal{P}(Y)$, the following statements hold:*

(i) $A \preccurlyeq_C^{(2L)} B \implies \inf_{a \in A} \sup_{b \in B} f(a-b) \leq 0$;

(ii) $\inf_{a \in A} \sup_{b \in B} f(a-b) < 0 \implies A \preccurlyeq_C^{(2L)} B$;

(iii) if A is compact and $a \mapsto f(a-b)$ is l.s.c. at any $a \in A$ for each $b \in B$, then

$$\inf_{a \in A} \sup_{b \in B} f(a-b) \leq 0 \implies A \preccurlyeq_C^{(2L)} B.$$

Theorem 13 (Type 2U; Theorem 3.7 in [7]). *Let $f : Y \to \mathbb{R} \cup \{+\infty\}$ be a characterizing function of $-C$. For $A, B \in \mathcal{P}(Y)$, the following statements hold:*

(i) $A \preccurlyeq_C^{(2U)} B \implies \inf_{b \in B} \sup_{a \in A} f(a-b) \leq 0$;

(ii) $\inf_{b \in B} \sup_{a \in A} f(a-b) < 0 \implies A \preccurlyeq_C^{(2U)} B$;

(iii) if B is compact and $b \mapsto f(a-b)$ is l.s.c. at any $b \in B$ for each $a \in A$, then

$$\inf_{b \in B} \sup_{a \in A} f(a-b) \leq 0 \implies A \preccurlyeq_C^{(2U)} B.$$

Theorem 14 (Type 3L; Theorem 3.9 in [7]). *Let $f : Y \to \mathbb{R} \cup \{+\infty\}$ be a characterizing function of $-C$. For $A, B \in \mathcal{P}(Y)$, the following statements hold:*

(i) $A \preccurlyeq_C^{(3L)} B \implies \sup_{b \in B} \inf_{a \in A} f(a-b) \leq 0$;

(ii) $\sup_{b \in B} \inf_{a \in A} f(a-b) < 0 \implies A \preccurlyeq_C^{(3L)} B$;

(iii) if A is compact and $a \mapsto f(a-b)$ is l.s.c. at any $a \in A$ for each $b \in B$, then

$$\sup_{b \in B} \inf_{a \in A} f(a-b) \leq 0 \implies A \preccurlyeq_C^{(3L)} B.$$

Theorem 15 (Type 3U; Theorem 3.10 in [7]). *Let $f : Y \to \mathbb{R} \cup \{+\infty\}$ be a characterizing function of $-C$. For $A, B \in \mathcal{P}(Y)$, the following statements hold:*

(i) $A \preccurlyeq_C^{(3U)} B \implies \sup_{a \in A} \inf_{b \in B} f(a-b) \leq 0$;

(ii) $\sup_{a \in A} \inf_{b \in B} f(a-b) < 0 \implies A \preccurlyeq_C^{(3U)} B$;

(iii) if B is compact and $b \mapsto f(a-b)$ is l.s.c. at any $b \in B$ for each $a \in A$, then

$$\sup_{a\in A}\inf_{b\in B} f(a-b) \le 0 \implies A \preccurlyeq_C^{(3U)} B.$$

Theorem 16 (Type 4; Theorem 3.13 in [7]). *Let $f : Y \to \mathbb{R} \cup \{+\infty\}$ be a characterizing function of $-C$. For $A, B \in \mathcal{P}(Y)$, the following statements hold:*

(i) $A \preccurlyeq_C^{(4)} B \implies \inf_{a\in A}\inf_{b\in B} f(a-b) \le 0 \quad \left(\textit{equivalently,} \quad \inf_{b\in A}\inf_{b\in B} f(a-b) \le 0\right)$*;*

(ii) $\inf_{a\in A}\inf_{b\in B} f(a-b) < 0 \quad \left(\textit{equivalently,} \quad \inf_{b\in A}\inf_{b\in B} f(a-b) \le 0\right) \implies A \preccurlyeq_C^{(4)} B$*;*

(iii) if A and B are compact and if $(a,b) \mapsto f(a-b)$ is l.s.c. at any $(a,b) \in A \times B$, then

$$\inf_{a\in A}\inf_{b\in B} f(a-b) \le 0 \implies A \preccurlyeq_C^{(4)} B.$$

These results would be useful to computational problems including set optimization. If we consider two subsets A and B in the paper as images of a set-valued map, it would be an interesting research direction. Finally, we would like to express our gratitude to Prof. S. Atsushiba and Prof. Y. Kimura as the organizers of this workshop and to the Research Institute for Mathematical Sciences, an International Joint Usage/Research Center located in Kyoto University for their supports.

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