Zeros of System of Sums of Two Maximally Monotone Operators and its Applications With Parallel Douglas-Rachford Algorithm

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1 Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let $M : H \multimap H$ be a multivalued mapping, let $dom(M) = \{x \in H : M(x) \neq \emptyset\}$ denotes the effective domain of M. M is monotone on H iff $\langle x-y, u-v \rangle \geq 0$ for all $x, y \in dom(M), u \in Mx$, and $v \in My$. M is maximally monotone on H iff M is a monotone operator on H and its graph

$$gra(M) = \{(x, y) \in H \times H : y \in M(x)\}$$

is not properly contained in the graph of any other monotone operator on H. In this case, J_β^M is defined by $J_\beta^M(x) := (I + \beta M)^{-1}(x)$ for each $\beta > 0$, and $x \in H$.

Let H be a real Hilbert space and let A and B be maximally monotone operators on H. The problem of finding a zero of $A + B$ is the problem:

(**Problem Q**) Find $x \in H$ such that $x \in (A + B)^{-1}0$.

(**Problem Q**) provides a powerful tool to study the problems in optimization theory, nonlinear analysis, medical science, data analysis and signal recovery. One can refer to sections 3-6 of this paper.

Let A and B be maximally monotone operators on H, let $\nu > 0$, let $R_{\nu A} = 2J_\nu^A - I$ be reflected Rachford resolvent of A and $T := T_{A,B} = \frac{1}{2}(I + R_{\nu B} R_{\nu A})$ be the Douglas-Rachford algorithm associated with (A, B). In 1956, the Douglas-Rachford algorithm was first introduced in [13] to numerical solution for certain heat equation. In 1976, Rockafellar [22]studied the zero solution of a maximally monotone operator. In 1979, Lions and Mercier [19] and Svaiter [23] used the Douglas-Rachford algorithm to establish the following celebrated result.

Theorem 1.1. [19, 23] Let H be a Hilbert space and let A and
B be set-valued maximally monotone operators from H to H with resolvent
$J_\nu^A = (I+\nu A)^{-1}, J_\nu^B = (I+\nu B)^{-1}$.
Suppose that $(A+B)^{-1}0 = \{x \in H : 0 \in Ax + Bx\} \neq \emptyset$,
let $x_0 \in H$ and iterate $\forall n \in \mathbb{N}$,
$y_n = J_\nu^B x_n$ and
$x_{n+1} = x_n + J_\nu^A(2y_n - x_n) - y_n$.
Then the following hold for some $(x, y) \in graJ_\nu^B$.
(i) $x = J_\nu^A(2y-x), x_n \rightharpoonup x$ and $y \in (A+B)^{-1}0$ for some $(x,y) \in graJ_\nu^B$;
(ii) $y_n \rightharpoonup y$.

It is well known that the Douglas-Rachford algorithm converges weakly to (**Problem Q)**. There are many results related to the solution of (**Problem Q)** and its applications. One can refer to [1, 2, 12, 14] and references therein.
For each $i \in \{1, 2, \cdots, m\}$,

(i) Let H_i (resp. H) be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in H and $x \in H$, $x_n \to x$ and $x_n \rightharpoonup x$ denote the strong convergence and the weak convergence of $\{x_n\}_{n\in\mathbb{N}}$ to x respectively.Let I_i (resp I) be the identity mapping on H_i (resp H);

(ii) Let $(\lambda_n)_{n\geq 0}$ and $(\beta_n)_{n\geq 0}$ be real sequence satisfying the conditions; (i) $0 < \beta_n \leq 1$ for any $n \geq 0$, $\lim\limits_{n\to\infty} \beta_n = 1, \sum\limits_{n\geq 0}(1-\beta_n) = \infty$ and
$\sum_{n\geq 1} |\beta_n - \beta_{n-1}| < \infty$;

(ii) $0 < \lambda_n \leq 1$ for any $n \geq 0, \lim\inf_{n\to\infty} \lambda_n > 0, \sum_{n\geq 1} |\lambda_n - \lambda_{n-1}| < \infty$;

(iii) Let$M_i : H \multimap H, G_i : H \multimap H$ be maximally monotone operators.

Throughout this paper,we use these notations unless specified otherwise.
Recently Bot et al. [3] establishes a strong convergence theorem for finding the solution for (**Problem Q)** with Douglas-Rachford algorithm.

Theorem 1.2. [3] Let $A, B : H \multimap H$ be maximally monotone operators such that $(A + B)^{-1}0 \neq \emptyset$, let $\nu > 0$. Let $x_0 \in H$ and $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ be sequences defined by
$y_n = J_\nu^{B_1}(\beta_n x_n)$;
$x_{n+1} = \beta_n x_n + \lambda_n[J_\nu^A(2y_n - \beta_n x_n) - y_n]$.

Then the following are true:
(i)$x = \lim_{n\to\infty} x_n = P_{Fix(R_{\nu A}R_{\nu B})}0$,
(ii) $\lim_{n\to\infty} y_n = J_\nu^B x \in (A + B)^{-1}0$.

□

Let $x_0^1 \in H$, $x_0^2 \in H$, $\cdots$, $x_0^m \in H$, $x_0 = (x_0^1, x_0^2, \cdots, x_0^m)$ and let the sequences $\{u_n^j\}_{n\in\mathbb{N}}, \{y_n^j\}_{n\in\mathbb{N}}$, and parallel Douglas-Rachford algorithm $\{x_n^j\}_{n\in\mathbb{N}}$, $j \in \{1, 2, \cdots, m\}$ be defined by

(i) $u_n^1 = x_n^1 - \frac{1}{m^2}((m-1)x_n^1 - x_n^2 - \cdots - x_n^m)$;
$u_n^m = x_n^m - \frac{1}{m^2}((m-1)x_n^m - x_n^1 - \cdots - x_n^{m-1})$;
$u_n^j = x_n^j - \frac{1}{m^2}((m-1)x_n^j - x_n^1 - \cdots - x_n^{j-1} - x_n^{j+1} - \cdots - x_n^m), 1 < j < m$;

(ii) $y_n^j = J_\nu^{G_j}(\beta_n u_n^j), j \in \{1, 2, \cdots, m\}$;

(iii) $x_{n+1}^j = \beta_n x_n^j + \lambda_n[\beta_n u_n^j + J_\nu^{M_j}(2y_n^j - \beta_n u_n^j) - y_n^j - \beta_n x_n^j]$.

Motivated and inspired by the above results, in this paper, we study zeros of system of sums of two maximally monotone operators with parallel Douglas-Rachford algorithm. We study the following problem:

(**Problem** P**)** Find $(\bar{x}^1, \bar{x}^1, \cdots, \bar{x}^1) \in \Lambda$ such that

(i) $\lim_{n\to\infty} x_n = (\bar{x}^1, \bar{x}^1, \cdots, \bar{x}^1) = P_\Lambda 0$, where $\Lambda = \{(x^1, x^1, \cdots, x^1) \in \bigotimes_{1\le i\le m} H : x^1 \in \bigcap_{i=1}^m Fix(R_{\nu M_i} R_{\nu G_i})\}$, and $x_n = (x_n^1, x_n^2, \cdots, x_n^m)$;

(ii) $\lim_{n\to\infty} J_\nu^{G_j}(x_n^j) = J_\nu^{G_j}\bar{x}^1 \in (M_j + G_j)^{-1}0$ for all $j \in \{1, 2, \cdots, m\}$.

By **Problem** P, we study the solution to the following problems:

(1) (**Problem Q**); (2) common zeros to finite maximally monotone operators; (3)solutions of convex feasibility problem, the method and algorithm of Problem (3) are different from Theorem 29.2(Dykstra's algorithm [1]); (4)solutions of system of mixed type variational inequalities; (5)solutions of system of constrained variational inequalities; (6) solutions of system of Stampachia variational inequalities; (7)solutions of system of Lax-Milgram type problems; (8)solutions of Stampachia variational inequality; (9)solutions of Lax-Milgram type problem; (10) solutions of system of optimization problems for the sum of two convex functions; (11) solutions of system of approximation problem; (12) solutions of system of constrained optimization problems for convex functions; (13) solution of system of signal recovery problems; (14) solutions of system of signal recovery problems; (15) solutions of system of split feasibility problem.

We study solution of system of feasibility problem without knowing the norms of the operator we consider.

Our result for solution to system of constrained variational inequalities is different from Theorem 3.6 [9]. In order to study zeros of system of sums of two maximally monotone operators (**Problem P**) , we introduce a new parallel algorithm to find a common fixed point for a family of α_i- averaged mappings. Then we apply this common fixed point theorem to study (**Problem P)** with parallel Douglas-Rachford algorithm and strong convergence theorems are established for (**Problem P**). We also apply our result for (**Problem P**) to study solutions of problems(1) to (15) with parallel Douglas-Rachford algorithm and strong convergence theorems are established for these problems.

2 Preliminaries

For each $x \in H$, there is a unique element $\bar{x} \in C$ such that $||x - \bar{x}|| = \min_{y \in C} ||x - y||$. In this study, we set $P_C x = \bar{x}$, and P_C is called the metric projection from H onto C. Let $T : C \to H$ be a mapping, we denote $Fix(T) := \{x \in C : Tx = x\}$. Let $T : C \to H$ be a mapping,

(i) T is nonexpansive iff $||Tx - Ty|| \leq ||x - y||$ for all $x, y \in C$;

(ii) T is demiclosed iff for each sequence $\{x_n\}$ and
x in C with $x_n \rightharpoonup x$ and $(I - T)x_n \to 0$ implies that $(I - T)x = 0$;

(iii) T is firmly nonexpansive iff $\|Tx - Ty\|^2 + \|(I - T)x - (I - T)y)\|^2 \leq \|x - y\|^2$ for all $x, y \in C$.

(iv) $T : C \to H$ is averaged or $\alpha-$ averaged if there exist $\alpha \in (0, 1)$ and a nonexpansive mapping $S : C \to H$ such that $T = (1 - \alpha)I + \alpha S$.

Let $I_i : H_i \to H_i$ (resp. $I : H \to H$) be the identity mapping on H_i(resp. H), Let $\Gamma_0(H_i)$(resp.$\Gamma_0(H)$) denote the set of all proper, lower semi-continuous, and convex functions from H_i(resp. H) into $(-\infty, \infty]$. Let $g : H \to]-\infty, \infty[$ and let $\gamma > 0$. The Moreau-Yosida approximation of g of order γ is

$$^{\gamma}g(x) = \inf_{y \in \mathbb{H}} g(y) + \frac{1}{2\gamma}\|x - y\|^2.$$

Let $g \in \Gamma_0(H)$ and $\lambda \in]0, \infty[$. The proximal operator of $g \in \Gamma_0(H)$ of order $\lambda \in]0, \infty[$ is

$$prox_{\lambda g}x = \arg\min_{v \in H}\{g(v) + \frac{1}{2\lambda}\|v - x\|^2\}, \ x \in H.$$

The variational inequality for an operator $F : C \to H$ is the following problem: Find $\bar{x} \in C$ such that $\langle F\bar{x}, y - \bar{x}\rangle \geq 0$ for all $y \in C$.
Let $f \in \Gamma_0(H)$. Then the subdifferential ∂f of f is defined by

$$\partial f(x) = \{u \in H : f(y) \geq f(x) + \langle y - x, u\rangle \text{ for all } y \in H\}.$$

The indicator function $\iota_C : H \to [0, \infty]$ is defined by

$$\iota_C(x) = \begin{cases} 0, & x \in C \\ \infty, & x \notin C \end{cases}$$

is a proper, lower semicontinuous, convex function.

Let $x \in H$, let $V(x)$ be the family of all neighborhood of x , let $W \in V(x)$ and let $f : W \to \mathbb{R}$. Then f is said to be Fréchet differentiable at $x \in H$ iff there exists an operator $\nabla f(x) \in H$ such that
$\lim\limits_{0 \neq \|y\| \to 0} \dfrac{|f(x+y) - f(x) - \langle y, \nabla f(x)\rangle|}{\|y\|} = 0.$
Here, $\nabla f(x) \in H$ is call the Fréchet derivative of f at x.

Lemma 2.1. [4] Let $T : C \to H$ be a nonexpansive mapping, and let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in C. If $x_n \rightharpoonup w$ and $\lim\limits_{n\to\infty} ||x_n - Tx_n|| = 0$, then $Tw = w$.

Proposition 2.1. [1] Let $T : C \to H$ be nonexpansive, and $\alpha \in (0, 1)$.Then the following are equivalent:
(i) T is α-averaged;
(ii)$(\forall x \in C)(\forall y \in C), \|Tx - Ty\|^2 \leq \|x - y\|^2 - \frac{1-\alpha}{\alpha}\|(I - T)x - (I - T)y\|^2$.

Lemma 2.2. [1] Let $T : H \to H$ be monotone and hemicontinuous, ie.,for every $(x, y, z) \in H^3, \lim_{\alpha\downarrow 0}\langle z, T(x + \alpha y)\rangle = \langle z, Tx\rangle$. Then T is maximally monotone.

Lemma 2.3. [21] Let $f \in \Gamma_0(H)$, then ∂f is maximally monotone.

Lemma 2.4. [1] Let $g \in \Gamma_0(H)$ and $\lambda \in]0, \infty[$. Then,
(i)$prox_{\lambda g} = (I + \lambda \partial g)^{-1} = J_{\lambda}^{\partial g}$;
(ii)$prox_{\lambda g}$ is firmly nonexpansive;
(iii) If C is a nonempty, closed and convex subset of H and $g = \iota_C$, then $prox_{\lambda g} = P_C$ for all $\lambda \in]0, \infty[$.

Lemma 2.5. [1] Let $A : H \multimap H$ and $B : \multimap H$ be maximally monotone operators such that one of the following holds:
(i) $domB = H$; (ii) $domA \bigcap int(domB) \neq \emptyset$; (iii)$0 \in (domA - domB)$.
Then $A + B$ is maximally monotone.

Lemma 2.6. [1] Let C be a nonempty closed convex subset of a Hilbert space H. Then the projector P_C is firmly nonexpansive.

Lemma 2.7. [1] Let $A : H \multimap A$ be maximally monotone and let $\nu > 0$. Then The reflection resolvent $R_{\nu A} : A \to H : x \to 2J_{\nu}^{A}x - x$ is nonexpansive.

Lemma 2.8. [1] Let $A : H \multimap H$ and $B : H \multimap H$ be monotone, let $R_{\nu A}$ and $R_{\nu B}$ be the reflection resolvents and let $\nu > 0$. Then $(A + B)^{-1}0 = J_{\nu}^{B} Fix(R_{\nu A} R_{\nu B})$.

Lemma 2.9. [10] Let T and $S : C \to H$ be mappings. Then the following are satisfied:
(i) If S and T are both averaged, then the product (composite) ST is averaged;
(ii) If the mappings $\{T_i\}_{i=1}^{n}$ are averaged and have a common fixed point, then $\bigcap_{i=1}^{n} Fix(T_i) = Fix(T_1 \cdots T_n)$.

Lemma 2.10. [1] Let $f \in \Gamma_0(H)$ and let $\gamma > 0$. Then ${}^{\gamma}f : H \to \mathbb{R}$ is Fréchet differentiable on H, and its gradient
$\nabla^{\gamma} f = \gamma^{-1}(I - prox_{\gamma f})$
is γ^{-1}-Lipschitz continuous.

Proposition 2.2. [1] Let $f : H \to]-\infty, \infty]$ be convex and let $\gamma > 0$.Then ${}^{\gamma}f$ is convex.

Lemma 2.11. [5] For each $i \in \{1, 2, \cdots, m\}$, let $\rho_i > 0, S_i : H \to H$ be a demiclosed ρ_i-strongly quasi-nonexpansive mapping, and let $S = S_1 S_2 \cdots S_m$. Suppose that $\bigcap_{i=1}^{m} Fix(S_i) \neq \emptyset$. Then S is a demiclosed $\rho-$ strongly quasi-nonexpansive mapping with $\rho = (\sum_{i=1}^{m} \rho_i^{-1})^{-1}$.

Proposition 2.3. [5] Let $A : H_1 \to H_2$ be a bounded linear operator with $\|A\| > 0$ and $T : H_2 \to H_2$ be an operator satisfying $TAw = Aw$ for some $w \in H_1$. Further let $V = I_1 - \frac{1}{\|A\|^2} A^*(I_2 - T)A$. If T is an $\alpha - SQNE$ operator for some $\alpha \geq 0$, then
(i) $Fix(V) = A^{-1} Fix(T)$;
(ii)V is $\alpha - SQNE$.
If T is demiclosed , then V is demiclosed .

Lemma 2.12. Stampachia [1] Let C be a nonempty closed convex subset of H, let $F : H \times H \to \mathbb{R}$ be a symmetric bilinear form, $\beta > 0, \alpha > 0$. Let $\ell \in B(H, \mathbb{R})$, and let $f : H \times H \to \mathbb{R}$ be defined by $f(x) = \frac{1}{2} F(x, x) - \ell(x)$, where $B(H, \mathbb{R})$ is the set of all bounded linear operators from H to $\mathbb{R}$. Suppose that
$|F(x, y)| \leq \beta \|x\| \|y\|$ and $F(x, x) \geq \alpha \|x\|^2$ for all $x \in H$ and for all $y \in H$.
Then there exists $\bar{x} \in H$ such that
$\bar{x} \in C$, $F(\bar{x}, y - \bar{x}) \geq \ell(y - \bar{x})$ for all $y \in C$.

Lemma 2.13. [1] Let $f, g \in \Gamma_0(H)$ such that one of the following holds:
(i) $dom \partial g = H$; (ii) $dom \partial f \bigcap int(dom \partial g) \neq \emptyset$; (iii)$0 \in (dom \partial f - dom \partial g)$.
Then $\partial(f + g) = \partial f + \partial g$.

3 Zeros of system of sums of two maximally monotone mappings

For each $i \in \mathbb{N}, j = 1, 2, \cdots, m + 1$, let
(i) $A_i : H_i \to H_{m+1}$ be a bounded linear operator with adjoint A_i^* and $\|A_i\| > 0$;
(ii)The product $\bigotimes_{1 \leq i \leq m} H_i = H_1 \times H_2 \times \cdots \times H_m$ is a Hilbert space with inner product

given by $\langle x, y\rangle = \sum_{i=1}^{m}\langle x_i, y_i\rangle$, and norm given by $\|x\| = (\sum_{i=1}^{m}\|x_i\|^2)^{\frac{1}{2}}$ for any $x = (x_1, x_2, \cdots, x_m) \in \bigotimes_{1\le i\le m} H_i, y = (y_1, y_2, \cdots, y_m) \in \bigotimes_{1\le i\le m} H_i$.

(iii) $M_i : H \multimap H, G_i : H \multimap H$ be maximally monotone operators, C_i ,$D_i$and C be nonempty closed convex subsets of H, $f_i \in \Gamma_0(H)$, and $g_i \in \Gamma_0(H)$;

(iv)$x_0^1 \in H,\ x_0^2 \in H,\ \cdots,\ x_0^m \in H$;

(v)$u_n^1 = x_n^1 - \frac{1}{m^2}((m-1)x_n^1 - x_n^2 - \cdots - x_n^m)$;

$u_n^m = x_n^m - \frac{1}{m^2}((m-1)x_n^m - x_n^1 - \cdots - x_n^{m-1})$;

$u_n^j = x_n^j - \frac{1}{m^2}((m-1)x_n^j - x_n^1 - \cdots - x_n^{j-1} - x_n^{j+1} - \cdots - x_n^m), 1 < j < m$.

Throughout this paper,we use these notations unless specified otherwise.

Theorem 3.1. [3]Let $x_0 \in H$, let $T : H \to H$ be a nonexpansive mapping such that $FixT \neq \emptyset$ and let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence defined by

$x_{n+1} = \beta_n x_n + \lambda_n(T(\beta_n x_n) - \beta_n x_n)$ for all $n \ge 0$.

Then $\lim\limits_{n\to\infty} x_n = P_{Fix(T)}0$.

$\square$

Theorem 3.2. For each $j \in \{1, 2, \cdots, m\}$, let $\rho_j > 0$, and let $T_j : H_j \to H_j$ be ρ_j-averaged. Suppose that

$\Lambda = \{(x^1, x^2, \cdots, x^m) \in \bigotimes_{1\le i\le m} H_i : x^j \in Fix(T_j), j \in \{1, 2, \cdots, m\},$

$A_1(x^1) = A_2(x^2) = \cdots = A_m(x^m)\} \neq \emptyset$.

Let $x_0^1 \in H_1,\ x_0^2 \in H_2,\ \cdots,\ x_0^m \in H_m,\ x_0 = (x_0^1, x_0^2, \cdots, x_0^m)$ and let the sequences $\{u_n^j\}_{n\in\mathbb{N}}$, and $\{x_n^j\}_{n\in\mathbb{N}}$, $j \in \{1, 2, \cdots, m\}$ be defined by

(i)$v_n^1 = x_n^1 - \frac{1}{m\sum_{i=1}^{m}\|A_i\|^2}A_1^*((m-1)A_1(x_n^1) - A_2(x_n^2) - \cdots - A_m(x_n^m))$;

$v_n^m = x_n^m - \frac{1}{m\sum_{i=1}^{m}\|A_i\|^2}A_m^*((m-1)A_m(x_n^m) - A_1(x_n^1) - \cdots - A_{m-1}(x_n^{m-1}))$;

$v_n^j = x_n^j - \frac{1}{m\sum_{i=1}^{m}\|A_i\|^2}A_j^*((m-1)A_j(x_n^j) - A_1(x_n^1) - \cdots - A_{j-1}(x_n^{j-1})$

$- A_{j+1}(x_n^{j+1}) - \cdots - A_m(x_n^m)), 1 < j < m$;

(ii)$x_{n+1}^j = \beta_n x_n^j + \lambda_n(T_j(\beta_n v_n^j) - \beta_n x_n^j)$, for all $n \ge 0$ and for all $j \in \{1, 2, \cdots, m\}$.

Let $x_n = (x_n^1, x_n^2, \cdots, x_n^m)$, then $\lim\limits_{n\to\infty} x_n = (\bar{x}^1, \bar{x}^2, \cdots, \bar{x}^m) = P_\Lambda 0$.

Theorem 3.3. For each $j \in \{1, 2, \cdots, m\}$, let $\rho_j > 0$, and let $T_j : H \to H$ be ρ_j-averaged. Suppose that
$\Lambda = \{(x^1, x^1, \cdots, x^1) \in \bigotimes_{1\leq i\leq m} H : x^1 \in \bigcap_{j=1}^{m} Fix(T_j)\} \neq \emptyset$.
Let sequences $\{x_n^j\}_{n\in\mathbb{N}}$, $j \in \{1, 2, \cdots, m\}$ be defined by:

(i)$x_{n+1}^j = \beta_n x_n^j + \lambda_n (T_j(\beta_n u_n^j) - \beta_n x_n^j), j \in \{1, 2, \cdots, m\}, n \geq 0$.
Let $x_n = (x_n^1, x_n^2, \cdots, x_n^m)$, then $\lim\limits_{n\to\infty} x_n = (\bar{x}^1, \bar{x}^1, \cdots, \bar{x}^1) = P_\Lambda 0$.

Theorem 3.4. Suppose that
$\Lambda = \{(x^1, x^1, \cdots, x^1) \in \bigotimes_{1\leq i\leq m} H : x^1 \in \bigcap_{i=1}^{m} Fix(R_{\nu M_i} R_{\nu G_i})\} \neq \emptyset$.
Let the sequences $\{x_n^j\}_{n\in\mathbb{N}}$, and $\{y_n^j\}_{n\in\mathbb{N}}$, $j \in \{1, 2, \cdots, m\}$ be defined by
(i)$y_n^j = J_\nu^{G_j}(\beta_n u_n^j)$;
(ii)$x_{n+1}^j = \beta_n x_n^j + \lambda_n[\beta_n u_n^j + J_\nu^{M_j}(2y_n^j - \beta_n u_n^j) - y_n^j - \beta_n x_n^j]$
Then there exists $(\bar{x}^1, \bar{x}^1, \cdots, \bar{x}^1)$ in Λ such that
(i) $\lim\limits_{n\to\infty} x_n = (\bar{x}^1, \bar{x}^1, \cdots, \bar{x}^1) = P_\Lambda 0$;
(ii) $\lim\limits_{n\to\infty} J_\nu^{G_j}(x_n^j) = J_\nu^{G_j}\bar{x}^1 \in (M_j + G_j)^{-1}0$ for all $j \in \{1, 2, \cdots, m\}$.

Theorem 3.5. [3]Let $\nu > 0$. Suppose that $\Lambda = (M + G)^{-1}0 \neq \emptyset$. Let $M : H \multimap H, G : H \multimap H$ be maximally monotone operators and $\nu > 0$. Suppose that
$\Lambda = (M + G)^{-1}0 \neq \emptyset$.
Let $x_0 \in H$ and $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ be sequences defined by
$y_n = J_\nu^{G_1}(\beta_n x_n)$;
$x_{n+1} = \beta_n x_n + \lambda_n[J_\nu^M(2y_n - \beta_n x_n) - y_n]$.
Then there exists $x \in Fix(R_{\nu M} R_{\nu G})$ such that the following hold:
(i) $J_\nu^B x \in (M + G)^{-1}0$;
(ii)$x = \lim\limits_{n\to\infty} x_n = P_{Fix(R_{\nu M} R_{\nu G})}0$.

Theorem 3.6. Let $\nu > 0$. Suppose that
$\Lambda = \{(x^1, x^1, \cdots, x^1) \in \bigotimes_{1\leq i\leq m} H : x^1 \in \bigcap_{j=1}^{m} M_j^{-1}0\} \neq \emptyset$.
Let sequences $\{x_n^j\}_{n\in\mathbb{N}}$, and $\{y_n^j\}_{n\in\mathbb{N}}$, $j \in \{1, 2, \cdots, m\}$ be defined by

(i)$y_n^j = \beta_n u_n^j$;

(ii) $x_{n+1}^j = \beta_n x_n^j + \lambda_n[J_\nu^{M_j}(\beta_n u_n^j) - \beta_n x_n^j)], j \in \{1, 2, \cdots, m\}$.

Let $x_n = (x_n^1, x_n^2, \cdots, x_n^m)$, then there exists $(\bar{x}^1, \bar{x}^1, \cdots, \bar{x}^1)$ in Λ such that

(i) $\lim\limits_{n\to\infty} x_n = (\bar{x}^1, \bar{x}^1, \cdots, \bar{x}^1) = P_\Lambda 0$;

(ii)$\bar{x}^1 \in \bigcap_{j=1}^m M_j^{-1}0$.

Theorem 3.7. Let $\nu > 0$. Suppose that

$\Lambda = \{(x^1, x^1, \cdots, x^1) \in \bigotimes_{1\le i\le m} H : x^1 \in \bigcap_{j=1}^m C_j\} \neq \emptyset,.$

Let sequences $\{x_n^j\}_{n\in\mathbb{N}}$, and $\{y_n^j\}_{n\in\mathbb{N}}$, $j \in \{1, 2, \cdots, m\}$ be defined by

(i) $y_n^j = \beta_n u_n^j$;

(ii) $x_{n+1}^j = \beta_n x_n^j + \lambda_n[P_{C_j}(\beta_n u_n^j) - \beta_n x_n^j)], j \in \{1, 2, \cdots, m\}$.

Let $x_n = (x_n^1, x_n^2, \cdots, x_n^m)$, then there exists $(\bar{x}^1, \bar{x}^1, \cdots, \bar{x}^1)$ in Λ such that

(i) $\lim\limits_{n\to\infty} x_n = (\bar{x}^1, \bar{x}^1, \cdots, \bar{x}^1) = P_\Lambda 0$;

(ii)$\bar{x}^1 \in \bigcap_{j=1}^m C_j$.

Theorem 3.8. Let $\nu > 0$. Suppose that

$\Lambda = \{(x^1, x^1, \cdots, x^1) \in \bigotimes_{1\le i\le m} H : x^1 \in \bigcap_{j=1}^m \arg\min_{y\in H} f_j(y)\} \neq \emptyset.$

Let sequences $\{x_n^j\}_{n\in\mathbb{N}}$, and $\{y_n^j\}_{n\in\mathbb{N}}$, $j \in \{1, 2, \cdots, m\}$ be defined by

(i) $y_n^j = \beta_n u_n^j$;

(ii)$x_{n+1}^j = \beta_n x_n^j + \lambda_n[J_\nu^{\partial f_j}(\beta_n u_n^j) - \beta_n x_n^j)], j \in \{1, 2, \cdots, m\}$.

Let $x_n = (x_n^1, x_n^2, \cdots, x_n^m)$, then there exists $(\bar{x}^1, \bar{x}^1, \cdots, \bar{x}^1)$ in Λ such that

(i) $\lim\limits_{n\to\infty} x_n = (\bar{x}^1, \bar{x}^1, \cdots, \bar{x}^1) = P_\Lambda 0$;

(ii)$\bar{x}^1 \in \bigcap_{j=1}^m \arg\min_{y\in H} f_j(y)$.

4 Solutions of system of variational inequalities, solutions of system of Stampachia variational inequalities

Theorem 4.1. Let $\nu > 0$. Suppose that

$\Lambda = \{(x^1, x^1, \cdots, x^1) \in \bigotimes_{1\le i\le m} H : x^1 \in \bigcap_{j=1}^m Fix((2P_{C_j} - I)R_{\nu G_j})\} \neq \emptyset$.
Let sequences $\{x_n^j\}_{n\in\mathbb{N}}$, and $\{y_n^j\}_{n\in\mathbb{N}}$, $j \in \{1, 2, \cdots, m\}$ be defined by
(i)$y_n^j = J_\nu^{G_j}(\beta_n u_n^j)$;
(ii)$x_{n+1}^j = \beta_n x_n^j + \lambda_n[\beta_n u_n^j + P_{C_j}(2y_n^j - \beta_n u_n^j) - y_n^j - \beta_n x_n^j], j \in \{1, 2, \cdots, m\}$.
Let $x_n = (x_n^1, x_n^2, \cdots, x_n^m)$, then there exists $(\bar{x}^1, \bar{x}^1, \cdots, \bar{x}^1)$ in Λ
such that $J_\nu^{G_j}\bar{x}^1 \in C_j$,
(i) $\lim\limits_{n\to\infty} x_n = (\bar{x}^1, \bar{x}^1, \cdots, \bar{x}^1) = P_\Lambda 0$;
(ii)$\langle y^1 - J_\nu^{G_j}\bar{x}^1, v_j\rangle \ge 0$ for all $y^1 \in C_j$ and for all $j \in \{1, 2, \cdots, m\}$
for some $v_j \in G_j(J_\nu^{G_j}\bar{x}_1)$.

Corollary 4.1. Let $\nu > 0$. For each $j \in \{1, 2, \cdots, m\}$, let $G_i : H \to H$ be a monotone hemicontinuous mapping. Suppose that
$\Lambda = \{(x^1, x^1, \cdots, x^1) \in \bigotimes_{1\le i\le m} H : x^1 \in \bigcap_{j=1}^m Fix(2P_{C_j} - I)R_{\nu G_j})\} \neq \emptyset$.
Let sequences $\{x_n^j\}_{n\in\mathbb{N}}$, and $\{y_n^j\}_{n\in\mathbb{N}}$, $j \in \{1, 2, \cdots, m\}$ be defined by
(i)$y_n^j = J_\nu^{G_j}(\beta_n u_n^j)$;
(ii) $x_{n+1}^j = \beta_n x_n^j + \lambda_n[\beta_n u_n^j + P_{C_j}(2y_n^j - \beta_n u_n^j) - y_n^j - \beta_n x_n^j], j \in \{1, 2, \cdots, m\}$.
Let $x_n = (x_n^1, x_n^2, \cdots, x_n^m)$, then there exists $(\bar{x}^1, \bar{x}^1, \cdots, \bar{x}^1)$ in Λ such that $J_\nu^{G_j}\bar{x}^1 \in C_j$,
(i) $\lim\limits_{n\to\infty} x_n = (\bar{x}^1, \bar{x}^1, \cdots, \bar{x}^1) = P_\Lambda 0$;
(ii)$\langle J_\nu^{G_j}\bar{x}^1 - y^1, G_j(J_\nu^{G_j}\bar{x}_1)\rangle \ge 0$ for all $y^1 \in C_j$ and for all $j \in \{1, 2, \cdots, m\}$.

Remark 4.1. Theorem 4.1 is different from Theorem 3.6 [9]. In Theorem 3.6 [9], Censor et al. assumed that $G_i : H \multimap H$ is a maximum monotone Lipschitz continuous mapping with bounded closed values for each $i \in \{1, 2, \cdots, m\}$, but in Theorem 4.1, for each $j \in \{1, 2, \cdots, m\}$, G_j does not assume to be Lipschitz continuous and bounded closed values. Nadezhkina and Takahashi [20], Kraikaew and Saejung [15], Censor, Gibali and Reich[8] studied variational inequalities for Lipschitz monotone mappings. The result, algorithm and technique of Corollary 4.1 are different from Nadezhkina and Takahashi [20], Kraikaew and Saejung [15], Censor, Gibali and Reich[8].

As consequence of Theorem 4.1, we study the solutions of system of the Stampachia variational inequalities.

Theorem 4.2. Let $\nu > 0$, for each $i \in \{1, 2, \cdots, m\}$, let $\gamma_i > 0$, let $F_i : H \times H \to \mathbb{R}$ be a symmetric bilinear form. For each $i \in \{1, 2, \cdots, m\}$, let $\ell_i \in B(H, \mathbb{R})$, and let $f_i : H \times H \to \mathbb{R}$ be defined by $f_i(x) = \frac{1}{2} F_i(x, x) - \ell_i(x)$, where $B(H, \mathbb{R})$ is the set of all bounded linear operator from $H :\to \mathbb{R}$. Suppose that
$|F_i(x, y)| \leq \gamma_i \|x\|\|y\|$ and $F_i(x, x) \geq 0$ for all $x \in H$ and for all $y \in H$, and for all $i \in \{1, 2, \cdots, m\}$,
and suppose that
$\Lambda = \{(x^1, x^1, \cdots, x^1) \in \bigotimes_{1 \leq i \leq m} H : x^1 \in \bigcap_{j=1}^{m} Fix((2P_{C_j} - I)R_{\nu \nabla f_j})\} \neq \emptyset$.
Let sequences $\{x_n^j\}_{n \in \mathbb{N}}$, and $\{y_n^j\}_{n \in \mathbb{N}}$, $j \in \{1, 2, \cdots, m\}$ be defined by
(i)$y_n^j = J_\nu^{\nabla f_j}(\beta_n u_n^j)$;
(ii)$x_{n+1}^j = \beta_n x_n^j + \lambda_n[\beta_n u_n^j + P_{C_j}(2y_n^j - \beta_n u_n^j) - y_n^j + \beta_n x_n^j], j \in \{1, 2, \cdots, m\}$.
Let $x_n = (x_n^1, x_n^2, \cdots, x_n^m)$, then there exists $(\bar{x}^1, \bar{x}^1, \cdots, \bar{x}^1)$ in Λ such that
$J_\nu^{\nabla f_j} \bar{x}^1 \in C_j$
(i) $\lim_{n \to \infty} x_n = (\bar{x}^1, \bar{x}^1, \cdots, \bar{x}^1) = P_\Lambda 0$;
(ii)$J_\nu^{\nabla f_j} \bar{x}^1 \in C_j$ and $\langle y - J_\nu^{\nabla f_j} \bar{x}^1, \nabla f_j(J_\nu^{\nabla f_j} \bar{x}^1). \rangle \geq 0$ for all $y \in C_j$ and for all $j \in \{1, 2, \cdots, m\}$;
(iii)$J_\nu^{\nabla f_j} \bar{x}^1 \in C_j$ and $F_j(J_\nu^{\nabla f_j} \bar{x}^1, y - J_\nu^{\nabla f_j} \bar{x}^1) \geq \ell_j(y - J_\nu^{\nabla f_j} \bar{x}^1))$ for all $y \in C_j$ and for all $j \in \{1, 2, \cdots, m\}$

By solutions of system of the Stampachia variational inequalities, we get the solutions of the Stampachia variational inequality.

Theorem 4.3. Let $\nu > 0$, $\gamma > 0, \alpha > 0$. Let $F : H \times H \to \mathbb{R}$ be a symmetric bilinear form , let $\ell \in B(H, \mathbb{R})$, and let $f : H \times H \to \mathbb{R}$ be defined by $f(x) = \frac{1}{2} F(x, x) - \ell(x)$, where $B(H, \mathbb{R})$ is the set of all bounded linear operators from H to $\mathbb{R}$. Suppose that
$|F(x, y)| \leq \gamma \|x\|\|y\|$ and $F(x, x) \geq 0$ for all $x \in H$ and for all $y \in H$. Let
$\Lambda = \{x \in H : x \in Fix((2P_C - I)R_{\nu \nabla f})\} \neq \emptyset$.
Let $x_0 \in H$ and $\{x_n\}_{n \in \mathbb{N}}$ and $\{y_n\}_{n \in \mathbb{N}}$ be sequences defined by
$y_n = J_\nu^{\nabla f}(\beta_n x_n)$;

$x_{n+1} = \beta_n x_n + \lambda_n[P_C(2y_n - \beta_n x_n) - y_n]$.

Then there exists $\bar{x} \in \Lambda$ such that $J_\nu^{\nabla f}\bar{x} \in C$ and

(i) $\lim_{n\to\infty} x_n = \bar{x} = P_\Lambda 0$;

(ii)$J_\nu^{\nabla f}\bar{x} \in C$, and $\langle y - J_\nu^{\nabla f}\bar{x}, \nabla f(J_\nu^{\nabla f}\bar{x})\rangle \geq 0$ for all $y \in C$;

(iii)there exists a unique $J_\nu^{\nabla f}\bar{x} \in C$ such that $J_\nu^{\nabla f}\bar{x} \in C$ and $F(J_\nu^{\nabla f}\bar{x}, y - J_\nu^{\nabla f}\bar{x}) \geq \ell(y - J_\nu^{\nabla f}\bar{x})$ for all $y \in C$.

Theorem 4.4. Let $\nu > 0$, $\gamma > 0, \alpha > 0$. Let $F : H \times H \to \mathbb{R}$ be a symmetric bilinear form , let $\ell \in B(H, \mathbb{R})$, and let $f : H \times H \to \mathbb{R}$ be defined by $f(x) = \frac{1}{2}F(x,x) - \ell(x)$, where $B(H, \mathbb{R})$ is the set of all bounded linear operators from H to $\mathbb{R}$. Suppose that $|F(x,y)| \leq \gamma\|x\|\|y\|$ and $F(x,x) \geq \alpha\|x\|^2$ for all $x \in H$ and for all $y \in H$. Let

$\Lambda = \{x \in H : x \in Fix((2P_C - I)R_{\nu\nabla f})\}$.

Let $x_0 \in H$ and $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ be sequences defined by

$y_n = J_\nu^{\nabla f}(\beta_n x_n)$;

$x_{n+1} = \beta_n x_n + \lambda_n[P_C(2y_n - \beta_n x_n) - y_n]$.

Then there exists $\bar{x} \in \Lambda$ such that $J_\nu^{\nabla f}\bar{x} \in C$ and

(i) $\lim_{n\to\infty} x_n = \bar{x} = P_\Lambda 0$;

(ii)$J_\nu^{\nabla f}\bar{x} \in C$, and $\langle y - J_\nu^{\nabla f}\bar{x}, \nabla f(J_\nu^{\nabla f}\bar{x})\rangle \geq 0$ for all $y \in C$;

(iii)there exists a unique $J_\nu^{\nabla f}\bar{x} \in C$ such that $J_\nu^{\nabla f}\bar{x} \in C$ and $F(J_\nu^{\nabla f}\bar{x}, y - J_\nu^{\nabla f}\bar{x}) \geq \ell(y - J_\nu^{\nabla f}\bar{x})$ for all $y \in C$.

Remark 4.2. In Theorem 4.2, we assume that $\Lambda \neq \emptyset$, and $F_i(x,x) \geq 0$ for all $x \in H$ and for all $i \in \{1, 2, \cdots, m\}$, while in Theorem 4.3, we assume that $F(x,x) \geq \alpha\|x\|^2$ for all $x \in H$, we don't assume $\Lambda \neq \emptyset$.

For the special case of solutions of system of the Stampachia variational inequalities , we study the solutions of the system of Lax-Milgram lemma.

Theorem 4.5. Let $\nu > 0$, for each $i \in \{1, 2, \cdots, m\}$, let $F_i : H \times H \to \mathbb{R}$ be a symmetric bilinear form, let $\gamma_i > 0$ and let $\ell_i \in B(H, \mathbb{R})$, let $f_i : H \times H \to \mathbb{R}$ be defined by $f_i(x) = \frac{1}{2}F_i(x,x) - \ell_i(x)$, where $B(H, \mathbb{R})$ is the set of all bounded linear operators from H to $\mathbb{R}$. Suppose that

$|F_i(x,y)| \leq \gamma_i \|x\|\|y\|$ and $F_i(x,x) \geq 0$ for all $x \in H$ and for all $y \in H$,

and suppose that

$\Lambda = \{(x^1, x^1, \cdots, x^1) \in \bigotimes_{1\leq i\leq m} H : x^1 \in \bigcap_{j=1}^{m} Fix((R_{\nu\nabla f_j})\} \neq \emptyset.$

Let sequences $\{x_n^j\}_{n\in\mathbb{N}}$, and $\{y_n^j\}_{n\in\mathbb{N}}$, $j \in \{1, 2, \cdots, m\}$ be defined by

(i) $y_n^j = J_\nu^{\nabla f_j}(\beta_n u_n^j)$;

(ii) $x_{n+1}^j = \beta_n x_n^j + \lambda_n(y_n^j - \beta_n x_n^j), j \in \{1, 2, \cdots, m\}$.

Let $x_n = (x_n^1, x_n^2, \cdots, x_n^m)$, then there exists $(\bar{x}^1, \bar{x}^1, \cdots, \bar{x}^1)$ in Λ such that

(i) $\lim\limits_{n\to\infty} x_n = (\bar{x}^1, \bar{x}^1, \cdots, \bar{x}^1) = P_\Lambda 0$;

(ii)$J_\nu^{\nabla f_j}\bar{x}^1 \in H$ and $F_j(J_\nu^{\nabla f_j}\bar{x}^1, y) = \ell_j(y)$ for all $y \in H$

and for all $j \in \{1, 2, \cdots, m\}$.

The special case of solutions of system of the Lax-Milgram lemma is the solutions of the Lax-Milgram lemma [16].

Theorem 4.6. Let $\nu > 0$, $\gamma > 0, \alpha > 0$, and let $F : H \times H \to \mathbb{R}$ be a symmetric bilinear form. Let $f : H \times H \to \mathbb{R}$ be defined by $f(x) = \frac{1}{2}F(x,x) - \ell(x)$, where $B(H,\mathbb{R})$ is the set of all bounded linear operators from H to $\mathbb{R}$.

Suppose that

$|F(x,y)| \leq \gamma\|x\|\|y\|$ and $F(x,x) \geq \alpha\|x\|^2$ for all $x \in H$ and for all $y \in H$. Let

$\Lambda = \{x \in H : x \in Fix(J_\nu^{\nabla f})\}$.

Let $x_0 \in H$ and $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ be sequences defined by

$y_n = J_\nu^{\nabla f}(\beta_n x_n)$;

$x_{n+1} = \beta_n x_n + \lambda_n(y_n - \beta_n x_n)$.

Then there $\bar{x} \in \Lambda$ such that

(i) $\lim\limits_{n\to\infty} x_n = \bar{x} = P_\Lambda 0$;

(ii)there exists unique $J_\nu^{\nabla f}\bar{x} \in H$ such that $F(J_\nu^{\nabla f}\bar{x}, y) = \ell(y)$ for all $y \in H$.

Proof. Take $C = H$ in Theorem 4.4, then by Theorem 4.4, there $\bar{x} \in \Lambda$ such that

(i) $\lim\limits_{n\to\infty} x_n = \bar{x} = P_\Lambda x_0$;

(ii)$J_\nu^{\nabla f}\bar{x} \in H$ and $F(J_\nu^{\nabla f}\bar{x}, y - J_\nu^{\nabla f}\bar{x}) \geq \ell(y - J_\nu^{\nabla f}\bar{x})$ for all $y \in H$. Argue as in Theorem 4.4, we can prove Theorem 4.6. $\square$

5 Solutions of system optimization problems for the sum of two convex functions

Theorem 5.1. Let $\nu > 0$. Suppose that
$\Lambda = \{(x^1, x^1, \cdots, x^1) \bigotimes_{1 \leq i \leq m} H : x^1 \in \bigcap_{j=1}^m Fix(R_{\nu\partial f_j} R_{\nu\partial g_j})\} \neq \emptyset$.
Let sequences $\{x_n^j\}_{n\in\mathbb{N}}$, and $\{y_n^j\}_{n\in\mathbb{N}}$, $j \in \{1, 2, \cdots, m\}$ be defined by
(i)$y_n^j = J_\nu^{\partial g_j}(\beta_n u_n^j)$;
(ii)$x_{n+1}^j = \beta_n x_n^j + \lambda_n[\beta_n u_n^j + J_\nu^{\partial f_j}(2y_n^j - \beta_n u_n^j) - y_n^j - \beta_n x_n^j], j \in \{1, 2, \cdots, m\}$.
Let $x_n = (x_n^1, x_n^2, \cdots, x_n^m)$, then there exists $(\bar{x}^1, \bar{x}^1, \cdots, \bar{x}^1)$ in Λ such that
(i) $\lim\limits_{n\to\infty} x_n = (\bar{x}^1, \bar{x}^1, \cdots, \bar{x}^1) = P_\Lambda 0$;
(ii) $\lim\limits_{n\to\infty} J_\nu^{\partial g_j}(x_n^j) = J_\nu^{\partial g_j}\bar{x}^1 \in \arg min_{y^1\in H}(f_j(y^1) + g_j(y^1))$ for all $j \in \{1, 2, \cdots, m\}$.

Proof. For each $i \in \{1, 2, ..., m\}$, let $M_i = \partial f_i$, and $G_i = \partial g_i$. We see ∂f_i, and ∂g_j are maximally monotone operator. Then by Theorem 3.4,
(i)$\lim\limits_{n\to\infty} x_n = (\bar{x}^1, \bar{x}^1, \cdots, \bar{x}^1) = P_\Lambda 0$;
(ii)$J_\nu^{\partial g_j}\bar{x}^1 \in (\partial f_j + \partial g_j)^{-1}0 \subseteq \partial(f_j + g_j)^{-1}0$ for all $j \in \{1, 2, \cdots, m\}$.
Therefore, $J_\nu^{\partial g_j}\bar{x}^1 \in \arg\min_{y\in H}(f_j + g_j)(y)$ for all $j \in \{1, 2, \cdots, m\}$.

Theorem 5.2. [3] Let $f \in \Gamma_0(H)$, and $g \in \Gamma_0(H)$. Suppose that
$\Lambda = \{x \in H : x \in Fix(R_{\nu\partial f} R_{\nu\partial g})\} \neq \emptyset$.
Let $x_0 \in H$ and the sequences $\{x_n\}_{n\in\mathbb{N}}$, and $\{y_n\}_{n\in\mathbb{N}}$ be defined by
(i)$y_n = J_\nu^{\partial g}(\beta_n x_n)$;
(ii)$x_{n+1} = \beta_n x_n + \lambda_n[J_\nu^{\partial f}(2y_n - \beta_n x_n) - y_n]$.
Then there exists $\bar{x} \in Fix(R_{\nu\partial f} R_{\nu\partial g})$ such that the following hold:
(i) $\lim\limits_{n\to\infty} x_n = \bar{x} = P_\Lambda 0$;
(ii) $\lim\limits_{n\to\infty} J_\nu^{\partial g}(x_n) = J_\nu^{\partial g}\bar{x} \in \arg min_{y\in H}(f(y) + g(y))$.

Theorem 5.3. Let $\nu > 0$. Suppose that

$\Lambda = \{(x^1, x^1, \cdots, x^1) \bigotimes_{1\leq i\leq m} H : x^1 \in \bigcap_{j=1}^m C_j \bigcap D_j)\} \neq \emptyset$.

Let sequences $\{x_n^j\}_{n\in\mathbb{N}}$, and $\{y_n^j\}_{n\in\mathbb{N}}$, $j \in \{1, 2, \cdots, m\}$ be defined by

(i)$y_n^j = P_{D_j}(\beta_n u_n^j)$;

(ii)$x_{n+1}^j = \beta_n x_n^j + \lambda_n[\beta_n u_n^j + P_{C_j}(2y_n^j - \beta_n u_n^j) - y_n^j - \beta_n x_n^j], j \in \{1, 2, \cdots, m\}$.

Let $x_n = (x_n^1, x_n^2, \cdots, x_n^m)$, then there exists $(\bar{x}^1, \bar{x}^1, \cdots, \bar{x}^1) \in \Omega$ such that

(i) $\lim_{n\to\infty} x_n = (\bar{x}^1, \bar{x}^1, \cdots, \bar{x}^1) = P_\Omega 0$, where

$\Omega = \{(x^1, x^1, \cdots, x^1) \bigotimes_{1\leq i\leq m} H : x^1 \bigcap_{j=1}^m Fix(R_{\nu\partial\iota_{C_j}} R_{\nu\partial\iota_{D_j}})\}$;

(ii) $P_{D_j}\bar{x}^1 \in C_j \bigcap D_j$ for all $j \in \{1, 2, \cdots, m\}$.

Theorem 5.4. Let $\nu > 0$, for each $i \in \{1, 2, \cdots, m\}$, let $g_i \in \Gamma_0(H)$ and let C_i be a nonempty closed convex subset of H. Suppose that

$\Lambda = \{(x^1, x^1, \cdots, x^1) \bigotimes_{1\leq i\leq m} H : x^1 \in \bigcap_{j=1}^m Fix((2P_{C_j} - I)R_{\nu\partial g_j})\} \neq \emptyset$.

Let sequences $\{x_n^j\}_{n\in\mathbb{N}}$, and $\{y_n^j\}_{n\in\mathbb{N}}$, $j \in \{1, 2, \cdots, m\}$ be defined by

(i) $y_n^j = J_\nu^{\partial g_j}(\beta_n u_n^j)$;

(ii)$x_{n+1}^j = \beta_n x_n^j + \lambda_n[\beta_n u_n^j + P_{C_j}(2y_n^j - \beta_n u_n^j) - y_n^j - \beta_n x_n^j], j \in \{1, 2, \cdots, m\}$.

Let $x_n = (x_n^1, x_n^2, \cdots, x_n^m)$, then there exists $(\bar{x}^1, \bar{x}^1, \cdots, \bar{x}^1)$ in Λ such that

(i) $\lim_{n\to\infty} x_n = (\bar{x}^1, \bar{x}^1, \cdots, \bar{x}^1) = P_\Lambda 0$;

(ii) $\lim_{n\to\infty} J_\nu^{\partial g_j}(x_n^j) = J_\nu^{\partial g_j}\bar{x}^1 \in \arg min_{y^1\in C_j} g_j(y^1)$ for all $j \in \{1, 2, \cdots, m\}$.

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