

LOCALLY DEFINABLE $C^\infty G$ APPROXIMATION OF LOCALLY DEFINABLE $C^r G$ MAPS

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ABSTRACT. Let G be a finite group. Consider an o-minimal expansion $\mathcal{R} = (\mathbb{R}, <, +, \cdot, \dots)$ of the field \mathbb{R} of real numbers and $0 \leq r < \infty$. If \mathcal{R} admits the C^∞ cell decomposition and is exponential, then we prove that every locally definable $C^r G$ map between affine locally definable $C^\infty G$ manifolds is approximated by locally definable $C^\infty G$ maps. Moreover we consider its applications.

1. INTRODUCTION

Let $\mathcal{R} = (\mathbb{R}, +, \cdot, <, e^x, \dots)$ be an exponential o-minimal expansion of the field \mathbb{R} of real numbers with C^∞ cell decomposition and let $0 \leq r < \infty$. Everything is considered in \mathcal{R} , the term “definable” is used in the sense of “definable with parameters in \mathcal{R} ”.

Locally definable C^r manifolds are defined in [6]. General references on o-minimal structures is [1].

Let G be a finite group. The definable version of definable C^∞ approximation of definable C^r maps is studied in [2].

We prove a locally definable version of [2]. It is published in [5] without proof. The proof is presented in this paper.

Theorem 1.1. *Let G be a finite group and $0 \leq r < \infty$. Let $\mathcal{R} = (\mathbb{R}, <, +, \cdot, e^x, \dots)$ be an exponential o-minimal expansion of the field \mathbb{R} of real numbers with C^∞ cell decomposition and X, Y affine locally definable $C^\infty G$ manifolds. Every locally definable $C^r G$ map $f : X \rightarrow Y$ is approximated by locally definable $C^\infty G$ maps in the C^r Whitney topology.*

Definable C^r manifolds are studied in [3], [7].

2. LOCALLY DEFINABLE C^r MANIFOLDS

We recall definitions in [6].

A subset of X of \mathbb{R}^n is *locally definable* if for any $x \in X$ there exists a definable open neighborhood U of x in \mathbb{R}^n such that $X \cap U$ is definable in \mathbb{R}^n . Note that every definable set is locally definable and any open set is locally definable.

Let $W \subset \mathbb{R}^n, V \subset \mathbb{R}^m$ be locally definable sets. A map $f : W \rightarrow V$ is locally definable if the graph of f is locally definable. For example, if $\mathcal{R} = \mathbf{R}_{an,exp}$, then $f : (0, 1] \rightarrow \mathbb{R}, f(x) = \sin \frac{1}{x}$ is locally definable but not definable.

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Note that for every locally definable continuous map f between locally definable sets X and Y , if X is compact, then $f(X)$ is a definable set and $f : X \rightarrow f(X)$ is a definable map. And that the field \mathbb{Q} is not locally definable subset of \mathbb{R} .

Let $W \subset \mathbb{R}^n, V \subset \mathbb{R}^m$ be open sets. We call a C^r map $f : W \rightarrow V$ a *locally definable C^r map* if f is locally definable. A locally definable C^r map is a *locally definable C^r diffeomorphism* if there exists a locally definable C^r map $h : V \rightarrow W$ such that $f \circ h = id$ and $h \circ f = id$.

Definition 2.1 ([6]). Let $0 \leq r \leq \infty$.

(1) A locally definable subset X of \mathbb{R}^n is a *d -dimensional locally definable C^r submanifold* of \mathbb{R}^n if for every $x \in X$ there exists a definable C^r diffeomorphism (a definable homeomorphism if $r = 0$) ϕ from some open definable neighborhood U of the origin in \mathbb{R}^n onto some definable open neighborhood V of x in \mathbb{R}^n such that $\phi(0) = x, \phi(\mathbb{R}^d \cap U) = X \cap V$, where $\mathbb{R}^d = \{x \in \mathbb{R}^n \mid \text{last } (n - d) \text{ components of } x \text{ are } 0\}$.

(2) A *locally definable C^r manifold of dimension d* is a Hausdorff space with a countable system of charts $\{\phi_i : U_i \rightarrow \mathbb{R}^d\}$ such that for each i, j $\phi_i(U_i \cap U_j)$ is a definable open subset of \mathbb{R}^d and the map $\phi_j \circ \phi_i^{-1} : \phi_i(U_i \cap U_j) \rightarrow \phi_j(U_i \cap U_j)$ is a definable C^r diffeomorphism. These charts are called *locally definable C^r* .

(3) Let h be a C^r map between locally definable C^r manifolds X, Y . The map h is a *locally definable C^r map* if for every $x \in M$, there exist definable open C^r neighborhoods U_1 of x in M and U_2 of $f(x)$ in N such that $f(U_1) \subset U_2$ and $f|_{U_1} : U_1 \rightarrow U_2$ is a definable C^r map.

(4) Let X, Y be locally definable C^r manifolds. We say that X and Y are locally definable C^r diffeomorphic if there exist locally definable C^r maps $f : X \rightarrow Y$ and $h : Y \rightarrow X$ such that $f \circ h = id$ and $h \circ f = id$.

(5) A locally definable C^r manifold is *affine* if it is locally definable C^r diffeomorphic to a locally definable C^r submanifold of some \mathbb{R}^l .

A representation map of G is a locally definable C^r group homomorphism from G to $O(n)$. A representation of G is a representation space of a representation map of G .

Definition 2.2 ([6]). Let $0 \leq r \leq \infty$.

(1) A locally definable C^r submanifold of a representation Ω of G is a *locally definable $C^r G$ submanifold* of Ω if it is G invariant.

(2) A *locally definable $C^r G$ manifold* is a pair (X, θ) consisting of a locally definable C^r manifold X and a group action $\theta : G \times X \rightarrow X$ is a locally definable C^r map. For simplicity of notation, we write X instead of (X, θ) .

(3) Let X, Y be locally definable $C^r G$ manifolds. A locally definable C^r map is a *locally definable $C^r G$ map* if it is a G map. We say that X and Y are *locally definable $C^r G$ diffeomorphic* if there exist locally definable $C^r G$ maps $f : X \rightarrow Y$ and $h : Y \rightarrow X$ such that $f \circ h = id$ and $h \circ f = id$.

(4) A locally definable $C^r G$ manifold is *affine* if it is locally definable $C^r G$ diffeomorphic to a locally definable $C^r G$ submanifold of some representation of G .

Recall existence of definable $C^\infty G$ tubular neighborhoods.

Proposition 2.3 ([4]). *Let G be a compact subgroup of $GL_n(\mathbb{R})$. Let X be a definable $C^\infty G$ submanifold of a representation of G . Then there exists a definable $C^\infty G$ tubular neighborhood (U, p) of X in Ω . That is U is a G invariant definable open neighborhood of X and $p : U \rightarrow X$ with $p|_X = id_X$.*

3. PROOF OF OUR RESULT

Proof of Theorem 1.1. Let X, Y be locally definable $C^\infty G$ submanifolds of representations Ω, Ξ of G , respectively. By replacing $\Omega \times \mathbb{R}, \Xi \times \mathbb{R}$, we may assume that X, Y are closed.

For $n \in \mathbb{N}$, $N_n = \{x \in \Omega \mid n - 1 \leq \|x\| \leq n + 3\}$.

Let $X_n = X \cap N_n$. Since X_n is compact, $X \cap N_n^\circ$ is a definable $C^\infty G$ submanifold of Ω , where N_n° denotes the interior of N_n . Thus there exists a polynomial map $H : X_n \rightarrow \Xi$ is a definable C^r approximation of $f|_{X_n}$. By averaging H , we may assume that H is a G map. Since $Y \cap f(X_n)$ is compact, $Y_n := (Y \cap f(X_n))^\circ$ is a definable $C^\infty G$ submanifold of Ξ . By Proposition 2.3, there exists a definable $C^\infty G$ tubular neighborhood (U, θ) of Y_n . If the approximation is sufficiently close, $H(X_n) \subset U$ and $h_n := \theta \circ H|_{X_n}$ is a definable $C^\infty G$ approximation of $f|_{X_n}$. We glue these h_n . We can find a locally definable C^∞ function $\phi : \mathbb{R} \rightarrow \mathbb{R}$ such that for each $n \in 4\mathbb{N} + 1$, $\phi|(n - 1, n) = 1, \phi|(n + 1, n + 2) = 0, 0 \leq \phi \leq 1$ because \mathcal{R} is exponential. Let $\psi(x) = \phi(\|x\|)$. Since we can glue h_n using ψ , we have a locally definable $C^\infty G$ map $F : X \rightarrow Y$. If the approximation is sufficiently close, F is the required locally definable $C^\infty G$ map. \square

4. APPLICATIONS

By a way similar to the proofs Theorem 1.1 and Theorem 1.2 [6], we have the following theorem.

Theorem 4.1. *Consider a finite group G and $0 < r < \infty$.*

(1) *Every affine locally definable $C^r G$ manifold is locally definable $C^r G$ diffeomorphic to a locally definable $C^\infty G$ manifold.*

(2) *For two affine locally definable $C^\infty G$ manifold, they are $C^1 G$ diffeomorphic if and only if they are locally definable $C^\infty G$ diffeomorphic.*

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