The universal family around the Klein curve and the homological quotient family of the Klein curve

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Abstract

The Klein curve is the most symmetric curve of genus 3 with maximal automorphism group $PSL_2(\mathbb{F}_7)$. We explicitly describe around the Klein curve, the (coarse) universal family over the moduli space of genus 3 curves by using a linearization method of the family. We also describe another family constructed from the homological representation of the automorphism group of the Klein curve.

1 Introduction

Throughout this paper, unless otherwise mentioned, Riemann surfaces and algebraic curves are assumed to be compact and nonsingular.

Linear quotient families are fibrations (with singular fibers), constructed from finite group actions together with linear representations of the groups. Many families in complex geometry, such as the universal families over the moduli spaces of Riemann surfaces, may be locally 'approximated' by linear quotient families [Tak]. The local descriptions of many families thus reduce to those of linear ones — the latter descriptions are carried out by linear algebra combined with group actions. We apply this procedure to the description of the universal family over the moduli space of Riemann surfaces of genus 3, around the Klein curve.

Let T_g be the Teichmüller space of marked Riemann surfaces of genus $g \geq 2$. The mapping class group Γ_g acts on T_g, and the quotient $M_g := T_g/\Gamma_g$ is the moduli space of Riemann surfaces of genus g. Let $\varphi : S_g \to T_g$ be the universal (or tautological) family over T_g. Here Γ_g naturally acts on S_g, and φ is Γ_g-equivariant. The quotient $\overline{\varphi} : U_g := S_g/\Gamma_g \to M_g := T_g/\Gamma_g$ is the (*coarse*) *universal family* over M_g (note: this is different from the universal family in the sense of Grothendieck — which is a functor). For each $[X] \in M_g$, $\overline{\varphi}^{-1}([X]) = X/\mathrm{Aut}(X)$ and M_g is, around $[X]$, isomorphic to $H^0(X, K^{\otimes 2})/\mathrm{Aut}(X)$, where $H^0(X, K^{\otimes 2})$ ($\cong \mathbb{C}^{3g-3}$) is the vector space of holomorphic quadratic differentials on X, and the $\mathrm{Aut}(X)$-action on it is induced from that on X. If $\mathrm{Aut}(X) = \{1\}$, then $\overline{\varphi}^{-1}([X]) = X$ and M_g is nonsingular around $[X]$ (note: for $g \geq 3$, $\mathrm{Aut}(X) = \{1\}$ for generic X, while for $g = 2$, the situation is subtle; see Remark 1.1). As $\mathrm{Aut}(X)$ becomes larger, the fiber $X/\mathrm{Aut}(X)$ becomes more 'folded' and the singularity $H^0(X, K^{\otimes 2})/\mathrm{Aut}(X)$ becomes more complicated.

Remark 1.1. If $g = 2$, then for *any* X, $\mathrm{Aut}(X) \supset \mathbb{Z}_2$, where $\mathbb{Z}_2$ is generated by the hyperelliptic involution of X, while for generic X, $\mathrm{Aut}(X) = \mathbb{Z}_2$, which acts trivially on $H^0(X, K^{\otimes 2})$, so $H^0(X, K^{\otimes 2})/\mathrm{Aut}(X)$ is nonsingular.

We next review linear quotient families. Suppose that a finite group G acts on a space X: depending on whether X is a complex analytic variety, an algebraic variety, a smooth manifold,

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or a topological space, the G-action is holomorphic, algebraic, smooth, or continuous. Now let $\rho : G \to GL(V)$ be a linear representation, via which G acts on V, and let G act on $X \times V$ diagonally: $(x, v) \mapsto (gx, \rho(g)v)$ (for $g \in G$), Then the projection pr : $X \times V \to V$ is G-equivariant, and the quotient $\eta := \overline{\mathrm{pr}} : (X \times V)/G \to V/G$ is called the *linear quotient family of X associated with ρ*.

We are concerned with the case that X is a Riemann surface of genus greater than 1, $G = \mathrm{Aut}(X)$ is the (holomorphic) automorphism group (which is finite, [FaKr] p.258), and $\rho : G \to GL(V)$ is the representation induced from the natural G-action on a (co)homology group $V = H_1(X, \mathbb{C})$, $H^0(X, K^{\otimes 2})$, where K denotes the canonical bundle of X. Associated with these representations, two linear quotient families — the *homological quotient family* and the *canonical quotient family* — are obtained:

$$(X \times H_1(X, \mathbb{C}))/\mathrm{Aut}(X) \to H_1(X, \mathbb{C})/\mathrm{Aut}(X),$$
$$(X \times H^0(X, K^{\otimes 2}))/\mathrm{Aut}(X) \to H^0(X, K^{\otimes 2})/\mathrm{Aut}(X).$$

By the *linearization theorem* [Tak] (p.385 Theorem 4.3), the latter family is orbi-diffeomorphic to the universal family $U_g \to M_g$ around $[X] \in M_g$ (where an orbi-diffeomorphism is an isomorphism between orbifold fibrations). Consequently the local description of $U_g \to M_g$ reduces to the description of the canonical quotient family of X (up to orbi-diffeomorphism).

This paper treats the case that X is the Klein curve, which is a plane curve defined by $x^3 y + y^3 z + z^3 x = 0$ in $\mathbb{CP}^2$. Note that X is a *nonhyperelliptic* Riemann surface of genus 3 with $\mathrm{Aut}(X) \cong PSL_2(\mathbb{F}_7)$ (see e.g.[Elk]). We will describe the canonical quotient family and the homological quotient family. These two families are very different (compare Figure 4.3 and Figure 4.2). Both the homological representation on $H_1(X, \mathbb{C})$ and the canonical representation on $H^0(X, K^{\otimes 2})$ are 6-dimensional and faithful, but they are distinct — as we see later the homological one is reducible, while the canonical one is irreducible (see Propositions 3.1 and 3.5).

2   Preparation

We first explain properties of linear quotient families. Suppose that a finite group G acts on a complex analytic variety X. Let $\rho : G \to GL(V)$ be a linear representation, via which G acts on a vector space V. Consider the diagonal action of G on $X \times V$: $(x, t) \mapsto (gx, \rho(g)t)$, $g \in G$. Then the projection $X \times V \to V$ is G-equivariant, and the quotient $\eta : (X \times V)/G \to V/G$ is defined, which is the *linear quotient family of X associated with ρ*.

A linear quotient family is a generalization of a degeneration: Suppose that X is a complex manifold and G is a cyclic group generated by a periodic automorphism f of order n. Let $\rho : G \to GL_1(\mathbb{C})$ be the homomorphism defined by $f \mapsto e^{2\pi i/n}$, then the linear quotient family $\eta : (X \times V)/G \to V/G$ is a degeneration of X, whose singular fiber $\eta^{-1}(0)$ is the multiple of X/G with multiplicity n, while the other fibers are X.

The fibers of a linear quotient family are determined by the following:

Theorem 2.1 (Quotient fiber theorem, [Tak] p.372 Theorem 2.1). $\eta^{-1}(s) = X/H_{\widetilde{s}}$, where $H_{\widetilde{s}}$ is the stabilizer of a lift $\widetilde{s} \in V$ of $s \in V$ (for the G-action on V via ρ).

Remark 2.2. For $s \in V/G$, its lift $\widetilde{s} \in V$ is generally not unique, however the quotient $X/H_{\widetilde{s}}$ does not depend on the choice of $\widetilde{s}$. Indeed for two lifts $\widetilde{s}, \widetilde{s}' \in V$, their stabilizers $H_{\widetilde{s}}$, $H_{\widetilde{s}'}$ are conjugate, say $H_{\widetilde{s}} = gH_{\widetilde{s}'}g^{-1}$ ($g \in G$), then g induces an isomorphism $X/H_{\widetilde{s}} \to X/H_{\widetilde{s}'}$, $[x] \mapsto [gx]$.

The *covering multiplicity* of a fiber X/H of η is the order $|H|$, which is equal to the covering degree of the quotient map $X \to X/H$. (In drawing the figure of a fiber, we attach this number on it; for a pure fiber, the covering multiplicity is always 1, and this number is not attached. See Figure 4.1.)

Remark 2.3. For the case $\dim V \geq 2$, a fiber of η is not of codimension 1 in $(X \times V)/G$, so *not* a divisor on $(X \times V)/G$, thus "algebro-geometric multiplicity" for it is not definable, but "covering multiplicity" for it is.

If $H_{\widetilde{s}} = \{1\}$, then $\eta^{-1}(s)$ is called a *pure fiber*, otherwise $\eta^{-1}(s)$ is called a *kaleido fiber*. In particular $\eta^{-1}(\overline{0}) = X/G$ is called the *crystal fiber* — the most folded fiber. Define the pure domain PD of V/G as the domain on which the pure fibers lie, and the kaleido locus KL of V/G as the locus on which the kaleido fibers lie. Then

$$\mathrm{PD} := \{s \in V/G \,:\, H_{\widetilde{s}} = \{1\}\} \quad \text{(pure domain)},$$
$$\mathrm{KL} := \{s \in V/G \,:\, H_{\widetilde{s}} \neq \{1\}\} \quad \text{(kaleido locus)}.$$

Now take the set $\mathcal{P} := \{H_{\widetilde{s}}\}$ of stabilizers for the G-action on V, and for $H \in \mathcal{P}$, define the *H-vein* V_H as the locus on which a kaleido fiber X/H lie. Explicitly,

$$V_H := \{s \in V/G \,:\, H_{\widetilde{s}} = H\}.$$

To explain properties of veins, we prepare notations. For each $H \in \mathcal{P}$, set

$$\begin{aligned}
A_H &:= \{t \in V \,:\, H_t = H\} \qquad \text{(the *H-artery*)}, \\
CA_H &:= \{t \in V \,:\, H_t \supset H\} \qquad \text{(the *H-conartery*)}.
\end{aligned} \tag{2.1}$$

The H-vein V_H is, by definition, the image of A_H under the quotient map $V \to V/G$. (Note: A_H is not equal to the fixed point set $\mathrm{Fix}(H)$, in fact $A_H \subset CA_H = \mathrm{Fix}(H)$.) The *$H$-convein* CV_H is the image of CA_H under $V \to V/G$, which satisfies $\mathrm{KL} = \bigcup_{H \in \mathcal{P} \setminus \{1\}} CV_H$.

For any $g \in G$ and $t \in V$, we have $H_{gt} = gH_t g^{-1}$. This implies that

$$\begin{cases} gA_H = A_{gHg^{-1}}, \\ gCA_H = CA_{gHg^{-1}}. \end{cases} \tag{2.2}$$

Note that if H and H' are conjugate, say $H' = gHg^{-1}$ ($g \in G$), then $A_{H'} = gA_H$. Thus the images of A_H and $A_{H'}$ under $V \to V/G$ coincide, that is, $V_H = V_{H'}$. If H and H' are not conjugate, then $A_{H'} \cap gA_H = \emptyset$ for any $g \in G$, and thus $V_H \cap V_{H'} = \emptyset$. We thus obtained the following:

Lemma 2.4. *For $H, H' \in \mathcal{P}$, if H and H' are conjugate then $V_H = V_{H'}$, and otherwise $V_H \cap V_{H'} = \emptyset$.*

We thus have the *vein decomposition*

$$\mathrm{KL} = \coprod_{H \in (\mathcal{P} \setminus \{1\})/\sim} V_H, \tag{2.3}$$

where $(\mathcal{P} \setminus \{1\})/\sim$ denotes the set of conjugacy classes of $\mathcal{P} \setminus \{1\}$. Note also that from the definition of vein and convein, we have

$$V_H = CV_H \setminus \left(\bigcup_{\substack{K \in \mathcal{P} \\ K \supsetneq H}} CV_K \right). \tag{2.4}$$

The determination of $(\mathcal{P} \setminus \{1\})/\sim$, V_H, CV_H and KL is a matter of linear algebra. [SaTa] gives an algorithm to determine them by computing the intersections of fixed point sets.

3

Conarteries for reducible representations

Consider the case that a linear quotient family $(X \times V)/G \to V/G$ is associated with a *reducible* represenation $\rho : G \to GL(V)$, say $\rho = \rho_1 \oplus \rho_2$. Let $V = W_1 \oplus W_2$ be the corresponding G-decomposition, where G acts on W_i as $\rho_i(G)$. For the H-action on W_i ($i = 1, 2$), let $\mathrm{Fix}_i(H)$ denote its fixed point set.

Lemma 2.5.   (1) *For any $g \in G$, $\mathrm{Fix}(g) = \mathrm{Fix}_1(g) \times \mathrm{Fix}_2(g)$.*

(2) *For any subgroup H of G, $\mathrm{Fix}(H) = \mathrm{Fix}_1(H) \times \mathrm{Fix}_2(H)$.*

Proof. (1): Each $v \in V$ is uniquely decomposed as $v = w_1 + w_2$ ($w_i \in W_i$), and on which $\rho(g)$ acts as $w_1 + w_2 \mapsto \rho_1(g)w_1 + \rho_2(g)w_2$. In particular $\rho(g)v = v$ if and only if $\rho_1(g)w_1 = w_1$ and $\rho_2(g)w_2 = w_2$. This means that $v \in \mathrm{Fix}(g)$ if and only if $w_1 \in \mathrm{Fix}_1(g)$ and $w_2 \in \mathrm{Fix}_2(g)$. Hence $\mathrm{Fix}(g) = \mathrm{Fix}_1(g) \times \mathrm{Fix}_2(g)$. (2) is confirmed as follows:

$$\mathrm{Fix}(H) = \bigcap_{h \in H} \mathrm{Fix}(h) = \bigcap_{h \in H} \big(\mathrm{Fix}_1(h) \times \mathrm{Fix}_2(h)\big) \quad \text{by (1)}$$

$$= \Big(\bigcap_{h \in H} \mathrm{Fix}_1(h)\Big) \times \Big(\bigcap_{h \in H} \mathrm{Fix}_2(h)\Big) = \mathrm{Fix}_1(H) \times \mathrm{Fix}_2(H).$$

$\square$

2.1   Veins and ridges of the total space

Let $\eta : (X \times V)/G \to V/G$ be a linear quotient family. For the G-action on V, let $\mathcal{P} = \mathcal{P}(G \curvearrowright V)$ denote the set of its stabilizers. For each $H \in \mathcal{P}$,

- $A_H = \{t \in V : H_t = H\}$ is the H-*artery*,

- V_H is the image (the H-*vein*) of A_H under the quotient map $V \to V/G$.

Similarly for the diagonal action of G on $X \times V$, let $\mathcal{P}(G \curvearrowright X \times V)$ denote the set of its stabilizers. For each $K \in \mathcal{P}(G \curvearrowright X \times V)$,

- $B_K := \big\{(x,t) \in X \times V : H_{(x,t)} = K\big\}$ is the K-*artery*,

- W_K is the image (the K-*vein*) of B_K under the quotient map $X \times V \to (X \times V)/G$.

We next decompose the veins of $(X \times V)/G$ into small pieces (this is *not* the case for the veins of V/G). This decomposition is based on the fact that $(X \times V)/G$ is a quotient of the *product* $X \times V$ (while V/G is not!). First for $(x,t) \in X \times V$, let J_x be the stabilizer of x for the G-action on X and let H_t be the stabilizer of t for the G-action on V. Then the stabilizer of (x,t) for the G-action on $X \times V$ is given by $J_x \cap H_t$, accordingly the K-artery B_K is given by

$$B_K = \big\{(x,t) \in X \times V : J_x \cap H_t = K\big\}.$$

We shall decompose B_K: For $J \in \mathcal{P}(G \curvearrowright X)$ and $H \in \mathcal{P}(G \curvearrowright V)$ such that $J \cap H = K$, set $\widetilde{R}_{J,H} := \big\{(x,t) \in X \times V : J_x = J, \, H_t = H\big\}$ (the (J, H)-*preridge*), then

$$B_K = \coprod_{J \cap H = K} \widetilde{R}_{J,H}. \tag{2.5}$$

The images of B_K and $\widetilde{R}_{J,H}$ under the quotient map $q : X \times V \to (X \times V)/G$ are denoted by W_K and $R_{J,H}$, respectively the K-*vein* and the (J, H)-*ridge* of $(X \times V)/G$. Then (2.5) descends to

$$W_K = \bigcup_{J \cap H = K} R_{J,H}, \tag{2.6}$$

where this union is generally *not* disjoint — possibly $R_{J,H} = R_{J',H'}$ for distinct pairs (J, H) and (J', H') (see (4) of the following).

Lemma 2.6. (1) *For any $g \in G$ and $(x, t) \in X \times V$, $J_{gx} = gJ_x g^{-1}$ and $H_{gt} = gH_t g^{-1}$.*

(2) *For any $g \in G$, $g\widetilde{R}_{J,H} = \widetilde{R}_{gJg^{-1}, gHg^{-1}}$.*

(3) *$\widetilde{R}_{J,H} = \widetilde{R}_{J',H'}$ if and only if $J = J'$ and $H = H'$.*

(4) *$R_{J,H} = R_{J',H'}$ if and only if (J, H) and (J', H') are simultaneously conjugate, that is, $J' = gJg^{-1}$ and $H' = gHg^{-1}$ for some $g \in G$.*

Proof. (1) is easy to show. (2) is confirmed as follows:

$$
\begin{aligned}
\widetilde{R}_{gJg^{-1}, gHg^{-1}} &= \{(x,\, v) \in X \times V \,:\, J_x = gJg^{-1},\, H_v = gHg^{-1}\} \\
&= \{(x,\, v) \in X \times V \,:\, g^{-1}J_x g = J,\, g^{-1}H_v g = H\} \\
&= \{(x,\, v) \in X \times V \,:\, J_{g^{-1}x} = J,\, H_{g^{-1}v} = H\} \quad \text{by (1)} \\
&= \{(gx',\, gv') \in X \times V \,:\, J_{x'} = J,\, H_{v'} = H\} \quad (x' := g^{-1}x,\, v' := g^{-1}v) \\
&= g\widetilde{R}_{J,H}.
\end{aligned}
$$

(3) is immediate from the definition of $\widetilde{R}_{J,H}$. (4) is confirmed as follows:

$$
\begin{aligned}
R_{J,H} = R_{J',H'} &\iff g\widetilde{R}_{J,H} = \widetilde{R}_{J',H'} \text{ for some } g \in G \\
&\iff \widetilde{R}_{gJg^{-1}, gHg^{-1}} = \widetilde{R}_{J',H'} \text{ by (2)} \\
&\iff J' = gJg^{-1} \text{ and } H' = gHg^{-1} \text{ by (3)}.
\end{aligned}
$$

$\square$

As seen from Lemma 2.6 (4), the union of ridges $W_K = \bigcup\limits_{J \cap H = K} R_{J,H}$ in (2.6) is generally *not* disjoint. Here $R_{J,H} \cap R_{J',H'} \neq \emptyset$ is just $R_{J,H} = R_{J',H'}$, and this occurs precisely when (J, H) and (J', H') are simultaneously conjugate. Let $[J, H]$ denote the simultaneous conjugacy class of (J, H) and let Λ be the set of all simultaneous conjugacy classes. Then

$$
W_K = \coprod_{\substack{[J,H] \in \Lambda \\ J \cap H = K}} R_{J,H} \qquad \text{(the *ridge decomposition*).} \tag{2.7}
$$

By abuse of terminology, a (J, H)-ridge is also called a $J \cap H$-ridge.

3 Representations of $PSL_2(\mathbb{F}_7)$

Throughout this section, X denotes the Klein curve.

To describe a linear quotient family $\eta : (X \times V)/PSL_2(\mathbb{F}_7) \to V/PSL_2(\mathbb{F}_7)$ associated with a representation $\rho : PSL_2(\mathbb{F}_7) \to GL(V)$, we have to determine the stabilizers for the $PSL_2(\mathbb{F}_7)$-actions on V and $X \times V$ up to conjugation. Before proceeding, we first review basic properties of $PSL_2(\mathbb{F}_7)$.

Generators, conjugacy classes, and the action on X

As an abstract group, $PSL_2(\mathbb{F}_7)$ is given by

$$
PSL_2(\mathbb{F}_7) = \langle a, b, c \,:\, a^2 = b^3 = c^7 = abc = 1,\, (c^4 a)^4 = 1 \rangle. \tag{3.1}
$$

(In fact $PSL_2(\mathbb{F}_7)$ is obtained from the $(2, 3, 7)$-triangle group $\langle a, b, c \,:\, a^2 = b^3 = c^7 = abc = 1 \rangle$ by adding one relation $(c^4 a)^4 = 1$ ([Bur] p.422).) Representatives of the conjugacy classes of

elements of $PSL_2(\mathbb{F}_7)$ are, for example, $1, a, b, c^4 a, c, c^{-1}$ ([Bur] p.422), whose orders are $1, 2, 3, 4, 7, 7$.

Set

$$\zeta := e^{2\pi i/7}, \quad s := \frac{\zeta^5 - \zeta^2}{\sqrt{7}i}, \quad t := \frac{\zeta^6 - \zeta}{\sqrt{7}i}, \quad u := \frac{\zeta^3 - \zeta^4}{\sqrt{7}i}. \tag{3.2}$$

Then the actions of generators a, c of $PSL_2(\mathbb{F}_7)$ ($\cong \mathrm{Aut}(X)$) on $X : x^3 y + y^3 z + z^3 x = 0$ in $\mathbb{CP}^2$ are as follows (see [Bur] p.311):

$$\begin{cases} a : [x : y : z] \longmapsto [sx + ty + uz : tx + uy + sz : ux + sy + tz], \\ c : [x : y : z] \longmapsto [\zeta x : \zeta^4 y : \zeta^2 z]. \end{cases} \tag{3.3}$$

From (3.3), the action $PSL_2(\mathbb{F}_7)$ ($\cong \mathrm{Aut}(X)$) $\curvearrowright X$ are obtained, and this action induces the homological representation $\rho_H : PSL_2(\mathbb{F}_7) \to GL(H_1(X, \mathbb{C}))$ and the canonical representation $\rho_C : PSL_2(\mathbb{F}_7) \to GL(H^0(X, K^{\otimes 2}))$. Both repersentations are 6-dimensional: $\dim(H_1(X, \mathbb{C})) = \dim(H^0(X, K^{\otimes 2})) = 6$. However ρ_H and ρ_C are distinct, in fact as we will show that while ρ_H is reducible, ρ_C is irreducible.

Subgroups

$PSL_2(\mathbb{F}_7)$ has 14 nontrivial subgroups up to conjugation ([Pfe] p.255):

$$\begin{aligned} &\mathbb{Z}_2, \ \mathbb{Z}_3, \ \mathbb{Z}_4, \ K_4, \ K_4', \ \mathfrak{S}_3, \ \mathbb{Z}_7, \ D_4, \\ &\mathfrak{A}_4, \ \mathfrak{A}_4', \ \mathbb{Z}_7 \rtimes \mathbb{Z}_3, \ \mathfrak{S}_4, \ \mathfrak{S}_4', \ PSL_2(\mathbb{F}_7), \end{aligned} \tag{3.4}$$

where
- K_4, K_4' ($\cong \mathbb{Z}_2 \times \mathbb{Z}_2$) are two nonconjugate Klein four groups,
- $\mathfrak{S}_3$ is the symmetric group of degree 3,
- D_4 is the dihedral group of degree 4 (the order $|D_4|$ is 8),
- $\mathfrak{A}_4$, $\mathfrak{A}_4'$ are two nonconjugate alternating groups of degree 4,
- $\mathfrak{S}_4$, $\mathfrak{S}_4'$ are two nonconjugate symmetric groups of degree 4.

The character table

As the number of conjugacy classes of the elements of $PSL_2(\mathbb{F}_7)$ is 6, $PSL_2(\mathbb{F}_7)$ has six irreducible representations — their dimensions are $1, 3, 3, 6, 7, 8$ (as seen from the character table of $PSL_2(\mathbb{F}_7)$; see e.g. [JaLi] p.318). Denote the irreducible representations of $PSL_2(\mathbb{F}_7)$ by ρ_1, ρ_3, ρ_3', ρ_6, ρ_7, ρ_8 respectively of dimension $1, 3, 3, 6, 7, 8$. Their characters are denoted by χ_1, χ_3, χ_3', χ_6, χ_7, χ_8. Table 1 gives the character table of $PSL_2(\mathbb{F}_7)$.

	1	a	b	$c^4 a$	c	c^{-1}
χ_1	1	1	1	1	1	1
χ_3	3	-1	0	1	$\frac{-1+\sqrt{7}i}{2}$	$\frac{-1-\sqrt{7}i}{2}$
χ_3'	3	-1	0	1	$\frac{-1-\sqrt{7}i}{2}$	$\frac{-1+\sqrt{7}i}{2}$
χ_6	6	2	0	0	-1	-1
χ_7	7	-1	1	-1	0	0
χ_8	8	0	-1	0	1	1

Table 1: The character table of $PSL_2(\mathbb{F}_7)$

Homological representation of $PSL_2(\mathbb{F}_7)$

We shall show that the homological representation $\rho_H : \mathrm{Aut}(X) \to GL(H_1(X, \mathbb{C}))$ is equivalent to the direct sum $\rho_3 \oplus \rho_3'$. Letting χ_H denote the character of ρ_H, it suffices to show that $\chi_H = \chi_3 + \chi_3'$, for which we first compute the values $\chi_H(a)$ and $\chi_H(c)$. Take the matrix representations of $\rho_H(a)$ and $\rho_H(c)$ in [RaLe] p.305 (with respect to the basis of $H_1(X, \mathbb{C})$ appearing in p.302 Table 2 therein):

$$
\rho_H(a) = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 \end{pmatrix}, \quad
\rho_H(c) = \begin{pmatrix} 0 & 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & -1 & -1 & -1 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 1 \\ -1 & 1 & 0 & 0 & -1 & -1 \\ 1 & -1 & 0 & 0 & 1 & 0 \end{pmatrix}.
$$

Thus

$$\chi_H(a) = -2, \qquad \chi_H(c) = -1. \tag{3.5}$$

Note also that as ρ_H is 6-dimensional, we have $\chi_H(1) = 6$. Comparing these with Table 1, it is shown that the unique possibility is $\chi_H = \chi_3 + \chi_3'$. We thus obtain the following:

Proposition 3.1. *The homological representation ρ_H of $\mathrm{Aut}(X)$ is equivalent to the direct sum $\rho_3 \oplus \rho_3'$ of two irreducible 3-dimensional representations ρ_3 and ρ_3'.*

Note the following:

Lemma 3.2. (1) *χ_3' is equal to the complex conjugate $\overline{\chi}_3$ of χ_3.* (2) *$\rho_3' \sim \overline{\rho}_3$ (equivalent).*

Lemma 3.2 (2) combined with Proposition 3.1 yields the following:

Corollary 3.3. *$\rho_H \sim \rho_3 \oplus \overline{\rho}_3$ (equivalent). That is, $H_1(X, \mathbb{C})$ admits a decomposition $H_1(X, \mathbb{C}) \cong V \oplus \overline{V}$ such that $PSL_2(\mathbb{F}_7)$ acts on V via ρ_3 and on $\overline{V}$ via $\overline{\rho}_3$.*

Remark 3.4. For any subgroup K of $PSL_2(\mathbb{F}_7)$,

$$\mathrm{Fix}\big(\rho_H(K)\big) \cong \mathrm{Fix}\big(\rho_3(K)\big) \times \mathrm{Fix}\big(\overline{\rho}_3(K)\big) \quad \text{(by Lemma 2.5 (2)).}$$

Thus to determine the fixed point sets of subgroups for the $PSL_2(\mathbb{F}_7)$-action on $H_1(X, \mathbb{C})$, it suffices to determine those for the $PSL_2(\mathbb{F}_7)$-actions on V and $\overline{V}$.

Canonical representation of $PSL_2(\mathbb{F}_7)$

We shall show that the (2-)canonical representation $\rho_C : PSL_2(\mathbb{F}_7) \to GL(H^0(X, K^{\otimes 2}))$ is equivalent to ρ_6. Note first that for any Riemann surface Y of genus $g \geq 2$, the following holds ([FaKr] p.80 Proposition III.5.2):

$$
\dim H^0(Y, K^{\otimes n}) = \begin{cases} g & (n = 1), \\ (2n-1)(g-1) & (n \geq 2). \end{cases}
$$

In particular if $g = 3$ and $n = 1, 2$, then

$$\dim H^0(Y, K) = 3, \qquad \dim H^0(Y, K^{\otimes 2}) = 6. \tag{3.6}$$

We consider the case that Y is the Klein curve X (i.e. $g = 3$ and $\mathrm{Aut}(X) = PSL_2(\mathbb{F}_7)$). The following is shown by taking a basis of $H^0(X, K)$ (resp. $H^0(X, K^{\otimes 2})$ ($\cong H^0(X, K) \odot H^0(X, K)$: the symetric tensor product)) and computing the matrix representation of $\mu_C : PSL_2(\mathbb{F}_7) \to GL(H^0(X, K))$ (resp. $\rho_C : PSL_2(\mathbb{F}_7) \to GL(H^0(X, K^{\otimes 2}))$) with respect to it:

Proposition 3.5. *Let X be the Klein curve; so $\mathrm{Aut}(X) = PSL_2(\mathbb{F}_7)$. Then the following holds*:

(1) ([Elk] p.67) *The 1-canonical representation $\mu_C : PSL_2(\mathbb{F}_7) \to GL(H^0(X, K))$ is equivalent to the irreducible representation $\overline{\rho}_3$.*

(2) *The 2-canonical representation $\rho_C : PSL_2(\mathbb{F}_7) \to GL(H^0(X, K^{\otimes 2}))$ is equivalent to the irreducible representation ρ_6.*

4 Main results

By using the properties of the reprentations ρ_3, ρ_3', ρ_H, ρ_C of $PSL_2(\mathbb{F}_7)$, we may determine the veins and ridges for their corresponding linear quotient families. Our results are summarized as follows:

Result 1. *Let $\eta, \eta' : (X \times \mathbb{C}^3)/\mathrm{Aut}(X) \to \mathbb{C}^3/\mathrm{Aut}(X)$ be the linear quotient families associated with ρ_3 and ρ_3'. The following then hold*:

(1) *η and η' are orbi-diffeomorphic and their base spaces are anti-biholomorphic.*

(2) (2.a) *The stabilizers of the $PSL_2(\mathbb{F}_7)$-action on $\mathbb{C}^3$ via ρ_3 (also ρ_3') are $PSL_2(\mathbb{F}_7)$, $\mathbb{Z}_4$, $\mathbb{Z}_3$; so η (also η') has three veins $V_{PSL_2(\mathbb{F}_7)}$, $V_{\mathbb{Z}_4}$, $V_{\mathbb{Z}_3}$ in $\mathbb{C}^3/PSL_2(\mathbb{F}_7)$, whose dimensions are 0, 1, 1, and*
$$\mathrm{Sing}\big(\mathbb{C}^3/PSL_2(\mathbb{F}_7)\big) = V_{PSL_2(\mathbb{F}_7)} \cup V_{\mathbb{Z}_4} \cup V_{\mathbb{Z}_3},$$
where $V_{PSL_2(\mathbb{F}_7)}$ $(= \overline{0})$ is an isolated singularity, and $V_{\mathbb{Z}_3}$ and $V_{\mathbb{Z}_4}$ are compound singularities along which quotient singularities $\mathbb{C}^2/\mathbb{Z}_3$ and $\mathbb{C}^2/\mathbb{Z}_4$ run respectively.

(2.b) *The stabilizers of the $PSL_2(\mathbb{F}_7)$-action on $X \times \mathbb{C}^3$ via ρ_3 (also ρ_3') are $\mathbb{Z}_7$, $\mathbb{Z}_3$, $\mathbb{Z}_2$; so η (also η') has three veins $W_{\mathbb{Z}_7}$, $W_{\mathbb{Z}_3}$, $W_{\mathbb{Z}_2}$ in $(X \times \mathbb{C}^3)/PSL_2(\mathbb{F}_7)$, whose dimensions are 0, 1, 1, and*
$$\mathrm{Sing}((X \times \mathbb{C}^3)/PSL_2(\mathbb{F}_7)) = W_{\mathbb{Z}_7} \cup W_{\mathbb{Z}_3} \cup W_{\mathbb{Z}_2},$$
where $W_{\mathbb{Z}_7}$ is an isolated singularity, and $W_{\mathbb{Z}_3}$ and $W_{\mathbb{Z}_2}$ are compound singularities along which quotient singularities $\mathbb{C}^2/\mathbb{Z}_3$ and $\mathbb{C}^2/\mathbb{Z}_2$ run respectively.

(3) *The kaleido fibers over $V_{PSL_2(\mathbb{F}_7)}$, $V_{\mathbb{Z}_4}$, $V_{\mathbb{Z}_3}$ are $168\mathbb{P}^1$, $4E_1$, $3E_2$, where E_1, E_2 are elliptic curves. These fibers intersect the ridges as illustrated in Figure 4.1.*
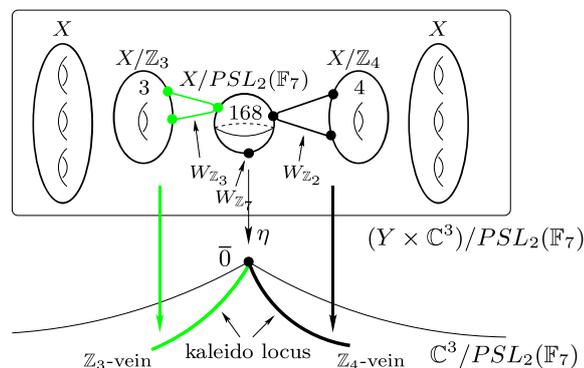

Figure 4.1: Linear quotient family of the Klein curve associated with ρ_3.

Result 2 (Homological quotient family η_H).
For $\eta_H : (X \times H_1(X, \mathbb{C}))/PSL_2(\mathbb{F}_7) \to H_1(X, \mathbb{C})/PSL_2(\mathbb{F}_7)$, the following hold:

(1) η_H has three veins $V_{PSL_2(\mathbb{F}_7)}$, $V_{\mathbb{Z}_4}$, $V_{\mathbb{Z}_3}$ in the base space, whose dimensions are $0, 2, 2$, and has three veins $W_{\mathbb{Z}_7}$, $W_{\mathbb{Z}_3}$, $W_{\mathbb{Z}_2}$ in the total space, whose dimensions are $0, 2, 2$.

(2) The kaleido fibers over $V_{PSL_2(\mathbb{F}_7)}$, $V_{\mathbb{Z}_4}$, $V_{\mathbb{Z}_3}$ are $168\mathbb{P}^1$, $4E_1$, $3E_2$, where E_1, E_2 are elliptic curves.

(3) η_H has 5 ridges. For the (J, H)-ridge $R_{J,H}$, $\dim R_{J,H}$ and $\#(R_{J,H} \cap X/H)$ (the number of intersection points of $R_{J,H}$ and each kaleido fiber X/H on V_H) are described in Table 2.

$\Gamma = J \cap H$	H	J	$\dim R_{J,H}$	$\#(R_{J,H} \cap X/H)$
$\mathbb{Z}_7$	$PSL_2(\mathbb{F}_7)$	$\mathbb{Z}_7$	0	1
$\mathbb{Z}_3$	$PSL_2(\mathbb{F}_7)$	$\mathbb{Z}_3$	0	1
$\mathbb{Z}_3$	$\mathbb{Z}_3$	$\mathbb{Z}_3$	2	2
$\mathbb{Z}_2$	$PSL_2(\mathbb{F}_7)$	$\mathbb{Z}_2$	0	1
$\mathbb{Z}_2$	$\mathbb{Z}_4$	$\mathbb{Z}_2$	2	2

Table 2: For each stabilizer Γ for the $PSL_2(\mathbb{F}_7)$-action on $X \times H_1(X, \mathbb{C})$, express $\Gamma = J \cap H$, where J and H are stabilizers for the $PSL_2(\mathbb{F}_7)$-actions on X and $H_1(X, \mathbb{C})$.

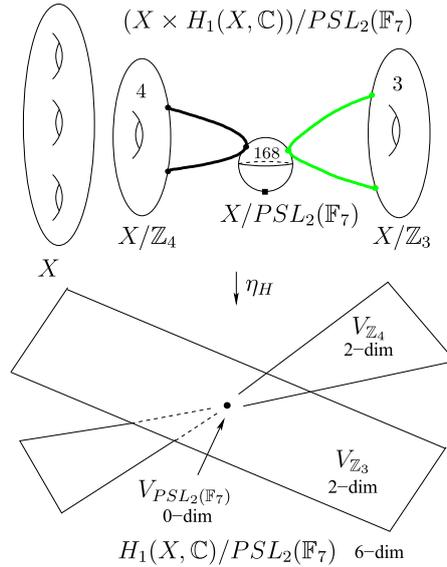
Figure 4.2: For the Klein curve X, its homological quotient family $(X \times H_1(X, \mathbb{C}))/PSL_2(\mathbb{F}_7) \to H_1(X, \mathbb{C})/PSL_2(\mathbb{F}_7)$ has the above configurations of veins, fibers, and ridges.

Result 3 (Canonical quotient family η_C).
For $\eta_C : (X \times H^0(X, K^{\otimes 2}))/PSL_2(\mathbb{F}_7) \to H^0(X, K^{\otimes 2})/PSL_2(\mathbb{F}_7)$, the following hold:

(1) η_C has eight veins. For each vein, its dimension and the kaleido fiber over it are described in Table 3 (k.f. means kaleido fiber):

	$V_{PSL_2(\mathbb{F}_7)}$	$V_{\mathfrak{S}_4}$	$V_{\mathfrak{S}_4'}$	V_{D_4}	$V_{\mathfrak{S}_3}$	V_{K_4}	$V_{K_4'}$	$V_{\mathbb{Z}_2}$
dim	0	1	1	2	2	3	3	4
k.f.	$168\mathbb{P}^1$	$24\mathbb{P}^1$	$24\mathbb{P}^1$	$8\mathbb{P}^1$	$6\mathbb{P}^1$	$4\mathbb{P}^1$	$4\mathbb{P}^1$	$2E$

Table 3: E is an elliptic curve.

(2) η_C has 13 ridges. For the (J,H)-ridge $R_{J,H}$, $\dim R_{J,H}$ and $\#(R_{J,H} \cap X/H)$ (the number of intersection points of $R_{J,H}$ and each kaleido fiber X/H on V_H) are described in Table 4.

$\Gamma = J \cap H$	H	J	$\dim R_{J,H}$	$\#(R_{J,H} \cap X/H)$
$\mathbb{Z}_7$	$PSL_2(\mathbb{F}_7)$	$\mathbb{Z}_7$	0	1
$\mathbb{Z}_3$	$PSL_2(\mathbb{F}_7)$	$\mathbb{Z}_3$	0	1
$\mathbb{Z}_3$	$\mathfrak{S}_4$	$\mathbb{Z}_3$	1	1
$\mathbb{Z}_3$	$\mathfrak{S}_4'$	$\mathbb{Z}_3$	1	1
$\mathbb{Z}_3$	$\mathfrak{S}_3$	$\mathbb{Z}_3$	2	1
$\mathbb{Z}_2$	$PSL_2(\mathbb{F}_7)$	$\mathbb{Z}_2$	0	1
$\mathbb{Z}_2$	$\mathfrak{S}_4$	$\mathbb{Z}_2$	1	3
$\mathbb{Z}_2$	$\mathfrak{S}_4'$	$\mathbb{Z}_2$	1	3
$\mathbb{Z}_2$	D_4	$\mathbb{Z}_2$	2	5
$\mathbb{Z}_2$	$\mathfrak{S}_3$	$\mathbb{Z}_2$	2	4
$\mathbb{Z}_2$	K_4	$\mathbb{Z}_2$	3	6
$\mathbb{Z}_2$	K_4'	$\mathbb{Z}_2$	3	6
$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	4	4

Table 4: For each stabilizer Γ for the $PSL_2(\mathbb{F}_7)$-action on $X \times H^0(X, K^{\otimes 2})$, express $\Gamma = J \cap H$, where J and H are stabilizers for the $PSL_2(\mathbb{F}_7)$-actions on X and $H^0(X, K^{\otimes 2})$.

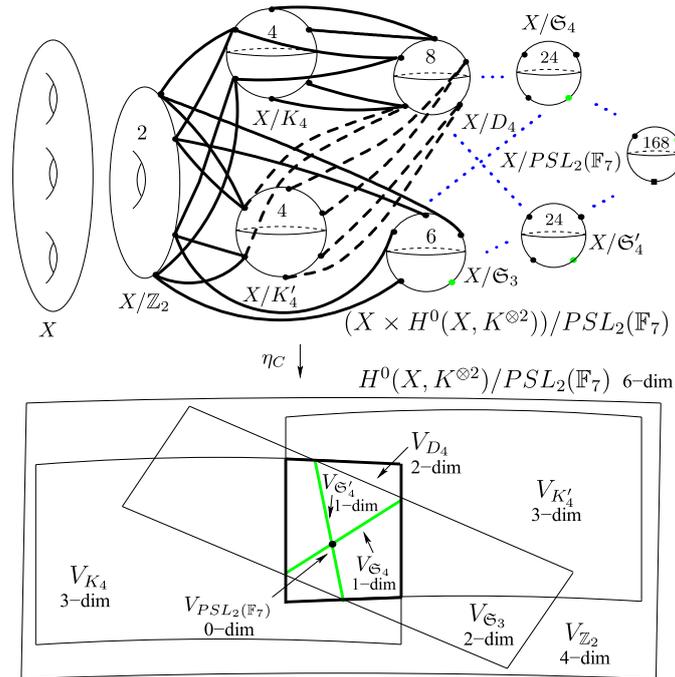
Figure 4.3: For the Klein curve X, its canonical quotient family $(X \times H^0(X, K^{\otimes 2}))/PSL_2(\mathbb{F}_7) \to H^0(X, K^{\otimes 2})/PSL_2(\mathbb{F}_7)$ has the above configurations of veins, fibers, and ridges.

Correspondence of stabilizers and conveins in the base space

For a linear quotient family $\eta : (X \times \mathbb{C}^n)/G \to \mathbb{C}^n/G$, the arrangement of the conveins in $\mathbb{C}^n/G$ is described by the Hasse diagram reversing the inclusion relation of the Hasse diagram for

the poset of stabilizers for the G-action on $\mathbb{C}^n$ with partial order by the inclusion relation, i.e. $K \leq H$ if $K \subset H$ (in this case, $CV_K \supset CV_H$).

For the $\mathrm{Aut}(X)$-actions on $H_1(X, \mathbb{C})$ and $H^0(X, K^{\otimes 2})$, the Hasse diagrams of the posets of stabilizers are given by (A) and (B) of Diagram 4.4 (note that (B) is much more complicated than (A)). The Hasse diagrams of the corresponding conveins to (A) and (B) are in turn given by (a) and (b) of Diagram 4.5, where the inclusions of (a), (b) are opposite to those of (A), (B) of Diagram 4.4, as $H \subset H' \implies CV_H \supset CV_{H'}$.
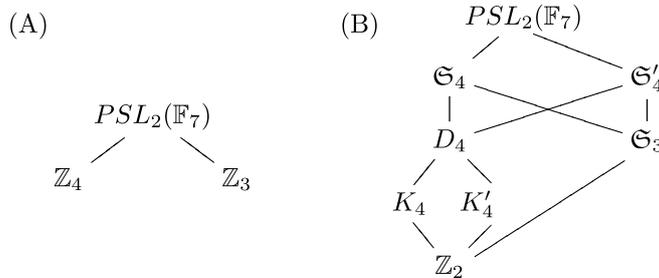

Diagram 4.4: (A) and (B) are the Hasse diagrams of stabilizers for the $\mathrm{Aut}(X)$-actions on $H_1(X, \mathbb{C})$ and $H^0(X, K^{\otimes 2})$. "$A - B$" means "$A \subset B$".
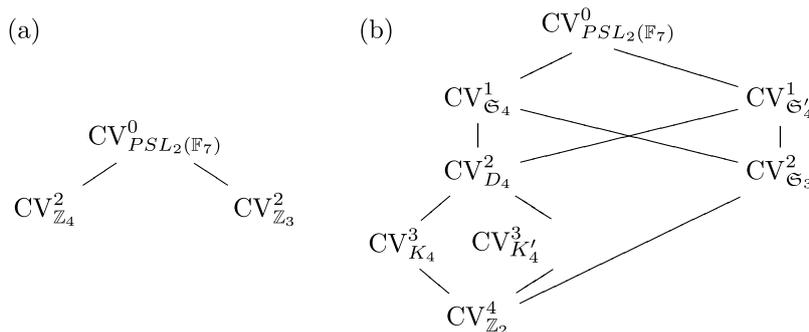

Diagram 4.5: (a) and (b) are the Hasse diagrams of conveins for the $\mathrm{Aut}(X)$-actions on $H_1(X, \mathbb{C})$ and $H^0(X, K^{\otimes 2})$. CV_H^d means that the H-convein CV_H is of dimension d.

Application to the local decription of the univeral family

For a finite subgroup H of the mapping class group Γ_g, the *global* convein CV_H^{global} is the image, under the quotient map $T_g \to M_g = T_g/\Gamma_g$, of the fixed point set by the H-action. Alternatively CV_H^{global} is the locus of M_g consisting of the points $[X]$ such that $\mathrm{Aut}(X) \supset H$. The *maximal* global conveins are the irreducible components of the singular locus of M_g for $g \geq 4$. For $g = 2, 3$, the situation is slightly different: let H be the cyclic group of order 2 generated by the hyperelliptic involution, then for $g = 2$, CV_H^{global} is the whole space M_2, and for $g = 3$, CV_H^{global} is a smooth locus of M_3.

According to the list in p.151 of [Cor], for $g = 3$, $CV_{\mathbb{Z}_7}^{\mathrm{global}}$, $CV_{\mathbb{Z}_3}^{\mathrm{global}}$ and $CV_{\mathbb{Z}_2}^{\mathrm{global}}$ are the irreducible components of the singular locus of M_3, where $CV_{\mathbb{Z}_2}^{\mathrm{global}}$ is not the hyperelliptic locus (i.e. this $\mathbb{Z}_2$ is not generated by the hyperelliptic involution) and the dimensions of $CV_{\mathbb{Z}_7}^{\mathrm{global}}$, $CV_{\mathbb{Z}_3}^{\mathrm{global}}$ and $CV_{\mathbb{Z}_2}^{\mathrm{global}}$ are 0, 2 and 4. Related to this, our result is as follows:

Result 4. *The point of M_3 corresponding to the Klein curve lies on $CV_{\mathbb{Z}_2}^{\mathrm{global}}$, but it does not lie on $CV_{\mathbb{Z}_3}^{\mathrm{global}}$ and $CV_{\mathbb{Z}_7}^{\mathrm{global}}$.*

We emphasize that we are interested in *all* conveins (not only maximal ones) — we determined them around the Klein point of M_3.

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