Computation of equivariant cohomology of certain $4n$-dimensional manifold with $(n+1)$-dimensional torus actions

岡山理科大学理学部応用数学科　黒木　慎太郎
Shintarô Kuroki
Dep. of Applied Math. Faculty of Science, Okayama University of Science

1. Introduction

This article is a progress report on joint work with Bidhan Paul.

1.1. $4n$-dimensional homogeneous space with a T^{n+1}-action. In [**KU24**], we computed a certain class of graph equivariant cohomology for GKM graphs corresponding to $4n$-dimensional manifolds with a T^{n+1}-action, motivating by the study of toric hyperKähler manifolds. In this paper, we regard such a space as a homogeneous manifold. More precisely, let G/K be a maximal rank homogeneous space, where G is a simply connected, simple compact Lie group. It is known that the action of a maximal torus T-action on such a space G/K gives a GKM manifold (see [**GHZ06**]). By [**Ku10**, Proposition 4.1] (also see [**Ku14**]), if $\dim G/K = 4n$ and $\mathrm{rank} G = \mathrm{rank} K = n+1$, then there are exactly three possible cases:

- the complex quadric $Q_{2n} = SO(2n+2)/SO(2n) \times SO(2)$;
- the quaternionic projective space $\mathbb{H}P^n = Sp(n+1)/Sp(1) \times Sp(n)$;
- the complex 2-plane Grassmaniann $G_2(\mathbb{C}^{n+2}) = SU(n+2)/S(U(2) \times U(n))$.

In [**Ku25**] (also see [**KP**]), we computed $H^*_T(Q_{2n})$ using generators and relations of the corresponding GKM subgraphs. Moreover, one easily checks that

$$H^*_T(\mathbb{H}P^n) \simeq \mathbb{Z}[x_1, \dots, x_{n+1}, p]/\langle \prod_{i=1}^{n+1} (x_i^2 - p) \rangle$$

where $\deg x_i = 2$ and $\deg p = 4$. Note that the relation $\prod_{i=1}^{n+1}(x_i^2 - p)$ is obtained from the combinatorial data of the GKM graph of $\mathbb{H}P^n$ with T-action. This paper aims to describe the equivariant cohomology of the Grassmannians

$$G_2(\mathbb{C}^{n+2}) = SU(n+2)/S(U(2) \times U(n))$$

for $n = 1, 2, 3, 4$ by using the generators and relations of the corresponding GKM graphs.

Remark 1.1. Knutson and Tao computed

$$H^*_{T^{n+2}}(G_2(\mathbb{C}^{n+2}))$$

by using the puzzle (see [**KT03**]). In contrast, our generators are obtained from certain subgraphs of the associated GKM graph. Thus, our approach is different from that of Knutson-Tao.

Note that $G_2(\mathbb{C}^3) = SU(3)/S(U(2) \times U(1)) \simeq \mathbb{C}P^2$ and $G_2(\mathbb{C}^4) = SU(4)/S(U(2) \times U(2)) \simeq Q_4$. Therefore, in what follows, we only introduce the cohomology rings for

- $G_2(\mathbb{C}^5) = SU(5)/S(U(2) \times U(3))$,
- $G_2(\mathbb{C}^6) = SU(6)/S(U(2) \times U(4))$.

2. The Grassmannian $G_2(\mathbb{C}^{n+2})$

We begin by recalling known results for the Grassmannian $G_2(\mathbb{C}^{n+2})$.

2.1. GKM graph of the Grassmannian $G_2(\mathbb{C}^{n+2})$. The GKM graph of the Grassmannian $G_2(\mathbb{C}^{n+2})$ with respect to the torus $T^{n+1}(=: T)$ is given by

$$J(2, n+2) = (\Gamma_n, \alpha_n)$$

as computed in [**Ku19**] (see [**Ku09**] about the basic facts for GKM theory). We briefly recall its structure.

The abstract graph Γ_n, called the *Johnson graph* (also see the *uniform matroid* $U_{2,[n+2]}$ in [**O06**]), is defined as follows:

- The set of vertices is $V(\Gamma_n) = \{ij \mid i, j \in [n+2], i \neq j\}$, where $[n+2] = \{1, \ldots, n+2\}$.
- The set of edges is $E(\Gamma_n) = \{(ij, jk) \mid k \neq i\}$.

The axial function $\alpha_n : E(\Gamma_n) \to H^2(BT^{n+1}) \simeq \mathbb{Z}\langle x_1, \ldots, x_{n+2}\rangle$ is defined by

$$\alpha_n(ij, jk) = x_k - x_i,$$

where $x_{n+2} = 0$.

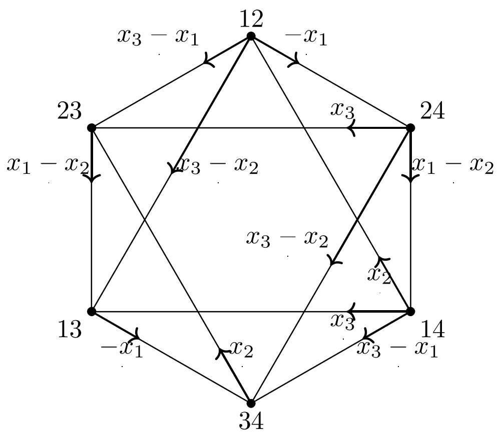

FIGURE 1. The GKM graph (Γ_2, α_2) of $G_2(\mathbb{C}^4)$ with T^3-action.

2.2. Equivariant cohomology of the Grassmannian $G_2(\mathbb{C}^{n+2})$. We denote the graph equivariant cohomology by $H_T^*(J(2, n+2))$. Since $H^{\mathrm{odd}}(G_2(\mathbb{C}^{n+2})) = 0$, it follows from [**GKM98**] that

$$H_T^*(G_2(\mathbb{C}^{n+2})) \simeq H_T^*(J(2, n+2)) := \{f : V(\Gamma_n) \to H^*(BT) \mid f(ij) - f(jk) \equiv 0 \mod x_k - x_i\}.$$

Thus, it suffices to compute $H_T^*(J(2, n+2))$. The defining condition in $H_T^*(J(2, n+2))$ is called a *congruence relation.*

In this subsection, we recall the equivariant cohomology of the Grassmannian $G_2(\mathbb{C}^{n+2})$, computed from the tautological bundles. Define elements $X \in H_T^2(J(2, n+2))$ and $Y \in H_T^4(J(2, n+2))$ by

$$X(ij) := x_i + x_j, \quad Y(ij) := x_i x_j.$$

It is straightforward to verify that both satisfy the congruence relations. Recall that the tautological vector bundle γ over $G_2(\mathbb{C}^{n+2})$ is defined by

$$\gamma := \{(V, x) \in G_2(\mathbb{C}^{n+2}) \times \mathbb{C}^{n+2} \mid x \in V(\simeq \mathbb{C}^2) \subset \mathbb{C}^{n+2}\}.$$

We also consider its orthogonal complement $\gamma^\perp$ inside the trivial bundle $\underline{\mathbb{C}}^{n+2}$ with the standard T-action. Thus, in $H_T^*(G_2(\mathbb{C}^{n+2}))$ we have

$$c^T(\gamma) \cdot c^T(\gamma^\perp) = c^T(\gamma \oplus \gamma^\perp) = c^T(\underline{\mathbb{C}}^{n+2}) = \prod_{i=1}^{n+1}(1 + x_i).$$

Hence,

$$c^T(\gamma) = 1 + X + Y, \quad c^T(\gamma^\perp) = \frac{\prod_{i=1}^{n+1}(1+x_i)}{1+X+Y} \text{ (mod } deg = 2n+2).$$

Since $\gamma^\perp$ is a rank n complex vector bundle, by the same argument as in [**EH13**, Theorem 5.26], we obtain the following theorem.

THEOREM 2.1. *The graph equivariant cohomology $H_T^*(J(2, n+2))$ is given by*

$$H_T^*(J(2, n+2)) \simeq \mathbb{Z}[x_1, \ldots, x_{n+1}, X, Y]/\mathcal{I},$$

where the ideal $\mathcal{I}$ is generated by the terms of total degree $2n+2$ and $2n+4$ in the power series expansion

$$\frac{\prod_{i=1}^{n+1}(1+x_i)}{1+X+Y} = \prod_{i=1}^{n+1}(1+x_i)(1-(X+Y)+(X+Y)^2 - \cdots) \in \mathbb{Z}[[X,Y]],$$

where $\deg x_i = \deg X = 2,\ \deg Y = 4$. *More explicitly, the degree $2n+2$ term is*

$$\sum_{i=0}^{n+1} f_{n+1-i}(X,Y)e_i(x),$$

and the degree $2n+4$ term is

$$\sum_{i=0}^{n+2} f_{n+2-i}(X,Y)e_i(x),$$

where $f_j(X,Y)$ is the $2j$ term in $(1-(X+Y)+(X+Y)^2 - \cdots)$ and $e_i(x)$ is the ith symmetric product of $x_1, \ldots, x_{n+1}$.

The ordinary cohomology can be obtained by setting $x_i = 0$ in the above formula. Thus, we obtain the following corollary (see [**EH13**, Theorem 5.26]).

COROLLARY 2.2. The ordinary cohomology $H^*(G_2(\mathbb{C}^{n+2}))$ is given by

$$H^*(G_2(\mathbb{C}^{n+2})) \simeq \mathbb{Z}[x,y]/\mathcal{J},$$

where the ideal $\mathcal{J}$ is generated by the terms of total degree $2n+2$ and $2n+4$ in the power series expansion

$$\frac{1}{1+x+y} = 1-(x+y)+(x+y)^2 - \cdots \in \mathbb{Z}[[x,y]],$$

where $\deg x = 2,\ \deg y = 4$.

3. Main result

In this section, we state the main result of the paper.

3.1. Subgraph classes of $H_T^*(J(2, n+2))$. We first recall several important subgraph classes in $J(2, n+2)$:

(1) For the vertex $ij \in V_{2,n+2}$, define

$$M_{ij} := x_i + x_j - X \in H_T^2(J(2, n+2))$$

with evaluation $M_{ij}(ab) = x_i + x_j - x_a - x_b$.

(2) For the subset $E_i = [n+2] \setminus \{i\}$, define

$$U_{2,E_i} \in H_T^4(J(2, n+2))$$

by $U_{2,E_i}(ab) := (x_i - x_a)(x_i - x_b)$.

(3) For the triangle $\Delta_{i,j,k} = \{ij, ik, jk\}$, defind

$$M_{\Delta_{i,j,k}} \in H^4_T(J(2, n+2))$$

by

$$M_{\Delta_{i,j,k}}(ab) = x_a^2 + x_b^2 + e_2(-x_a, -x_b, x_i, x_j, x_k),$$

where e_2 denotes the second elementary symmetric polynomial.

(4) For the n-simplex $\Delta_i^{(n)} = \{ij \mid j \in [n+2] \setminus \{i\}\}$, defined

$$\Delta_i^{(n)} \in H^{2n}_T(J(2, n+2))$$

by $\Delta_i^{(n)}(jk) = 0$ if $j, k \neq i$, and

$$\Delta_i^{(n)}(ij) = \prod_{a \in [n+2] \setminus \{i,j\}} (x_a - x_i).$$

It is straightforward to verify that all of these elements satisfy the congruence relations; hence they indeed define classes in $H^*_T(J(2, n+2))$.

3.2. The equivariant cohomology ring of $G_2(\mathbb{C}^5)$. Let $[5] = \{i, j, k, l, m\}$. Then, $J(2, 5)$ is the one-skeleton of the hypersimplex $\Delta_{2,5}$. In this case, we have the following classes:

- $M_{ij} \in H^2_T(J(2, 5))$ (the singular Thom class of one point removing subgraph);
- $U_{2,E_i} \in H^4_T(J(2, 5))$ (the Thom class of $J(2, 4) \subset J(2, 5)$);
- $M_{\Delta_{i,j,k}} \in H^4_T(J(2, 5))$ (the singular Thom class of the triangle removing subgraph);
- $\Delta_i^{(3)} := M_{\Delta_{j,k,l}} M_{ml} - M_{jk} U_{2,E_l} \in H^6_T(J(2, 5))$ (the class of a 3-simplex);
- $\Delta_i^{(2)} := M_{\Delta_{j,k,l}} \in H^4_T(J(2, 4))$ (the class of a 2-simplex in $[5] \setminus \{p\}$)

We obtain the following theorem.

THEOREM 3.1. *For $n = 3$,*

$$H^*_T(J(2, 5)) \simeq \mathbb{Z}[M_{ij}, U_{2,E_i}, M_{\Delta_{i,j,k}} \mid i, j, k \in [5]]/\mathcal{I}$$

where $\mathcal{I}$ is the ideal generated by the following (R1)–(R4):

(R1): $M_{ij} + M_{kl} = M_{ik} + M_{jl}$;
(R2): $\Delta_i^{(3)} U_{2,E_i} = 0$;
(R3): $M_{ij} M_{jk} = M_{\Delta_{i,j,k}} + U_{2,E_j}$;
(R4): $\Delta_i^{(3)} M_{im} = U_{2,E_m} M_{\Delta_{i,j,k}} (= U_{2,E_m} \Delta_i^{(2)})$.

3.3. The equivariant cohomology ring of $G_2(\mathbb{C}^6)$. Let $[6] = \{i, j, k, l, m, p\}$. Then, $J(2, 6)$ is the one-skeleton of the hypersimplex $\Delta_{2,6}$). In this case, we have the following classes:

- $M_{ij} \in H^2_T(J(2, 6))$ (the singular Thom class of one point removing subgraph);
- $U_{2,E_i} \in H^4_T(J(2, 6))$ (the Thom class of $J(2, 5) \subset J(2, 6)$);
- $M_{\Delta_{i,j,k}} \in H^4_T(J(2, 6))$ (the singular Thom class of the triangle removing subgraph);
- $\Delta_i^{(4)} = M_{\Delta_{p,j,m}} M_{\Delta_{k,l,m}} - U_{2,E_m} M_{pj} M_{kl} \in H^8_T(J(2, 6))$ (the class of a 4-simplex);
- $\Delta_i^{(3)} = M_{\Delta_{j,k,l}} M_{ml} - M_{jk} U_{2,E_l}$ (which is $\Delta_i^{(3)}$ as above in $[6] \setminus \{p\}$).

We obtain the following theorem:

THEOREM 3.2. *For $n = 4$,*

$$H^*_T(J(2, 6)) \simeq \mathbb{Z}[M_{ij}, U_{2,E_i}, M_{\Delta_{i,j,k}} \mid i, j, k \in [6]]/\mathcal{I}$$

where $\mathcal{I}$ is the ideal generated by the following (R1)–(R4):

(R1): $M_{ij} + M_{kl} = M_{ik} + M_{jl}$ *and* $M_{ij} + M_{jk} + M_{lm} = M_{ik} + M_{jm} + M_{lj}$;
(R2): $\Delta_i^{(4)} U_{2,E_i} = 0$;
(R3): $M_{ij} M_{jk} = M_{\Delta_{i,j,k}} + U_{2,E_j}$;
(R4): $\Delta_i^{(4)} M_{ip} = U_{2,E_p} \Delta_i^{(3)}$.

Note that $\Delta_m^{(n)}$ for $n = 1, 2, 3, 4$ is as follows:

$n=0$: $\Delta_i^{(0)} = 1$;
$n=1$: $\Delta_i^{(1)} = M_{jk} \in H_T^2(J(2,3))$ for $\{i,j,k\} = [3]$;
$n=2$: $\Delta_i^{(2)} = M_{\Delta_{j,k,l}} \in H_T^4(J(2,4))$ for $\{i,j,k,l\} = [4]$;
$n=3$: $\Delta_i^{(3)} = M_{\Delta_{j,k,l}} M_{ml} - M_{jk} U_{2,E_l} \in H_T^6(J(2,5))$ for $\{i,j,k,l,m\} = [5]$;
$n=4$: $\Delta_i^{(4)} = M_{\Delta_{p,j,m}} M_{\Delta_{k,l,m}} - U_{2,E_m} M_{pj} M_{kl} \in H_T^8(J(2,6))$ for $\{i,j,k,l,m,p\} = [6]$;

From the above observations, in order to obtain a more general formula for $H_T^*(J(2,n+2))$, it may be necessary to employ certain *recursive relations*.

Further details of this article, together with more general results for $H_T^*(G_2(\mathbb{C}^{n+2})) = H_T^*(J(2,n+2))$, will appear elsewhere.

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Department of Applied Mathematics Faculty of Science, Okayama University of Science, 1-1 Ridai-cho Kita-ku Okayama-shi Okayama 700-0005, Okayama, Japan

Email address: kuroki@ous.ac.jp