The open cell property in linearly ordered structures

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Abstract

A linearly ordered structure $(M, <, \dots)$ is said to be o-minimal if every definable subset of M is a finite union of points and intervals. A linearly ordered structure $(M, <, \dots)$ is said to be weakly o-minimal if every definable subset of M is a finite union of points and definable convex sets. Under suitable conditions, o-minimal structures satisfy the open cell property. In this paper, we study the open cell property in weakly o-minimal structures.

Throughout this paper, "definable" means "definable possibly with parameters" and we assume that a structure $\mathcal{M} = (M, <, \dots)$ is a dense linear ordering $<$ without endpoints.

A subset A of M is said to be *convex* if $a, b \in A$ and $c \in M$ with $a < c < b$ then $c \in A$. Moreover if $A = \emptyset$ or $\inf A, \sup A \in M \cup \{-\infty, +\infty\}$, then A is called an *interval* in M. We say that $\mathcal{M}$ is *o-minimal* (*weakly o-minimal*) if every definable subset of M is a finite union of points and intervals (definable convex sets), respectively. A theory T is said to be *weakly o-minimal* if every model of T is weakly o-minimal. The reader is assumed to be familiar with fundamental results of o-minimality and weak o-minimality; see, for example, [3], [4], [5], or [7].

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For any subsets C, D of M, we write $C < D$ if $c < d$ whenever $c \in C$ and $d \in D$. A pair $\langle C, D \rangle$ of non-empty subsets of M is called a *cut* in M if $C < D, C \cup D = M$ and D has no lowest element. A cut $\langle C, D \rangle$ is said to be *definable* in $\mathcal{M}$ if the sets C, D are definable in $\mathcal{M}$. The set of all cuts definable in $\mathcal{M}$ will be denoted by $\overline{M}$. Note that we have $M = \overline{M}$ if $\mathcal{M}$ is o-minimal. We define a linear ordering on $\overline{M}$ by $\langle C_1, D_1 \rangle < \langle C_2, D_2 \rangle$ if and only if $C_1 \subsetneq C_2$. Then we may treat $(M, <)$ as a substructure of $(\overline{M}, <)$ by identifying an element $a \in M$ with the definable cut $\langle (-\infty, a], (a, +\infty) \rangle$.

We equip M $(\overline{M})$ with the *interval topology* (the open intervals form a base), and each product M^n $(\overline{M}^n)$ with the corresponding product topology, respectively.

We recall the notion of definable functions from [7]. Let n be a positive integer and $A \subseteq M^n$ definable. A function $f : A \to \overline{M}$ is said to be *definable* if the set $\{\langle x, y \rangle \in M^{n+1} : x \in A,\ y < f(x)\}$ is definable. A function $f : A \to \overline{M} \cup \{-\infty, +\infty\}$ is said to be *definable* if f is a definable function from A to $\overline{M}$, $f(x) = -\infty$ for all $x \in A$, or $f(x) = +\infty$ for all $x \in A$.

We recall the notion of strong cells from [8].

Definition 1. Suppose that $\mathcal{M} = (M, <, \ldots)$ is a weakly o-minimal structure. For each positive integer n, we inductively define *strong cells* in M^n and their completions in $\overline{M}^n$.

(1) A one-element subset of M is called a *strong* 0-*cell* in M. If $C \subseteq M$ is a strong 0-cell, then its completion $\overline{C} := C$.

(2) A non-empty definable convex open subset of M is called a *strong* 1-*cell* in M. If $C \subseteq M$ is a strong 1-cell, then its completion $\overline{C} := \{x \in \overline{M} : (\exists a, b \in C)(a < x < b)\}$.

Assume that k is a non-negative integer, and strong k-cells in M^n and their completions in $\overline{M}^n$ are already defined.

(3) Let $C \subseteq M^n$ be a strong k-cell in M^n and $f : C \to M$ is a definable continuous function which has a continuous extension $\overline{f} : \overline{C} \to \overline{M}$. Then the graph $\Gamma(f)$ is called a *strong k-cell* in M^{n+1} and its completion $\overline{\Gamma(f)} := \Gamma(\overline{f})$.

(4) Let $C \subseteq M^n$ be a strong k-cell in M^n and $g, h : C \to \overline{M} \cup \{-\infty, +\infty\}$ are definable continuous functions which have continuous extensions $\overline{g}, \overline{h} : \overline{C} \to \overline{M} \cup \{-\infty, +\infty\}$ such that

   (a) each of the functions g, h assumes all its values in one of the sets M, $\overline{M} \setminus M$, $\{\infty\}$, $\{-\infty\}$,

   (b) $\overline{g}(x) < \overline{h}(x)$ for all $x \in \overline{C}$.

   Then the set

$$(g, h)_C := \{\langle a, b\rangle \in C \times M : g(a) < b < h(a)\}$$

   is called a *strong* $(k+1)$*-cell* in M^{n+1}. The completion of $(g, h)_C$ is defined as

$$\overline{(g, h)_C} := \{\langle a, b\rangle \in \overline{C} \times \overline{M} : \overline{g}(a) < b < \overline{h}(a)\}.$$

(5) Let C be a subset of M^n. The set C is called a *strong cell* in M^n if there exists some non-negative integer k such that C is a strong k-cell in M^n.

The notion of strong cells in weakly o-minimal settings is consistent with that of cells in o-minimal settings.

Let C be a strong cell of M^n. A definable function $f : C \to \overline{M}$ is said to be *strongly continuous* if f has a continuous extension $\overline{f} : \overline{C} \to \overline{M}$. A function which is identically equal to $-\infty$ or $+\infty$, and whose domain is a strong cell is also said to be *strongly continuous*.

Definition 2. Let $\mathcal{M} = (M, <, \ldots)$ be a weakly o-minimal structure. For each positive integer n, we inductively define a *strong cell decomposition* (or a *decomposition into strong cells*) in M^n of a non-empty definable set $A \subseteq M^n$.

(1) If $A \subseteq M$ is a non-empty definable set and $\mathcal{D} = \{C_1, \ldots, C_k\}$ is a partition of A into strong cells in M, then $\mathcal{D}$ is called a *decomposition of A into strong cells* in M.

(2) Suppose that $A \subseteq M^{n+1}$ is a non-empty definable set and $\mathcal{D} = \{C_1, \dots, C_k\}$ is a partition of A into strong cells in M^{n+1}. Then $\mathcal{D}$ is called a *decomposition of A into strong cells* in M^{n+1} if $\{\pi(C_1), \dots, \pi(C_k)\}$ is a decomposition of $\pi(A)$ into strong cells in M^n, where $\pi : M^{n+1} \to M^n$ is the projection on the first n coordinates.

Definition 3. Let $\mathcal{M} = (M, <, \dots)$ be a weakly o-minimal structure and n a positive integer. Suppose that $A, B \subseteq M^n$ are definable sets, $A \neq \emptyset$ and $\mathcal{D}$ is a decomposition of A into strong cells in M^n. We say that $\mathcal{D}$ *partitions* B if for each strong cell $C \in \mathcal{D}$, we have either $C \subseteq B$ or $C \cap B = \emptyset$.

Definition 4. A weakly o-minimal structure $\mathcal{M} = (M, <, \dots)$ is said to have the *strong cell decomposition property* if for any positive integers k, n and any definable sets $A_1, \dots, A_k \subseteq M^n$, there exists a decomposition of M^n into strong cells partitioning each of the sets $A_1, \dots, A_k$.

Let $\mathcal{C}$, $\mathcal{D}$ be strong cell decompositions of M^m. We denote $\mathcal{C} \prec \mathcal{D}$ if every strong cell of $\mathcal{D}$ is a subset of some strong cell of $\mathcal{C}$. Then, the relation $\prec$ is a partial order on the family of all strong cell decompositions of M^m.

Lemma 5 ([8, Fact 2.1])**.** *Suppose that $\mathcal{M} = (M, <, \dots)$ is a weak o-minimal structure with the strong cell decomposition property. If $X_1, \dots, X_k \subseteq M^m$ are definable sets, then there exists the smallest strong cell decomposition $\mathcal{C}$ of M^m partitioning each of $X_1, \dots, X_k$.*

Definition 6 ([6, Definition 3.1])**.** Suppose that $\mathcal{M} = (M, <, \dots)$ is a weak o-minimal structure with the strong cell decomposition property. Let X be a definable subset of M^m and $\mathcal{C}$ the smallest strong cell decomposition of M^m partitioning X. Then we set the completion of X in $\overline{M}^m$ as $\overline{X} := \bigcup\{\overline{C} : C \in \mathcal{C} \wedge C \subseteq X\}$.

An o-minimal structure $\mathcal{M} = (M, <, \dots)$ is said to have the open cell property if every non-empty definable open subset of M^n is a union of finitely many open cells. A weakly o-minimal structure $\mathcal{M} = (M, <, \dots)$ is said to have the open cell property if every non-empty definable open subset of M^n is a union of finitely many open strong cells. In o-minimal settings, the following results hold.

Theorem 7 ([9, Theorem 1.3])**.** *Let $\mathcal{M} = (M, <, +, \cdot, \dots)$ be an o-minimal expansion of a real closed field. Then any definable, bounded open subset of M^n is a union of finitely many open cells.*

Remark 8. *Wilkie's aforementioned result requires boundedness. For example, we suppose that $\mathcal{M} = (M, <, +, \cdot, \dots)$ be an o-minimal expansion of a real closed field. Let $C_1 = \{\langle x, y\rangle \in M^2 : x > 0, y < 1/x\}$,*
$C_2 = \{\langle x, y\rangle \in M^2 : x < 0, y < -1/x\}$, and $C_3 = \{\langle x, y\rangle \in M^2 : x = 0\}$. Then $C_1 \cup C_2 \cup C_3$ is a definable open subset of M^2. However C_3 is not open in M^2.

Theorem 9 ([1, Theorem 2])**.** *Let $\mathcal{M} = (M, <, +, \dots)$ be an o-minimal expansion of an ordered group. If $\mathcal{M}$ is linear, then $\mathcal{M}$ have the open cell property.*

Theorem 10 ([2, Theorem 1.1])**.** *Let $\mathcal{M} = (M, <, +, \dots)$ be an o-minimal expansion of an ordered group. If $\mathcal{M}$ is semi-bounded, then $\mathcal{M}$ have the open cell property.*

Let $\mathcal{M} = (M, <, +, \ldots)$ be a weakly o-minimal expansion of an ordered group $(M, <, +)$. Then, the weakly o-minimal structure $\mathcal{M}$ is said to be *non-valuational* if for any definable cut $\langle C, D\rangle$ we have $\inf\{d - c : c \in C,\ d \in D\} = 0$.

Fact 11 ([7, Corollary 2.16])**.** *Let $\mathcal{M} = (M, <, +, \ldots)$ be a weakly o-minimal expansion of an ordered group $(M, <, +)$. Then the following conditions are equivalent.*

1. *$\mathcal{M}$ is non-valuational.*
2. *$\mathcal{M}$ has the strong cell decomposition property.*

Using the same method as in the proof of Proposition 6.1.2 of [4], we can prove the following proposition.

Proposition 12. *Suppose that $\mathcal{M} = (M, <, +, \dots)$ is a non-valuational weakly o-minimal expansion of an ordered group. Let $S \subseteq M^{n+1}$ be definable, $\pi : M^{n+1} \to M^n$ the projection on the first n coordinates. Then, there exists a definable map $f : \pi(S) \to \overline{M}$ such that $\Gamma(f) \subseteq \overline{S}$.*

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