

Ordinary cohomology groups as higher codimensional birational invariants

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Abstract

As a sequel to [M25], I shall report on my recent investigation of algebro-geometric invariants observed in topology. Here the focus is the bare ordinary cohomologies.

1 Background

The kind of algebro-geometric invariance I have been interested in is the following very naive one, which interpolates the biregular invariance and the birational invariance:

Definition 1.1. *Let us say smooth projective equi-dimensional k -schemes X, Y are codimension $> c$ birational equivalent (or isomorphism in codimension c), if there are closed subsets $Z_X \subset X$ and $Z_Y \subset Y$ s.t.*

- $\text{codim}_X Z_X > c, \quad \text{codim}_Y Z_Y > c.$
- $X \setminus Z_X \xrightarrow{f} Y \setminus Z_Y.$

(Whereas the case $c = \dim X = \dim Y$ is the biregular invariance, the case $c = 0$ is the usual birational equivalence.)

Then, for a smooth projective complex variety X , we obtain various quantities as the cokernels of the cycle map $cl_{Hdg}^i(X)$ and related maps, as follows:

$$\begin{array}{ccc}
 CH^i(X) & \xrightarrow{cl_{Hdg}^i(X)} & \text{Hdg}^{2i}(X, \mathbb{Z}) := \\
 & & H^{2i}(X, \mathbb{Z}) \cap H^{i,i}(X, \mathbb{C}) \\
 \downarrow cl_{Tot}^i(X) & \searrow & \downarrow \\
 (MU^*(X) \otimes_{MU^*} \mathbb{Z})^{2i} & \xrightarrow{cl_{Betti}^i(X)} & H^{2i}(X, \mathbb{Z}) \\
 & \xrightarrow{Thom^{2i}(X)} & \\
 \Rightarrow & & \\
 Z_{Hdg}^{2i}(X)\{tors\} \hookrightarrow Z_{Hdg}^{2i}(X) := \text{Coker}(cl_{Hdg}^i(X)) & & \\
 \parallel & & \downarrow \\
 Z_{Betti}^{2i}(X)\{tors\} \hookrightarrow Z_{Betti}^{2i}(X) := \text{Coker}(cl_{Betti}^i(X)) \twoheadrightarrow Z_{Thom}^{2i}(X) := \text{Coker}(Thom^{2i}(X)) & &
 \end{array}$$

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In my previous report [M25], I gave the following algebro-geometric invariant interpretations for these quantities, amongst of all:

Theorem 1.2. (i) $Z_{Hdg}^{2i}(X)$ is codimension $> (\dim_{\mathbb{C}} X - 1) - i$ birational invariant.

(When $i = \dim_{\mathbb{C}} X - 1$, this recovers the birational invariance of $Z_{Hdg}^{2 \dim_{\mathbb{C}} X - 2}(X)$, which was first observed by Soulé-Voisin and Voisin.)

(ii) $Z_{Hdg}^{2i}(X) \otimes \mathbb{Q}$ is codimension $> (\dim X_{\mathbb{C}} - 2) - i$ birational invariant.

(When $i = \dim_{\mathbb{C}} X - 2$, this recovers the birational invariance of $Z_{Hdg}^{2 \dim_{\mathbb{C}} X - 4}(X)_{\mathbb{Q}}$, which should be well-known to experts as this too can be immediately observed from the blowup formula, under the weak factorization theorem.)

(iii) Each p -primary component $Z_{Thom}^j(X)_{(p)}$ of the finite abelian group $Z_{Thom}^j(X)$ is codimension $> \left[\dim_{\mathbb{C}} X - \frac{(2p+1)+j}{2} \right]$ birational invariant.

(When $j = 2 \dim_{\mathbb{C}} X - 2p, 2 \dim_{\mathbb{C}} X - (2p + 1)$, this recovers the birational invariance of $Z_{Thom}^{2 \dim_{\mathbb{C}} X - 2p}(X)_{(p)}, Z_{Thom}^{2 \dim_{\mathbb{C}} X - (2p+1)}(X)_{(p)}$, which appear to be new.)

Whereas these results should be of independent interest, especially in their connections with the counter-examples of the integral Hodge conjectures, they have obvious drawbacks:

Quanties $Z_{Hdg}^*(X)$, $Z_{Hdg}^*(X) \otimes \mathbb{Q}$, and $Z_{Thom}^*(X)_{(p)}$, especially the first two, are usually extremely difficult to compute!

So, it is highly desirable to have more tractable invariants as our disposal of algebro-geometric invariants.

Presumably, the most well-known classical result in this direction is the following theorem of Artin-Mumford [AM72]:

Theorem 1.3. $H^3(X)\{\text{tos}\}$ is a birational invariant.

Now, my main result in this report hierarchically generalize this result of Artin-Mumford (with some little bobus) as follows:

Theorem 1.4. • $H^{2c}(X), H^{2c+1}(X)$ are both codimension $> c$ birational invariants.

• $H^{2c+2}(X)\{\text{tors}\}, H^{2c+3}(X)\{\text{tors}\}$ are both codimension $> c$ birational invariants.

We also have the corresponding results of $H^j(X), H^j(X)\{\text{tors}\}$ for large j , by the universal coefficient theorem and the Poincaré duality.

Both Theorem 1.2 and Theorem 1.4 follow from some general theorem of higher codimensional birational invariants.

For its precise statement and its proof, please wait for my arxiv upload in a very near futute.

References

- [AM72] M. Artin, D. Mumford, *Some elementary examples of unirational varieties which are not rational*, Proc. London Math. Soc. (3) 25 (1972), 75–95.
- [AH62] M.F. Atiyah, F. Hirzebruch, *Analytic cycles on complex manifolds*, Topology 1 (1962), 25–45.
- [M25] Norihiko Minami, *Algebro-geometric invariants reflected in counter-examples of the (integral) Hodge conjecture*, RIMS Kôkyûroku No.2303, 13, 5p, Jan. 2025.