# $p$-TORAL GROUPS WITHOUT THE WEAK BORSUK-ULAM PROPERTY

Abstract. In our recent research, we constructed certain $S^1 \times C_p$-maps that serve as counterexamples to the weak Borsuk-Ulam property in the class of $p$-toral groups. In this note, we present our main results together with an outline of the proof. We also discuss some related results on the right Borsuk-Ulam invariant.

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## 1. Introduction

Let $G$ be a compact Lie group. In this note, a $G$-representation $V$ is assumed to be orthogonal and fixed-point-free, that is, $V^G = 0$, unless otherwise stated. $S(V)$ denotes the unit sphere of $V$, called a $G$-representation sphere. All maps between $G$-spaces are assumed to be continuous.

Borsuk [2] proved the celebrated Borsuk-Ulam theorem, which was later extended in several directions. One extension provides the non-existence of equivariant maps. The following result is known as a generalization of the Borsuk-Ulam theorem.

**Theorem 1.1** (Borsuk-Ulam theorem for $G = C_p^k$, $T^k$)**.** *Let $G$ be an elementary abelian group $C_p^k$ of rank $k$ or a $k$-dimensional torus $T^k$. Then if there exists a $G$-map $f : S(V) \to S(W)$, then the inequality* $\dim V \leq \dim W$ *holds.*

A natural question is: for which groups does the Borsuk-Ulam theorem hold? We say that $G$ has the *Borsuk-Ulam property* if the Borsuk-Ulam theorem holds for $G$. After many contributions by various authors, we finally proved the following:

**Theorem 1.2** (see [6, 7])**.** *$G$ has the Borsuk-Ulam property if and only if $G$ is $C_p^k$ or $T^k$.*

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On the other hand, Bartsch [1] studied a weaker version of the Borsuk-Ulam theorem, referred to here as the *weak* Borsuk-Ulam theorem. We say that $G$ has the *weak Borsuk-Ulam property* if the weak Borsuk-Ulam theorem holds for $G$. Bartsch [1] showed that finite $p$-groups have the weak Borsuk-Ulam theorem. It is reasonable to consider extending this result to $p$-toral groups, where we say that $G$ is a $p$-toral group if $G$ has an extension $1 \to T^k \to G \to P \to 1$, $P$ a finite $p$-group. However, this expectation turned out to be incorrect. We have the following result.

**Theorem 1.3.** *The group $S^1 \times C_p$ does not have the weak Borsuk-Ulam property.*

By lifting the group action, we immediately obtain the following.

**Corollary 1.4.** *If $G$ has a quotient group isomorphic to $S^1 \times C_p$, then $G$ does not have the weak Borsuk-Ulam property.*

In the following sections, we first recall the weak Borsuk-Ulam property and known results, and we then proceed to outline the proof of the theorem. Further details can be found in [8].

## 2. The weak Borsuk-Ulam property

We recall the Borsuk-Ulam function $b_G : \mathbb{N} \to \mathbb{N}$, introduced by Bartsch [1].

**Definition**. The value $b_G(n)$ is defined as the minimum of $\dim W$ such that there exists a $G$-map $f : S(V) \to S(W)$ for some $V$ with $\dim V \geq n$.

Clearly $b_G$ is non-decreasing. Moreover, if there exists a $G$-map $f : S(V) \to S(W)$, then $b_G(\dim V) \leq \dim W$.

**Definition**. We say that $G$ has the weak Borsuk-Ulam property if $\lim_{n\to\infty} b_G(n) = \infty$.

Clearly, the Borsuk-Ulam property implies the weak Borsuk-Ulam property. Hence $C_p^k$ and $T^k$ have the weak Borsuk-Ulam property. Furthermore, the following result is known.

**Proposition 2.1** ([1]). *Let $G$ be a compact Lie group.*

1. *If $G$ has the weak Borsuk-Ulam property, then $G$ is a $p$-toral group.*
2. *Weak Borsuk-Ulam theorem. If $G$ is a finite $p$-group, then $G$ has the weak Borsuk-Ulam property.*

Therefore, in the case of finite groups, a group $G$ has the weak Borsuk-Ulam property if and only if $G$ is a $p$-group. These results might lead one to expect that all $p$-toral groups have the weak Borsuk-Ulam property. However, this is not the case. In fact, as mentioned in the Introduction, the group $S^1 \times C_p$ does not have the weak Borsuk-Ulam property.

## 3. Outline of the proof

A full proof of the main result can be found in [8]. Here, we provide an outline of the argument. Set

$$G := S^1 \times C_p = \{ta^i \mid t \in S^1, i \in \mathbb{Z}\}.$$

We denote by $V_{k,l}, (= \mathbb{C})$ the complex irreducible $G$-representations, where $k \in \mathbb{Z}$ and $l \in \mathbb{Z}/p$. The action of $G$ on $V_{k,l}$ is defined by

$$t \cdot z = t^k z, \quad a \cdot z = \xi_p^l z \quad (\xi_p = e^{2\pi i/p},\ z \in V_{k,l}).$$

For $n \geq 1$ and $m \geq 1$, set

$$V_n := V_{1,1} \oplus V_{p,1} \oplus \cdots \oplus V_{p^{n-1},1}, \textit{and}$$
$$W_m := V_{p^m,0} \oplus V_{0,1}.$$

Note $\dim S(V_n) = 2n-1$, and $\dim S(W_m) = 3$. To prove the main result, we suffice to show the following.

**Theorem 3.1.** *For every $n \geq 1$, there exist $m \geq n$ and a $G$-map $f_n : S(V_n) \to S(W_m)$.*

*Outline of proof.* (i) The Case of $n = 1$. Let $m = 1$. There exists a $G$-map

$$f_1 : S(V_{1,1}) \to S(V_{p,1} \oplus V_{0,1})$$

given by

$$f_1(z) = (z^p, 0) \quad \text{for } z \in S(V_1).$$

(ii) The Case of $n = 2$. Let $m = 1$. Inspired by Crabb [4], we can define a $G$-map

$$\bar{f}_2 : V_2 = V_{1,1} \oplus V_{p,1} \to W_2 = V_{p^2,0} \oplus V_{0,1}$$

by

$$\bar{f}_2(z, w) = (z^{p^2} - w^p, \bar{z}^p w).$$

Since $\bar{f}_2^{-1}(0,0) = (0,0)$, we obtain a $G$-map

$$f_2 : S(V_2) \to S(W_2), \quad f_2(z, w) = \frac{\bar{f}_2(z, w)}{\|\bar{f}_2(z, w)\|}.$$

**Remark**. The degree of $f_2$ is zero, since $f_2$ is not surjective.

(iii) The Case of $n = 3$. We show that there exists a $G$-map $f_3 : S(V_3) \to S(W_3)$. Decompose $S(V)$ into

$$S(V_3) \cong X_0 \cup X \times I \cup X_1,$$

where $X_0 = S(V_{1,1}) \times D(V_{p,1} \oplus V_{p^2,1})$, $X = S(V_{1,1}) \times S(V_{p,1} \oplus V_{p^2,1})$, $X_1 = D(V_{1,1}) \times S(V_{p,1} \oplus V_{p^2,1})$. Define a $G$-map $g_0 : X_0 \to S(W_3)$ by $g_0(u, z, w) = (u^{p^3}, 0)$. Lifting $f_2$, we have a $G$-map

$$\tilde{f}_2 : S(V_{p,1} \oplus V_{p^2,1}) \to S(W_3) = S(V_{p^3,0} \oplus V_{0,1}).$$

Set a $G$-map $g_1 : X_1 \to S(W_3)$ を $g_1(u, z, w) = \tilde{f}_2(z, w)$, $h_0 = g_0|_{\partial X_0} : \partial X_0 = X \times \{0\} \to S(W_3)$ and $h_1 = g_1|_{\partial X_1} : \partial X_1 = X \times \{1\} \to S(W_3)$. We construct a $G$-homotopy $H : X \times I \to S(W_3)$ between $h_0$ and $h_1$. This implies that there exists a $G$-map $f_3$.

Let $Y = S(V_{p,1} \oplus V_{p^2,1})$. Then the map

$$\bar{F} : \mathrm{Map}_G(X, S(W_3)) \to \mathrm{Map}_{C_p}(Y, S(W_3)),$$
$$\bar{F}(\alpha)(z_1, z_2) = \alpha(1, (z_1, z_2))$$

is bijective for $\alpha : X \to S(W_3)$, $X = S(V_{1,1}) \times S(V_{p,1} \oplus V_{p^2,1})$, where $1 \in S(V_{1,1}) = S^1 \subset \mathbb{C}$. Since $C_p$ acts freely on $Y$, the equivariant Hopf theorem [3, 5] shows that

$$\deg : [Y, S(W_3)]_{C_p} \to [Y, S(W_3)]_{\{1\}} = [S^3, S^3] \cong \mathbb{Z}$$

is injective, Since $\deg F([h_0]) = \deg F([h_1]) = 0$, it is seen that $\bar{F}(h_1)$ and $\bar{F}(h_1)$ is $C_p$-homotopic and so $h_0$ and $h_1$ is $G$-homotopic. Therefore there exists a $G$-homotopy $H$ between $h_0$ and $h_1$. □

Next consider the case of $n \geq 4$. We show the existence $f_n$ by induction on $n$. Namely we show that if there exists a $G$-map $f_{n-1} : S(V_{n-1}) \to S(W_{m'})$ for some $m' \geq n - 1 \geq 3$, then there exists a $G$-map $f_n : S(V_n) \to S(W_m)$ for some $m \geq n$.

As before, decompose $S(V_n)$ into

$$S(V_n) = X_0 \cup X \times I \cup X_1,$$

where $X_0 = S(V_{1,1}) \times D(V_{p,1} \oplus \cdots \oplus V_{p^{n-1},1})$, $X = S(V_{1,1}) \times S(V_{p,1} \oplus \cdots \oplus V_{p^{n-1},1})$, $X_1 = D(V_{1,1}) \times S(V_{p,1} \oplus \cdots \oplus V_{p^{n-1},1})$. Define a $G$-map $g_0 : X_0 \to S(W_{m'+1})$: $g_0(u, \boldsymbol{x}) = (u^{p^{m'+1}}, 0)$ for $\boldsymbol{x} \in Y = S(V_{p,1} \oplus \cdots \oplus V_{p^{n-1},1})$ and a $G$-map $g_1 : X_1 \to S(W_{m'+1})$, $g_1(u, \boldsymbol{x}) = \tilde{f}_{n-1}(\boldsymbol{x})$, where $\tilde{f}_{n-1}$ is the lift of $f_{n-1}$ via $\pi : G \to G$, $ta^i \mapsto t^p a^i$. Set $h_0 = g_0|_{\partial X_0} : X \times \{0\} \to S(W_{m'+1})$ and $h_1 = g_1|_{\partial X_1} : X \times \{1\} \to S(W_{m'+1})$. In this case, there might be an obstruction to the existence of a G-homotopy. However, the following result ensures that the construction can proceed.

**Lemma 3.2.** *There exists a $G$-map $R_m : S(W_{m'+1}) \to S(W_m)$ for some $m \geq m'+1$ such that $R_m \circ h_0$ and $R_m \circ h_1$ are $G$-homotopic.*

If there exists such a $G$-homotopy $H$ between $R_m \circ h_0$ and $R_m \circ h_1$, then we obtain a $G$-map

$$f_n = R_m \circ g_0 \cup H \cup R_m \circ g_1 : S(V_n) \to S(W_m).$$

□

*Sketch of the proof of Lemma* 3.2. This lemma is proved by using equivariant obstruction theory. The key facts are as follows:

First, the equivariant obstruction groups

$$\mathfrak{H}^k_G(S(V_n), \pi_k) \cong H^k(L^3(p); \pi_k),$$

where $L^3(p)$ is a lens space and $\pi_k = \pi_k(S^3)$, are torsion groups.

Second, for any $G$-map $R_m : S(W_{m'}) \to S(W_m)$, the obstruction class of the composition $R_m \circ f_{n-1}$ on the $k$-skeleton of $X$ satisfies

$$o_G(R_m \circ f_{n-1}) = (\deg R_m) o_G(f_{n-1}).$$

Therefore by choosing a suitable $m$ such that $\deg(R_m)$ annihilates the class $o_G(f_{n-1})$, we see that the obstruction $o_G(R_m \circ f_{n-1})$ vanishes. By repeating this process inductively, we obtain the desired $G$-map $f_n$. □

## 4. Related results

We give some remarks on the Borsuk-Ulam function.

**Proposition 4.1.** *Let $G = S^1 \times C_p$, where $p$ is an odd prime. Then $b_G(n) = 4$ for any $n \geq 4$.*

*Proof.* By Theorem 3.1, we have $b_G(n) \leq 4$. It remains to show that $b_G(n) = 4$.

Suppose, for contradiction, that $b_G(n) < 4$. Then $b_G(n) \leq 2$, since $p$ is odd and $\dim V$ is even. Now assume $\dim V \geq 4$ and $\dim W = 2$. We claim that there is no $G$-maps $f : S(V) \to S(W)$. Indeed, if $W = V_{k,l}$ for some $(k,l) \neq (0,0) \in \mathbb{Z} \times \mathbb{Z}/p$, then we have $W^{S^1} = 0$ or $W^{C_p} = 0$. Accordingly, $V^{S^1} = 0$ or $V^{C_p} = 0$. By the Borsuk-Ulam theorem for $S^1$ or $C_p$, it follows that $\dim V \leq \dim W$, which contradicts $\dim V \geq 4$. □

Next, let $G$ be a compact Lie group. We define

$$r_G(W) = \sup\{\dim V \mid \exists\, f : S(V) \to S(W) \ G\text{-map }\} \leq \infty,$$

which is called the *right Borsuk-Ulam invariant* of $W$. As before, we assume that $V$, $W$ are fixed-point-free representations. We observe the following:

**Lemma 4.2.** *Let $H \leq G$, $W$ be a $G$-representation $W$ with $W^H = 0$. Then $r_G(W) \leq r_H(W)$.*

Determining the right Borsuk-Ulam invariant is a difficult problem. Nevertheless, the following finiteness result holds, which may be regarded as a Borsuk-Ulam type theorem.

**Proposition 4.3.** *If $G$ is a $p$-toral group, then $r_G(W) < \infty$ for every $G$-representation $W$ with $W^G = 0$.*

*Proof.* Consider the short exact sequence

$$1 \to T^k \to G \to P \to 1,$$

where $P$ is a finite $p$-group. By [5, Chapter IV (3.8), 3)], there exists a sequence $\{P_n\}$ of finite $p$-subgroups of $G$ such that $\lim_{n\to\infty} P_n = G$ with respect to the

Hausdorff metric. For sufficiently large $n$, we have $W^{P_n} = W^G = 0$. By Lemma 4.2, we have $r_G(W) \leq r_{P_n}(W)$. Since finite $p$-groups have the weak Borsuk-Ulam property, we conclude that $r_{P_n}(W) < \infty$, and thus $r_G(W) < \infty$. □

Conversely, if $G$ is not a $p$-toral group, then, as shown by Bartsch [1], there exists a $G$-representation $W$ such that $r_G(W) = \infty$. Note also that, by Theorem 3.1, we have $\lim_{m\to\infty} r_G(W_m) = \infty$, where $G = S^1 \times C_p$ and $W_m$ is the representation given in Theorem 3.1.

Finally, we propose the following conjecture:

**Conjecture**. Positive-dimensional $p$-toral groups other than $T^k$ do not have the weak Borsuk-Ulam property.

## References


[1] T. Bartsch, *On the existence of Borsuk-Ulam theorems*, Topology **31** (1992), 533–543.
[2] K. Borsuk, *Drei Sätze über die n-dimensionale Sphäre*, Fund. Math, **20** (1933), 177–190.
[3] C. Bowszyc, *On the winding number and equivariant homotopy classes of maps of manifolds with some finite group actions*, Fund. Math. **115** (1983), 235–247.
[4] M. C. Crabb, *Equivariant maps between spheres: Nagasaki's examples*, J. Fixed Point Theory Appl. **24** (2022), article number 78.
[5] T. tom Dieck, *Transformation groups*, Walte de Gruyter, Berlin, New York 1987.
[6] I. Nagasaki, *Elementary abelian p-groups are the only finite groups with the Borsuk-Ulam property,* J. Fixed Point Theory Appl. **21** (2019), article number 16.
[7] I. Nagasaki, *Determination of compact Lie groups with the Borsuk-Ulam property* , op. Appl. **319** (2022), article number 108216.
[8] I. Nagasaki, The existence of $S^1 \times C_p$-maps between representation spheres and its applications, J. Fixed Point Theory Appl. **27** (2025), article number 25.



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