On the Grossman representations of the automorphism groups of free groups

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Abstract

In this announcement, we show that for any $k \geq 2$ the subgroup of the automorphism group of a free group consisting of all automorphisms which act trivially on the $(k+1)$-st derived quotient of the free group is not finitely generated in some stabele range. In order to show it, we consider a slight generalization of results of Grossman in [6], and a sequence of descending filtrations of the IA-automorphism group of a free group.

Let F_n be the free group of rank $n \geq 2$ with basis $x_1, \ldots, x_n$, and $\mathrm{Aut}\, F_n$ the automorphism group of F_n. Let F_n^{ab} be the abelianization of F_n. The kernel of the natural homomorphism $\mathrm{Aut}\, F_n \to \mathrm{Aut}\, F_n^{\mathrm{ab}}$ is called the IA-automorphism group of F_n, and is denoted by $\mathrm{IA}(F_n)$, or simply IA_n. Over the last three decades, IA_n has actively been studied with the Andreadakis-Johnson filtration and the Johnson homomorphisms by many authors. By using verbal subgroups of F_n, Grossman [6] introduced a descending filtration of IA_n, which is essentially different from the Andreadakis-Johnson filtration. For any $k \geq 1$, let $V_{n,k}$ be the verbal subgroups of F_n defined by $V_{n,1} := F_n$ and $V_{n,k} := V_{n,k-1}(X^2)$ for $k \geq 2$. (For details for verbal subgroups, see Subsection **??**.) Denote by $K_{n,k}$ by the kernel of the natural homomorphism $\mathrm{Aut}\, F_n \to \mathrm{Aut}\, V_{n,k}^{\mathrm{ab}}$, where for a group G, the abelianization of G is written as G^{ab}. Then Grossman showed that the subgroups $K_{n,k}$ defines a descending filtration $\mathrm{IA}_n \supset K_{n,1} \supset K_{n,2} \supset \cdots$, and they have the trivial intersection $\bigcap_{k \geq 1} K_{n,k} = \{1\}$. Since each of $V_{n,k}^{\mathrm{ab}}$ is a free abelian group of finite rank, the homomorphisms $\mathrm{Aut}\, F_n \to \mathrm{Aut}\, V_{n,k}^{\mathrm{ab}}$ are considered as linear representations of $\mathrm{Aut}\, F_n$. As is well-known due to Formanek and Procesi [5], $\mathrm{Aut}\, F_n$ is not linear. Grossman's result $\bigcap_{k \geq 1} K_{n,k} = \{1\}$ shows that $\mathrm{Aut}\, F_n$ is residually linear.

In this announcement, to begin with, we slightly generalize Grossman's results with the same arguments as hers by replacing 2-torsion with d-torsion. We crucially use the d-torsion generalization to prove Theorem 4. For any $d \geq 2$ and $k \geq 1$, let $V_{n,k}$ be the verbal subgroups of F_n defined by

$$V_{n,d,1} := F_n, \quad V_{n,d,k} := V_{n,d,k-1}(X^d, XYX^{-1}Y^{-1}) \quad (k \geq 2).$$

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Denote by $K_{n,d,k}$ by the kernel of the natural homomorphism $\rho_{n,d,k} : \mathrm{Aut}\, F_n \to \mathrm{Aut}\, V_{n,d,k}^{\mathrm{ab}}$. Grossman's subgroup $K_{n,k}$ is nothing but $K_{n,2,k}$. We call $\rho_{n,d,k}$ the Grossman representation of $\mathrm{Aut}\, F_n$. Remark that for a group G the verbal subgroup $G(X^2)$ contains the commutator subgroup of G. For $d \geq 3$, the verbal subgroup $G(X^d)$ does not always contain the commutator subgroup of G. By using the same arguments as those by Grossman, we obtain the following.

Theorem 1. (1) *For any $n, d \geq 2$,*

$$\mathrm{IA}_n = K_{n,d,1} \supset K_{n,d,2} \supset K_{n,d,3} \supset \cdots .$$

(2) *For any $n, d \geq 2$,*

$$\bigcap_{k \geq 1} K_{n,d,k} = \{1\}.$$

We remark that the successive quotient group $K_{n,d.k}/K_{n,d,k+1}$ is not abelian in general.

Since the successive quotient group $V_{n,d,k}/V_{n,d,k+1}$ for any $k \geq 1$ is a finite abelian group consisting of d-torsions, each $V_{n,d,k}$ is a free group of finite rank. We see the faithfulness of the action of $\mathrm{Aut}\, F_n$ on $V_{n,d,k}$.

Theorem 2. *For any $n \geq 2$, $d \geq 2$ and $k \geq 1$, the map $\mathrm{Aut}\, F_n \to \mathrm{Aut}\, V_{n,d,k}$ is injective.*

From this theorem, we can consider $\mathrm{Aut}\, F_n$ as a subgroup of $\mathrm{Aut}\, V_{n,d,k}$. From this viewpoint, as a corollary, we show that $K_{2,d,k}$ coincide with the inner automorphism group of $V_{2,d,k}$. Moreover, by using the first Johnson homomorphism of $\mathrm{Aut}\, V_{n,d,k}$, we show that the abelianization of $K_{n,d,k}$ contains a free abelian group which is isomorphic to $V_{n,d,k}^{\mathrm{ab}}$.

In general, the action of $\mathrm{Aut}\, F_n$ on $V_{n,d,k}^{\mathrm{ab}}$ does not factor through $\mathrm{Aut}\, F_n^{\mathrm{ab}}$. In other words, IA_n does not act on $V_{n,d,k}^{\mathrm{ab}}$ trivially. In order to describe the action of $\mathrm{Aut}\, F_n$, we need a basis of $V_{n,d,k}$. However, the rank of $V_{n,d,k}$ is too large to write down a basis of $V_{n,d,k}$ in general. In this announcement, we consider the case where $k = 2$ as a first step. We study the $\mathrm{Aut}\, F_n$-module structure of $V_{n,d,2}^{\mathrm{ab}}$ In particular, we obtain the following.

Theorem 3. *For any $n \geq 2$ and $d \geq 2$, the module $V_{n,d,2}^{\mathrm{ab}}$ decomposes to $W_{n,d} \oplus W_{n,d}^{\perp}$ as an $\mathrm{Aut}\, F_n$-module. Moreover, the module $W_{n,d}$ is isomorphic to F_n^{ab}.*

We have obtained explicit basis of $W_{n,d}$ and $W_{n,d}^{\perp}$.

Let $F_n = D_n(1) \supset D_n(2) \supset D_n(3) \supset \cdots$ be the derived series of F_n. Denote by $\mathcal{K}_n^{(k)}$ the kernel of the natural homomorphism $\mathrm{Aut}\, F_n \to \mathrm{Aut}\,(F_n/D_n(k+1))$ for any $k \geq 2$. Notice that $\mathcal{K}_n^{(1)} = \mathrm{IA}_n$ and $\mathcal{K}_n^{(k)} \subset K_{n,d,k}$ for any $d \geq 2$. We show the following theorem.

Theorem 4. *If $\binom{n-1}{2} \geq 2^{k-1}$, then $(\mathcal{K}_n^{(k)})^{\mathrm{ab}}$ contains a free abelian group of infinite rank. In particular, $\mathcal{K}_n^{(k)}$ is not finitely generated.*

In order to show the above theorem, we use the first Johnson homomorphism of $\mathrm{Aut}\, V_{n,d,k}$ to detect infinitely many linearly independent elements in $(\mathcal{K}_n^{(k)})^{\mathrm{ab}}$. In our previous paper [15], we obtained the result in the case where $k = 2$. This theorem is a generalization of our result in [15] for general $k \geq 2$. At the present stage, we cannot give any comment on whether the statement in Theorem 4 holds or not in the unstable range.

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