Boundedness of diffeomorphism groups of manifold pairs

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1 Backgroud

Study of boundedness of groups aims, in a wide sense, to understand quantitative behavior of various norms and metrics on groups. In [6] D. Burago, S. Ivanov and L. Polterovich (2008) introduce a strict formulation of boundedness of groups and make a systematic study of bounded groups. More recently, in [14, 15] J. Kędra, A. Libman and B. Martin (2018 -) introduced some refinements of the notion of boundedness, strong and uniform boundedness, for finitely normally generated groups and studied some classes of uniformly bounded groups in semisimple Lie groups, arithmetic groups and linear algebraic groups. In geometry, the algebraic properties of automorphism groups of manifolds (with some structure) have been studied by many people. For groups of diffeomorphisms of closed manifolds, the main concern has been uniform perfectness and uniform simplicity (related to the commutator lengh norms and the conjugation-generated norms on one word). In [6] they studied the sphere case and the closed 3-manifold case (2008) and in [17, 18, 19] T. Tsuboi obtained complehensive results for closed manifolds of any dimension $\neq 2, 4$ (and also in dimesions 2 and 4 when the manifolds have a handle desomposition with no handles of middle index) (2008 - 2012).

In this subject our present goal is a complehensive study of boundedness of bundle diffeomorphism groups and diffeomorphism groups of manifold pairs. In [11, 12] we have studied the cases of bundle diffeomorphism groups over a circle and diffeomorphism groups of manifold pairs in which the submanifold is a finite disjoint union of circles. These cases are both studied using the rotation angle of paths on the circles. This expository article includes main resutls in [12] for the circle case and also some statements from [13] (in preparation) for the higher dimensional cases.

2 Diffeomorphism groups of manifold pairs

A manifold pair means a pair (M, N) of a C^∞ manifold M and a submanifold N. A diffeomorphism of the pair (M, N) is a C^∞ diffeomorphism f of M with $f(N) = N$. Let $\mathrm{Diff}_c(M, N)$ and $\mathrm{Isot}_c(M, N)$ denote the groups of diffeomorphisms and isotopies of (M, N) with compact support and we set $\mathrm{Isot}_c(M, N)_0 := \{F \in \mathrm{Isot}_c(M, N) \mid F_0 = \mathrm{id}\}$ and $\mathrm{Diff}_c(M, N)_0 := \{F_1 \mid F \in \mathrm{Isot}_c(M, N)_0\}$. (One should be careful for definitions of these symbols here and in any articles to avoid any ambiguity.) As usual, the subscript c is

removed, when M is compact. Let $P : \mathrm{Diff}_c(M, N)_0 \longrightarrow \mathrm{Diff}_c(N)_0$ denote the restriction map.

K. Abe and K. Fukui studied the uniform perfectness of the group $\mathrm{Diff}_c(M, N)_0$ in [?, 2, 3, 4] (2009 - 2013). Their results are summarized as follows ([2, 4]).

Theorem AF. Suppose M is a connected closed C^∞ manifold and N is a closed C^∞ submanifold of M with $\dim N \geq 1$.

(1) (i) The group $\mathrm{Diff}(M, N)_0$ is perfect.   (ii) $cld\,\mathrm{Diff}_c(\mathbb{R}^m, \mathbb{R}^n)_0 \leq 2$ for $1 \leq n \leq m$.

(2) (i) If $\mathrm{Diff}(N)_0$ and $\mathrm{Diff}_c(M - N)_0$ are uniformly perfect and $|\pi_0 \mathrm{Ker}\,P| < \infty$, then $\mathrm{Diff}(M, N)_0$ is uniformly perfect.

  (ii) In the case $\dim N = 1$ (i.e, N is a finite disjoint union of circles), if $|\pi_0 \mathrm{Ker}\,P| = \infty$, then $\mathrm{Diff}(M, N)_0$ admits a unbounded quasimorphism, so that it is not uniformly perfect.

Using the criterion (2) on $\pi_0 \mathrm{Ker}\,P$, they showed that for a knot K of the 3-sphere $\mathbb{S}^3$ the group $\mathrm{Diff}(\mathbb{S}^3, K)_0$ is uniformly perfect if and only if K is a torus knot ([1]).

Based on these results, in [12] we continue the comprehensive study of conjugation invariant norms cl, clb, η and ζ on the group $\mathrm{Diff}(M, N)_0$. Here, cl is the commutator length, clb is the commutator length supported in balls, η is the fragmentation norm and ζ_g is the conjugation-generated norm on a word $g \in \mathrm{Diff}(M, N)_0^\times$. In order to define the norms clb and η on $\mathrm{Diff}(M^n, N^\ell)_0$, we use the family of finite disjoint unions of n-balls $D = \cup_i D_i$ in M such that each n-ball D_i satisfies either $D_i \subset M - N$ or $(D_i, D_i \cap N)$ is diffeomorphic to the standard ball pair $(\mathbb{B}^n, \mathbb{B}^\ell)$. The norm cl is related to the uniform perfectness and ζ_g is related to the uniform (weak) simplicity. The norm clb is important for our purpose since it dominates the norms cl and ζ_g and is dominated by η.

First we observe the relations and finiteness of these norms on the group $\mathrm{Diff}_c(M, N)_0$.

Theorem 2.1. Suppose M is a connected C^∞ manifold without boundary and N is a proper C^∞ submanifold of M with $\dim N \geq 1$.

(1) $cl\,f \leq clb\,f \leq 2\eta(f) < \infty$ $(^\forall f \in \mathrm{Diff}_c(M, N)_0)$.

(2) Suppose N has finitely many connected components N_i $(i \in [m])$. Then,

$$(*) \quad \zeta_g(f) \leq 4clb\,f \quad (^\forall f \in \mathrm{Diff}_c(M, N)_0 - \mathcal{KP})$$

and $\mathrm{Diff}_c(M, N)_0$ is weakly simple relative to $\mathcal{KP}$.

The statement (1) follows from Fragmentation Lemma for the manifold pair and Theorem AF (1)(ii). For the statement (2), consider the group epimorphisms

$$P_i : \mathrm{Diff}_c(M, N)_0 \longrightarrow \mathrm{Diff}_c(N_i)_0 : P_i(f) = f|_{N_i} \quad (i \in [m]).$$

Then, the group $\mathrm{Diff}_c(M, N)_0$ includes the normal subgroups $\mathrm{Ker}\,P_i$ $(i \in [m])$ and any element of $\mathcal{KP} := \bigcup_{i \in [m]} \mathrm{Ker}\,P_i$ does not normally generate $\mathrm{Diff}_c(M, N)_0$. On the other hand, for any $g \in \mathrm{Diff}_c(M, N)_0 - \mathcal{KP}$, Epstein's classical argument yields the inequality $(*)$ and the finiteness of ζ_g means that g normally generates $\mathrm{Diff}_c(M, N)_0$. The notion of weak simplicity is introduced to describe this situation. In the case $m = 1$, the subset

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$\mathcal{KP}$ itself is a normal subgroup of $\mathrm{Diff}_c(M, N)_0$ and this notion reduces to the notion of relative simplicity (cf. [16]).

The group $\mathrm{Diff}(M, N)_0$ contains the normal subgroup

$$\mathcal{G} \equiv \mathcal{G}(M, N) := \mathrm{Diff}(M; \mathrm{rel}\,\mathcal{U}(N))_0 \cong \mathrm{Diff}_c(M - N)_0,$$

where $\mathcal{U}(N)$ denotes the neighborhood system of N in M. It is natural to separate the evaluation of the norms cl and clb into one localized around N and another away from N. For this purpose we introduce the norms $\underline{cl}$ and $\underline{clb}$ modulo $\mathcal{G}$, which are denoted by $cl_{/\mathcal{G}}$ and $clb_{/\mathcal{G}}$ and satisfy the following inequalities : $\quad {}^\forall f \in \mathrm{Diff}(M, N)_0$

$$(\sharp) \quad cl_{/\mathcal{G}}\,f \leq cl\,f \leq cl_{/\mathcal{G}}\,f + cld\,\mathcal{G} \quad \text{and} \quad clb_{/\mathcal{G}}\,f \leq clb\,f \leq clb_{/\mathcal{G}}\,f + clbd\,\mathcal{G}.$$

In [10] we have studied the norms cl and clb on the group

$$\mathcal{D}(L) := \mathrm{Diff}(L; \mathrm{rel}\,\mathcal{U}(\partial L))_0 \cong \mathrm{Diff}_c(\mathrm{Int}\,L)_0 \quad \text{for a compact } n\text{-manifold } L,$$

by an appropriate modification of Tsuboi's arguments in [17, 18, 19].

Theorem FRY ([10]).

(1) In the case $n = 2m + 1 \;\; (m \geq 0) : \quad cld\,\mathcal{D}(L) \leq 4 \quad clbd\,\mathcal{D}(L) \leq 2n + 4$

(2) In the case $n = 2m \;\; (m \geq 1) :$

    (i) $cld\,\mathcal{D}(L) < \infty \quad clbd\,\mathcal{D}(L) < \infty \quad$ for $m \geq 3$.

       ∘ Some upper bounds are obtained in terms of triangulations of L or handle decompositions of $\mathrm{Int}\,L$.

    (ii) If $\mathrm{Int}\,L$ admits a handle decomposition without m-handles, then

$$cld\,\mathcal{D}(L) \leq 3 \quad clbd\,\mathcal{D}(L) \leq 2n + 1.$$

These results were a basis for the study of these norms on the diffeomorphism groups of open manifolds in [10]. We also note that

(3) $cld\,\mathcal{D}(L) \leq 2$ if $\mathrm{Int}\,L$ is portable ([6]).

Here we can apply these results to the group $\mathcal{G}(M, N)$ so to obtain some upper bounds for $cld\,\mathcal{G}$ and $clbd\,\mathcal{G}$. Therefore, it remains to obtain some estimates on $cl_{/\mathcal{G}}$, $clb_{/\mathcal{G}}$, and so we can localize the discussion around N.

3   The case $\dim N = 1$

In this section we focus on the case that $\dim N = 1$ and N is a disjoint union of circles. In this case we can use the rotation angle of paths on the circles to obtain some explicit evaluation of the norms $cl_{/\mathcal{G}}$ and $clb_{/\mathcal{G}}$.

Suppose M is a connected C^∞ manifold with $\dim M \geq 2$ and N is a disjoint union of circles S_i $(i \in [m])$ in $\mathrm{Int}\,M$. For each $i \in [m]$ take a universal cover $\pi_i : \mathbb{R} \longrightarrow \mathbb{R}/\mathbb{Z} \approx S_i$ and fix a point $p_i \in S_i$. The rotation angle $\theta(c)$ of a path $c : [0, 1] \to S_i$ is defined by

$$\theta(c) := \widetilde{c}(1) - \widetilde{c}(0) \qquad (\text{Here, } \widetilde{c} : [0, 1] \to \mathbb{R} \text{ is any lift of } c.)$$

The behavior of each $F \in \mathrm{Isot}(M, N)_0$ around N is detected by the surjective quasimorphism

$$\nu = (\nu_i)_{i \in [m]} : \mathrm{Isot}(M, N)_0 \longrightarrow \mathbb{R}^m : \quad \nu_i(F) := \theta(F(p_i, *))$$

It restricts to a group homomorphism

$$\nu| : \mathrm{Isot}(M, N)_{\mathrm{id,id}} \longrightarrow \mathbb{Z}^m : \quad \nu_i(F) = [\text{the degree of the loop } F(p_i, *) \text{ in } S_i].$$

This determines a subgroup $A := \mathrm{Im}\,\nu|$ of $\mathbb{Z}^m$ and ν induces a surjective map

$$\widehat{\nu} : \mathrm{Diff}(M, N)_0 \longrightarrow \mathbb{R}^m / A, \quad \widehat{\nu}(f) = \nu(F) + A \subset \mathbb{R}^m \quad (F \in \mathrm{Isot}(M, N)_{\mathrm{id,}f}).$$

The subgroup A is related to $\pi_0 \mathrm{Ker}\, P$ in Theorem AF (2) by $\pi_0 \mathrm{Ker}\, P = \mathbb{Z}^m / A$. However, A itself and the affine lattice $\widehat{\nu}(f)$ in $\mathbb{R}^m$ include more informations than the quotient $\mathbb{Z}^m / A$.

For $\mathbb{R}^m$ we consider the norm $\quad \|x\| := \max_{i \in [m]} |x_i| \quad (x = (x_i)_{i \in [m]} \in \mathbb{R}^m)$. For a subset $C \subset \mathbb{R}^m$ let $\|C\| := \{\|x\| \mid x \in C\} \subset \mathbb{R}$. We have the following estimates of the norms $cl_{/\mathcal{G}}$ and $clb_{/\mathcal{G}}$.

Theorem 3.1. Suppose M is a connected closed C^∞ n-manifold ($n \geq 2$) and N is a finite disjoint union of circles in M. Then, for any $f \in \mathrm{Diff}(M, N)_0$ the following holds.

(1) If $\min \|\widehat{\nu}(f)\| < \ell \in \mathbb{Z}_{\geq 1}$, then

$$cl_{/\mathcal{G}}\, f \leq clb_{/\mathcal{G}}\, f \leq 2\ell + 1 \quad \text{and} \quad cl\, f \leq 2\ell + 1 + cld\,\mathcal{G}, \quad clb\, f \leq 2\ell + 1 + clbd\,\mathcal{G}.$$

(2) $cl\, f \geq cl_{/\mathcal{G}}\, f \geq \frac{1}{4}\big(\min \|\widehat{\nu}(f)\| + 1\big) \quad$ if $f \in \mathrm{Diff}(M, N)_0 - \mathcal{G}$.

The boundedness of the norms cl and clb on the whole group $\mathrm{Diff}(M, N)_0$ is detected by rank A.

[I] In the case rank $A < m$:

We can find a group epimorphism $\varrho : \mathbb{R}^m / A \to \mathbb{R}$ such that $\varphi := \varrho\,\widehat{\nu} : \mathrm{Diff}(M, N)_0 \longrightarrow \mathbb{R}$ is a surjective quasimorphism. This implies the following conclusion.

Proposition 3.1. Suppose M is a connected closed C^∞ n-manifold ($n \geq 2$) and N is a disjoint union of m circles in M. If rank $A < m$, then the group $\mathrm{Diff}(M, N)_0$ admits a surjective quasimorphism, so that it is neither bounded nor uniformly perfect.

[II] In the case that rank $A = m$:

The quotient group $\mathbb{Z}^m / A$ is a finite abelian group and we can introduce the following quantities : $\quad k := \max_{i \in [m]} k_i \in \mathbb{Z}_{\geq 1} \quad$ and $\quad \widehat{k} := 2\lfloor k/2 \rfloor + 3$

where e_i ($i \in [m]$) is the standard basis of $\mathbb{R}^m$ and $k_i := \mathrm{ord}\,[e_i] \in \mathbb{Z}_{\geq 1}$ in $\mathbb{Z}^m / A$ for each $i \in [m]$. Since any lattice $x + A \subset \mathbb{R}^m$ ($x \in \mathbb{R}^m$) meets the rectangle $\prod_{i \in [m]} (-k_i/2, k_i/2]$, we obtain the following upper bounds for cl, clb and $clb_{/\mathcal{G}}$.

Theorem 3.2. Suppose M is a connected closed C^∞ n-manifold ($n \geq 2$) and N is a disjoint union of m circles in M. Let $\mathcal{D} \equiv \mathrm{Diff}(M, N)_0$. If rank $A = m$, then

$$clb_{/\mathcal{G}}d\,\mathcal{D} \leq \widehat{k} \quad \text{and} \quad cld\,\mathcal{D} \leq \widehat{k} + cld\,\mathcal{G}, \quad clbd\,\mathcal{D} \leq \widehat{k} + clbd\,\mathcal{G}.$$

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Therefore, if $n \neq 2, 4$ (or in some special cases for $n = 2, 4$ as in Complement ($\flat$)), then $clbd\,\mathcal{G} < \infty$ and the group $\mathcal{D}$ is uniformly weakly simple relative to $\mathcal{KP}$ and is bounded.

Complement ($\flat$).

(1) $cld\,\mathcal{G} \leq 4,\ \ clbd\,\mathcal{G} \leq 2n + 4$ in the case $n = 2k + 1$ $(k \geq 1)$.

(2) $cld\,\mathcal{G} \leq clbd\,\mathcal{G} < \infty$ in the case $n = 2k$ $(k \geq 3)$

(3) $cld\,\mathcal{G} \leq 3,\ \ clbd\,\mathcal{G} \leq 2n + 1$

   if $n = 2k$ $(k \geq 1)$ and $M - N$ admits a handle decomposition without k-handles.

(4) $cld\,\mathcal{G} \leq 2$ if $M - N$ is portable.

In the case $m = 1$ (i.e., N consists of a circle), A is just a subgroup of $\mathbb{Z}$ and $A = k\mathbb{Z}$ for a unique $k \in \mathbb{Z}_{\geq 0}$. Then, rank $A < 1$ (i.e. $A = 0$) if and only if $k = 0$. In [1] it is shown that for a knot K in the 3-sphere $\mathbb{S}^3$, the group $\mathrm{Diff}(\mathbb{S}^3, K)_0$ is uniformly perfect if and only if K is a torus knot. This example extends to the following form.

Example 3.1. Suppose M is a connected closed C^∞ n-manifold $(n \geq 2)$ and K is a circle in M.

(1) Suppose M admits an $\mathbb{S}^1$ action ϱ such that $K = \mathbb{S}^1 \cdot p$ for some point p of K and the orbit map $\varrho_p : \mathbb{S}^1 \to K$, $\varrho_p(z) = z \cdot p$, has degree ℓ (up to $\pm$). Then, $\ell \in A$ and $k|\ell$.

   (i) In the case M is a Seifert fibered 3-manifold, if K is a regular fiber, then $k = 1$ and if K is a (p, q) multiple fiber, then $k|p$.

   (ii) In particular, if K is a torus knot in $\mathbb{S}^3$, then $k = 1$, since K is a regular fiber of a standard Seifert fibering of $\mathbb{S}^3$ with two singular fibers.

(2) $A = \{0\} \subset \mathbb{Z}$ and $k = 0$ in the following cases :

   ($\dagger$) (a) $\pi_1(M)$ has a trivial center and (b) $\pi_1(K) \to \pi_1(M)$ is injective.

   ($\ddagger$) $n = 3$, (a) $\pi_1(M - K)$ has a trivial center and (b) $\pi_1(U - K) \to \pi_1(M - K)$ is injective for a tubular neighborhood U of K in M.

   (i) For example, a non-torus knot K in $\mathbb{S}^3$ satisfies the condition ($\ddagger$) ([7, Ch 3]).

   (ii) The following example (M, K) satisfies the condition ($\dagger$) for $n \geq 4$. Suppose G is a finitely presented group such that $Z(G) = 1$ and G includes an element a of infinite order. Since $n \geq 4$, there exists a connected closed C^∞ n-manifold M with $\pi_1(M) \cong G$. Take a circle K in M which represents the element a in $\pi_1(M)$, so that the inclusion $i : K \subset M$ induces an isomorphism $i_* : \pi_1(K) \cong \langle a \rangle < \pi_1(M)$.

Some unbounded examples for $m \geq 2$ are obtained using parallel circles. Suppose M is a connected manifold and S_1, S_2 are disjoint circles in Int M. For the pair $(M, N = S_1 \cup S_2)$, we have the associated quasimorphisms $\nu = (\nu_1, \nu_2)$ on Isot$(M, N)_0$. We say that the circles S_1 and S_2 are parallel if there is an embedded annulus E in M such that $\partial E = S_1 \cup S_2$. In this case the components ν_1 and ν_2 are seemed to be not independent (under some additional assumption).

Lemma 3.1. Suppose S_1 and S_2 are parallel with respect to an annulus E in M, $E \subset U \subset M$ and $i_{1*} : \pi_1(S_1, p_1) \to \pi_1(U, p_1)$ is injective. If $F \in \mathrm{Isot}^r(M, L)_{\mathrm{id},\mathrm{id}}$ and $F(E \times I) \subset U$, then $\nu_1(F) = \pm \nu_2(F)$. (The sign depends on the choice of orientations of S_1 and S_2.)

Example 3.2. (Hopf link) Consider the Hopf fibration $\pi : \mathbb{S}^3 \to \mathbb{S}^2$. Let $m \in \mathbb{Z}_{\geq 1}$ and consider a link $L^{(m)} := \bigcup_{i \in [m]} S_i \subset \mathbb{S}^3$ consisting of m distinct circle fibers $S_i := \pi^{-1}(q_i)$ ($i \in [m]$). Note that S_i and S_j ($i < j$) are parallel in $\mathbb{S}^3 - \bigcup_{k \neq i,j} S_k$.

For the pair $(\mathbb{S}^3, L^{(m)})$ the lattice $A = A_{(\mathbb{S}^3, L^{(m)})} < \mathbb{Z}^m$ takes the following form.

m	1	2	≥ 3
A	$\mathbb{Z}$	$\mathbb{Z}^2$	$\mathbb{Z}(1, \cdots, 1)$
$\mathrm{rank}\, A$	1	2	1

Hence, the group $\mathrm{Diff}^\infty(\mathbb{S}^3, L^{(m)})_0$ is bounded for $m = 1, 2$ and unbounded for $m \geq 3$.

Lemma 3.1 also has the following applications to the group $\mathrm{Diff}^r_L(M, \mathcal{F})_0$ of leaf-preserving diffeomorphisms on a manifold foliated by circles and the group $\mathrm{Aut}_\pi(M)_0$ of fiber-preserving bundle isomorphisms of a circle bundle π.

Example 3.3. (Foliation) Suppose $(M, \mathcal{F})$ is an n-manifold ($n \geq 2$) equipped with a 1-dimensional foliation $\mathcal{F}$. If $\mathcal{F}$ includes two circle leaves $S_1, S_2 \subset \mathrm{Int}\, M$ which are parallel with respect to an $\mathcal{F}$-saturated annulus E in M, then the group $\mathrm{Diff}^r_L(M, \mathcal{F})_0$ admits a surjective quasimorphism $\varphi : \mathrm{Diff}^r_L(M, \mathcal{F})_0 \longrightarrow \mathbb{R}$.

Example 3.4. (Circle bundle) Suppose $\pi : M \to B$ is a $(\mathbb{S}^1, \Gamma)$ bundle with $\mathrm{SO}(2) < \Gamma < \mathrm{Diff}^r(\mathbb{S}^1)$ and $\dim B \geq 1$. Then the group $\mathrm{Aut}_\pi(M)_0$ admits a surjective quasimorphism $\varphi : \mathrm{Aut}_\pi(M)_0 \longrightarrow \mathbb{R}$.

4 The case $\dim N \geq 2$

This final section includes some comments on the case $\dim N \geq 2$ ([13]).

4.1 The case $\dim N = 2$

Recall the restriction map $P : \mathrm{Diff}_c(M, N)_0 \longrightarrow \mathrm{Diff}_c(N)_0$. Since it is a surjective group homomorphism, if $\nu : \mathrm{Diff}_c(N)_0 \longrightarrow \mathbb{R}$ is a unbounded quasimorphism, then so is the composition $\varphi P : \mathrm{Diff}_c(M, N)_0 \longrightarrow \mathbb{R}$.

Proposition 4.1. When N is a connected closed surface $\neq \mathbb{S}^2, \mathbb{P}^2$, it is known that $\mathrm{Diff}(N)_0$ admits a unbounded quasimorphism. Hence, the group $\mathrm{Diff}(M, N)_0$ also admits a unbounded quasimorphism and so it is unbounded and not uniformly perfect.

4.2 The higher-dimensional case

By the inequality ($\sharp$) and Theorem FRY in Section 2, our concern is reduced to the evaluation of the norms $cl_{/\mathcal{G}}$ and $clb_{/\mathcal{G}}$. If Tsuboi's strategy on factorization of isotopies in

[17, 18, 19] can be applied to the submanifold N, then we can extend this factorizations in N to that in a neighborhood of N and obtain some estimates on $cl_{/\mathcal{G}}$ and $clb_{/\mathcal{G}}$.

Consider the following conditions on a manifold pair (M, N) :

(i) M is a connected compact C^∞ n-manifold

(ii) N is a closed ℓ-submanifold of $\mathrm{Int}\, M$ or $N = \partial M$ (iii) $n > \ell \geq 2$

(iv) $\ell = 2$ and $N = \mathbb{S}^2$ and

(v) $\ell = 4$ and N has a handle decomposition without 2-handles.

Under these conditions, we have the following estimates for the group $\mathcal{D} \equiv \mathrm{Diff}(M, N)_0$.

Theorem 4.1 ([13])**.** Under the conditions (i), (ii), (iii) the following holds.

(1) In the case $\ell = 2k + 1$: $cl_{/\mathcal{G}} d\, \mathcal{D} \leq 4,$ $clb_{/\mathcal{G}} d\, \mathcal{D} \leq 2\ell + 4$

(2) In the case $\ell = 2k$: $cl_{/\mathcal{G}} d\, \mathcal{D} < \infty$ $clb_{/\mathcal{G}} d\, \mathcal{D} < \infty$ for $k \geq 3$.

     ◦ In the cases (iv) and (v) : $cl_{/\mathcal{G}} d\, \mathcal{D} \leq 3$ $clb_{/\mathcal{G}} d\, \mathcal{D} \leq 2\ell + 1$.

In the case $\ell = 4$ we need a new approach for the evaluation of $cl_{/\mathcal{G}}$.

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