

Note on the space of real non-resultant systems determined by a toric variety

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Abstract

For a fan Σ in \mathbb{R}^m , a field \mathbb{F} and an r -tuple $D = (d_1, \dots, d_r)$ of positive integers, let $\text{Poly}_n^{D, \Sigma}(\mathbb{F})$ denote the space of non-resultant systems of bounded multiplicity n determined by a toric variety X_Σ , where r is the number of one dimensional cones in Σ and X_Σ denotes the toric variety over \mathbb{C} corresponding to the fan Σ . Since $\text{Poly}_n^{D, \Sigma}(\mathbb{C}) = \text{Hol}_D^*(S^2, X_\Sigma)$ when $n = 1$ and the certain condition is satisfied, this space may be also regarded as one of generalization of the space $\text{Hol}_D^*(S^2, X_\Sigma)$ of all based rational curves from the Riemann surface S^2 to X_Σ of degree D . As one of real analogues of this space, we shall study the space $\text{Q}_n^{D, \Sigma}(\mathbb{F})$ for $\mathbb{F} = \mathbb{R}$ or \mathbb{C} . In particular, we report that the Atiyah-Jones-Segal type homotopy stability holds for the space $\text{Q}_n^{D, \Sigma}(\mathbb{F})$ (cf. [2], [21]). This note is based on the joint works with A. Kozłowski ([12], [17], [18]).

1 Basic definitions and notations

Let \mathbb{N} be a set of all positive integers. For connected spaces X and Y , let $\text{Map}(X, Y)$ denote the space consisting of all continuous maps $f : X \rightarrow Y$ with the compact open topology. Let $\text{Map}^*(X, Y) \subset \text{Map}(X, Y)$ be the subspace of all base point preserving maps $f : (X, *) \rightarrow (Y, *)$. For a based homotopy class $D \in \pi_0(\text{Map}^*(X, Y)) = [X, Y]$, we denote by $\text{Map}_D^*(X, Y) \subset \text{Map}^*(X, Y)$ the path component containing the homotopy class D .

Now recall several basic definitions and notations concerning to toric varieties.

Definition 1.1. (i) A convex rational polyhedral cone in \mathbb{R}^m is a subset of \mathbb{R}^m of the form

$$(1.1) \quad \sigma = \text{Cone}(S) = \text{Cone}(\mathbf{m}_1, \dots, \mathbf{m}_s) = \left\{ \sum_{k=1}^s \lambda_k \mathbf{m}_k : \lambda_k \geq 0 \right\}$$

for some finite set $S = \{\mathbf{m}_1, \dots, \mathbf{m}_s\} \subset \mathbb{Z}^m$. The dimension of σ is the dimension of the smallest subspace of \mathbb{R}^m which contains σ .

(ii) A convex rational polyhedral cone σ is called *strongly convex* if $\sigma \cap (-\sigma) = \{\mathbf{0}_m\}$, where we set $\mathbf{0}_m = \mathbf{0} = (0, 0, \dots, 0) \in \mathbb{R}^m$. A *face* τ of a convex rational polyhedral cone

σ is a subset $\tau \subset \sigma$ of the form $\tau = \sigma \cap \{\mathbf{x} \in \mathbb{R}^m : L(\mathbf{x}) = 0\}$ for some linear form L on \mathbb{R}^m , such that $\sigma \subset \{\mathbf{x} \in \mathbb{R}^m : L(\mathbf{x}) \geq 0\}$.

(iii) Let Σ be a finite collection of strongly convex rational polyhedral cones in \mathbb{R}^m . Then it is called a *fan* (in \mathbb{R}^m) if the following two conditions (1.1.1) and (1.1.2) are satisfied:

(1.1.1) Every face τ of $\sigma \in \Sigma$ belongs to Σ .

(1.1.2) If $\sigma_1, \sigma_2 \in \Sigma$, $\sigma_1 \cap \sigma_2$ is a common face of each σ_k ($k = 1, 2$) and so $\sigma_1 \cap \sigma_2 \in \Sigma$.

(iv) An m dimensional irreducible normal variety X (over \mathbb{C}) is called a *toric variety* if it has a Zariski open subset $\mathbb{T}_{\mathbb{C}}^m = (\mathbb{C}^*)^m$ and the action of $\mathbb{T}_{\mathbb{C}}^m$ on itself extends to an action of $\mathbb{T}_{\mathbb{C}}^m$ on X . The most significant property of a toric variety is that it is characterized up to isomorphism entirely by its associated fan Σ . We denote by X_{Σ} the toric variety associated to a fan Σ . \square

Definition 1.2. Let K be a simplicial complex on the index set $[r] = \{1, 2, \dots, r\}$,¹ and let (X, A) be a pairs of based spaces.

(i) Let $I(K)$ denote the collection of subsets $\sigma \subset [r]$ defined by

$$(1.2) \quad I(K) = \{\sigma \subset [r] : \sigma \notin K\}.$$

(ii) Define the *polyhedral product* $\mathcal{Z}_K(X, A)$ of the pair (X, A) with respect to K by

$$(1.3) \quad \mathcal{Z}_K(X, A) = \bigcup_{\sigma \in I(K)} (X, A)^{\sigma}, \quad \text{where} \\ (X, A)^{\sigma} = \{(x_1, \dots, x_r) \in X^r : x_k \in A \text{ if } k \notin \sigma\}.$$

Definition 1.3. Let $\mathbb{K} = \mathbb{R}$ or \mathbb{C} , let Σ be a fan in \mathbb{R}^m such that $\{\mathbf{0}_m\} \subsetneq \Sigma$, and let

$$(1.4) \quad \Sigma(1) = \{\rho_1, \dots, \rho_r\}$$

denote the set of all one dimensional cones in Σ .

(i) For each $1 \leq k \leq r$, we denote by $\mathbf{n}_k \in \mathbb{Z}^m$ the *primitive generator* of ρ_k , such that $\rho_k \cap \mathbb{Z}^m = \mathbb{Z}_{\geq 0} \cdot \mathbf{n}_k$. Note that $\rho_k = \text{Cone}(\mathbf{n}_k)$.

(ii) Let \mathcal{K}_{Σ} denote the *underlying simplicial complex* of Σ defined by

$$(1.5) \quad \mathcal{K}_{\Sigma} = \left\{ \{i_1, \dots, i_s\} \subset [r] : \mathbf{n}_{i_1}, \mathbf{n}_{i_2}, \dots, \mathbf{n}_{i_s} \text{ span a cone in } \Sigma \right\}.$$

It is easy to see that \mathcal{K}_{Σ} is a simplicial complex on the index set $[r]$.

¹Let K be some set of subsets of $[r]$. Then the set K is called an *abstract simplicial complex* on the index set $[r]$ if the following condition holds: if $\tau \subset \sigma$ and $\sigma \in K$, then $\tau \in K$. In this paper by a simplicial complex K we always mean an *abstract simplicial complex*, and we always assume that a simplicial complex K contains the empty set \emptyset .

(iii) Define the subgroup $G_{\Sigma, \mathbb{K}} \subset \mathbb{T}_{\mathbb{K}}^r = (\mathbb{K}^*)^r$ by

$$(1.6) \quad G_{\Sigma, \mathbb{K}} = \{(\mu_1, \dots, \mu_r) \in \mathbb{T}_{\mathbb{K}}^r : \prod_{k=1}^r (\mu_k)^{\langle \mathbf{n}_k, \mathbf{m} \rangle} = 1 \text{ for all } \mathbf{m} \in \mathbb{Z}^m\},$$

where $\langle \mathbf{u}, \mathbf{v} \rangle = \sum_{k=1}^m u_k v_k$ for $\mathbf{u} = (u_1, \dots, u_m)$ and $\mathbf{v} = (v_1, \dots, v_m) \in \mathbb{R}^m$.

(iv) Let

$$(1.7) \quad (\mathbb{K}^n)^* = \mathbb{K}^n \setminus \{\mathbf{0}_n\}$$

and consider the natural $G_{\Sigma, \mathbb{K}}$ -action on $\mathcal{Z}_{\mathcal{K}_{\Sigma}}(\mathbb{K}^n, (\mathbb{K}^n)^*)$ which is given by coordinate-wise multiplication, i.e.

$$(1.8) \quad (\mu_1, \dots, \mu_r) \cdot (\mathbf{x}_1, \dots, \mathbf{x}_r) = (\mu_1 \mathbf{x}_1, \dots, \mu_r \mathbf{x}_r)$$

for $((\mu_1, \dots, \mu_r), (\mathbf{x}_1, \dots, \mathbf{x}_r)) \in G_{\Sigma, \mathbb{K}} \times (\mathbb{K}^n)^r$, where we set

$$(1.9) \quad \mu \mathbf{x} = (\mu x_1, \dots, \mu x_n) \quad \text{if } (\mu, \mathbf{x}) = (\mu, (x_1, \dots, x_n)) \in \mathbb{K} \times \mathbb{K}^n.$$

Then define the space $X_{\Sigma, \mathbb{K}}(n)$ by the corresponding orbit space

$$(1.10) \quad X_{\Sigma, \mathbb{K}}(n) = \mathcal{Z}_{\mathcal{K}_{\Sigma}}(\mathbb{K}^n, (\mathbb{K}^n)^*) / G_{\Sigma, \mathbb{K}}.$$

Moreover, let

$$(1.11) \quad q_{n, \mathbb{K}} : \mathcal{Z}_{\mathcal{K}_{\Sigma}}(\mathbb{K}^n, (\mathbb{K}^n)^*) \rightarrow X_{\Sigma, \mathbb{K}}(n) = \mathcal{Z}_{\mathcal{K}_{\Sigma}}(\mathbb{K}^n, (\mathbb{K}^n)^*) / G_{\Sigma, \mathbb{K}}$$

denote the corresponding canonical projection.

Theorem 1.4 ([4], Theorem 2.1). *If the set $\{\mathbf{n}_k\}_{k=1}^r$ of all primitive generators spans \mathbb{R}^m (i.e. $\sum_{k=1}^r \mathbb{R} \cdot \mathbf{n}_k = \mathbb{R}^m$), there is a natural isomorphism*

$$(1.12) \quad X_{\Sigma} \cong \mathcal{Z}_{\mathcal{K}_{\Sigma}}(\mathbb{C}, \mathbb{C}^*) / G_{\Sigma, \mathbb{C}} = X_{\Sigma, \mathbb{C}}(1).$$

Hence, we can identify $X_{\Sigma, \mathbb{C}}(n)$ with the toric variety X_{Σ} if $n = 1$. □

Remark 1.5. Let $\mathbb{K} = \mathbb{R}$ or \mathbb{C} . For each $1 \leq i \leq r$, let $F_i = (f_{1;i}, \dots, f_{n;i}) \in \mathbb{K}[z_0, \dots, z_s]^n$ be an n -tuple of homogenous polynomials of the same degree d_i satisfying the following condition:

(1.12)* For each $\sigma \in I(\mathcal{K}_{\Sigma})$, the homogenous polynomials $\{f_{k;i}\}_{k \in \sigma}$ have no common *real* root except $\mathbf{0}_{s+1} \in \mathbb{R}^{s+1}$.

In this situation, consider the map

$$(1.13) \quad F = (F_1, \dots, F_r) : \mathbb{R}^{s+1} \setminus \{\mathbf{0}_{s+1}\} \rightarrow (\mathbb{K}^n)^r = \mathbb{K}^{rn} \quad \text{given by}$$

$$(1.14) \quad \begin{cases} F(\mathbf{x}) &= (F_1(\mathbf{x}), \dots, F_r(\mathbf{x})) \quad \text{for } \mathbf{x} \in \mathbb{R}^{m+1} \setminus \{\mathbf{0}_{m+1}\}, \\ F_i(\mathbf{x}) &= (f_{1;i}(\mathbf{x}), f_{2;i}(\mathbf{x}), \dots, f_{n;i}(\mathbf{x})) \quad \text{for } 1 \leq i \leq r. \end{cases}$$

By the assumption (1.12)*, homogenous polynomials $\{f_{k;i}\}_{k \in \sigma}$ have no common real root except $\mathbf{0}_{s+1} \in \mathbb{R}^{s+1}$ for each $1 \leq i \leq r$ and $\sigma \in I(\mathcal{K}_\Sigma)$. Thus, we see that the image of the map F is contained in $\mathcal{Z}_{\mathcal{K}_\Sigma}(\mathbb{K}^n, (\mathbb{K}^n)^*)$, and we may regard the map F as

$$(1.15) \quad F = (F_1, \dots, F_r) : \mathbb{R}^{s+1} \setminus \{\mathbf{0}_{s+1}\} \rightarrow \mathcal{Z}_{\mathcal{K}_\Sigma}(\mathbb{K}^n, (\mathbb{K}^n)^*). \quad \square$$

Lemma 1.6 (cf. [5], Theorem 3.1). *Suppose that the set $\{\mathbf{n}_k\}_{k=1}^r$ of all primitive generators spans \mathbb{R}^m . For each $1 \leq i \leq r$ and $\sigma \in I(\mathcal{K}_\Sigma)$, let $F_i = (f_{1;i}, \dots, f_{n;i}) \in \mathbb{K}[z_0, \dots, z_s]^n$ be an n -tuple of homogenous polynomials of the same degree d_i satisfying the condition (1.12)*.*

Then there is a unique map $f : \mathbb{RP}^s \rightarrow X_{\Sigma, \mathbb{K}}(n)$ such that the diagram

$$(1.16) \quad \begin{array}{ccc} \mathbb{R}^{s+1} \setminus \{\mathbf{0}_{s+1}\} & \xrightarrow{(F_1, \dots, F_r)} & \mathcal{Z}_{\mathcal{K}_\Sigma}(\mathbb{K}^n, (\mathbb{K}^n)^*) \\ \gamma_s \downarrow & & \downarrow q_{n, \mathbb{K}} \\ \mathbb{RP}^s & \xrightarrow{f} & \mathcal{Z}_{\mathcal{K}_\Sigma}(\mathbb{K}^n, (\mathbb{K}^n)^*) / G_{\Sigma, \mathbb{K}} = X_{\Sigma, \mathbb{K}}(n) \end{array}$$

is commutative if and only if the following condition holds:

$$(1.17) \quad \sum_{k=1}^r d_k \mathbf{n}_k = \mathbf{0}_m.$$

Here, $\gamma_s : \mathbb{R}^{s+1} \setminus \{\mathbf{0}_{s+1}\} \rightarrow \mathbb{RP}^s$ denotes the canonical double covering, and the map $F = (F_1, \dots, F_r)$ is given by (1.15). \square

Assumptions. From now on, let Σ be a fan in \mathbb{R}^m as in Definition 1.3, and we always assume that X_Σ is simply connected and non-singular. Moreover, we shall assume the following condition holds.

(1.17)* There is an r -tuple $D_* = (d_1^*, \dots, d_r^*) \in \mathbb{N}^r$ such that $\sum_{k=1}^r d_k^* \mathbf{n}_k = \mathbf{0}_m$.

Remark 1.7. (i) It follows from [6, Theorem 12.1.10] that X_Σ is simply connected if and only if the following condition (††) holds:

(††) The set $\{\mathbf{n}_k\}_{k=1}^r$ of all primitive generators spans \mathbb{Z}^m over \mathbb{Z} , i.e.
 $\sum_{k=1}^r \mathbb{Z} \cdot \mathbf{n}_k = \mathbb{Z}^m$.

Thus, one can easily see that the set $\{\mathbf{n}_k\}_{k=1}^r$ of all primitive generators spans \mathbb{R}^m if X_Σ is simply connected. In particular, we can see that X_Σ is simply connected if X_Σ is a compact smooth toric variety.

(ii) Let Σ be a fan in \mathbb{R}^m such that $\Sigma(1) = \{\rho_1, \dots, \rho_r\}$, where $\Sigma(1)$ denotes the set of all one dimensional cones in Σ as in (1.4). In this situation, consider the case $s = 1$ for

Lemma 1.6. We make the identification $\mathbb{RP}^1 = S^1 = \mathbb{R} \cup \infty$ and choose the points ∞ and $[1, 1, \dots, 1]$ as the base points of \mathbb{RP}^1 and X_Σ , respectively. Then, by setting $z = \frac{z_0}{z_1}$, for each $1 \leq k \leq r$, we can view f_k as a monic polynomial $f_k(z) \in \mathbb{K}[z]$ of degree d_k in the real variable z . \square

Now it is ready to define the space $Q_n^{D, \Sigma}(\mathbb{K})$ for $\mathbb{K} = \mathbb{R}$ or \mathbb{C} .

Definition 1.8. Let $\mathbb{K} = \mathbb{R}$ or \mathbb{C} . For each r -tuple $D = (d_1, \dots, d_r) \in \mathbb{N}^r$, let $Q_n^{D, \Sigma}(\mathbb{K})$ denote the space of all r -tuples $(f_1(z), \dots, f_r(z)) \in \mathbb{K}[z]^r$ of \mathbb{K} -coefficients monic polynomials satisfying the following two conditions:

- (1.17.1) For each $1 \leq i \leq r$, $f_i(z) \in \mathbb{K}[z]$ is an \mathbb{K} -coefficients monic polynomial of the degree d_i .
- (1.17.2) For each $\sigma = \{i_1, \dots, i_s\} \in I(\mathcal{K}_\Sigma)$, polynomials $f_{i_1}(z), \dots, f_{i_s}(z)$ have no common real root $\alpha \in \mathbb{R}$ of multiplicity $\geq n$ (but may have a common root $\alpha \in \mathbb{C} \setminus \mathbb{R}$ of any multiplicity).

(ii) For a monic polynomial $f(z) \in \mathbb{K}[z]$ of degree d , let $F_n(f)(z)$ denote the n -tuple of monic polynomials of the same degree d defined by

$$(1.18) \quad F_n(f)(z) = (f(z), f(z) + f'(z), f(z) + f''(z), \dots, f(z) + f^{(n-1)}(z)).$$

Note that a monic polynomial $f(z) \in \mathbb{K}[z]$ has a root $\alpha \in \mathbb{C}$ of multiplicity $\geq n$ iff $F_n(f)(\alpha) = \mathbf{0}_n \in \mathbb{C}^n$.

Remark 1.9. (i) Recall that the space $Q_n^{D, \Sigma}(\mathbb{C})$ was already investigated for the case $n = 1$ in [15],² and that the space $Q_n^{D, \Sigma}(\mathbb{K})$ was already studied well in [16] for the the case $(X_\Sigma, D) = (\mathbb{CP}^{m-1}, D_m(d))$,³ where $D_m(d) \in \mathbb{N}^m$ denotes the m -tuple of positive integers defined by

$$(1.19) \quad D_m(d) = (d, d, \dots, d) \quad (m\text{-times}).$$

(ii) Let $\mathbb{Z}_2 = \{\pm 1\}$ denote the multiplicative cyclic group of order 2, and we denote by $X^{\mathbb{Z}_2}$ the \mathbb{Z}_2 -fixed point set of a \mathbb{Z}_2 -space X .

Then the complex conjugation on \mathbb{C} easily extends to the \mathbb{Z}_2 -actions on the spaces $\mathcal{Z}_{\mathcal{K}_\Sigma}(\mathbb{C}^n, (\mathbb{C}^n)^*)$ and $Q_n^{D, \Sigma}(\mathbb{C})$ such that

$$(1.20) \quad \mathcal{Z}_{\mathcal{K}_\Sigma}(\mathbb{R}^n, (\mathbb{R}^n)^*) = \mathcal{Z}_{\mathcal{K}_\Sigma}(\mathbb{C}^n, (\mathbb{C}^n)^*)^{\mathbb{Z}_2}, \quad Q_n^{D, \Sigma}(\mathbb{R}) = Q_n^{D, \Sigma}(\mathbb{C})^{\mathbb{Z}_2}.$$

It is easy to see that the complex conjugation on \mathbb{C} also naturally extends to the \mathbb{Z}_2 -action on $X_{\Sigma, \mathbb{C}}(n) = \mathcal{Z}_{\mathcal{K}_\Sigma}(\mathbb{C}^n, (\mathbb{C}^n)^*)/G_{\Sigma, \mathbb{C}}$ such that

$$(1.21) \quad X_{\Sigma, \mathbb{R}}(n) = X_{\Sigma, \mathbb{C}}(n)^{\mathbb{Z}_2}. \quad \square$$

²It is written as $\text{Pol}_D^*(S^1, X_\Sigma) = Q_n^{D, \Sigma}(\mathbb{C})$ if $n = 1$ in [15].

³It is written as $Q_n^{d, m}(\mathbb{K}) = Q_n^{D, \Sigma}(\mathbb{K})$ in [16] for $(X_\Sigma, D) = (\mathbb{CP}^{m-1}, D_m(d))$.

Definition 1.10. Suppose that the condition (1.17)* holds, and let $D = (d_1, \dots, d_r) \in \mathbb{N}^r$ be an r -tuple of positive integers satisfying the condition $\sum_{k=1}^r d_k \mathbf{n}_k = \mathbf{0}_m$.

(i) First, consider the case $\mathbb{K} = \mathbb{C}$. By Lemma 1.6, one can define the map

$$(1.22) \quad i_{D,n,\mathbb{C}} : \mathbb{Q}_n^{\mathbf{D},\Sigma}(\mathbb{C}) \rightarrow \Omega X_\Sigma(\mathbf{n}) \quad \text{by}$$

$$i_{D,n,\mathbb{C}}(f)(\alpha) = \begin{cases} [F_n(f_1)(\alpha), F_n(f_2)(\alpha), \dots, F_n(f_r)(\alpha)] & \text{if } \alpha \in \mathbb{R} \\ [\mathbf{e}, \mathbf{e}, \dots, \mathbf{e}] & \text{if } \alpha = \infty \end{cases}$$

for $f = (f_1(z), \dots, f_r(z)) \in \mathbb{Q}_n^{\mathbf{D},\Sigma}(\mathbb{C})$ and $\alpha \in \mathbb{R} \cup \infty = S^1$, where we set $\mathbf{e} = (1, 1, \dots, 1) \in \mathbb{C}^n$.

Since the space $\mathbb{Q}_n^{\mathbf{D},\Sigma}(\mathbb{C})$ is simply connected and $\Omega q_{n,\mathbb{C}}$ is a universal covering, the map $i_{D,n,\mathbb{C}}$ lifts to the space $\Omega \mathcal{Z}_{\mathcal{K}_\Sigma}(\mathbb{C}^n, (\mathbb{C}^n)^*)$, and there is a based map

$$(1.23) \quad j_{D,n,\mathbb{C}} : \mathbb{Q}_n^{\mathbf{D},\Sigma}(\mathbb{C}) \rightarrow \Omega \mathcal{Z}_{\mathcal{K}_\Sigma}(\mathbb{C}^n, (\mathbb{C}^n)^*)$$

such that the following equality holds:

$$(1.24) \quad \Omega q_{n,\mathbb{C}} \circ j_{D,n,\mathbb{C}} = i_{D,n,\mathbb{C}}.$$

(ii) Next, consider the case $\mathbb{K} = \mathbb{R}$. Recall the \mathbb{Z}_2 -action on the spaces $\mathbb{Q}_n^{\mathbf{D},\Sigma}(\mathbb{C})$ and X_Σ induced from the complex conjugation on \mathbb{C} , and remark that the map $i_{D,n,\mathbb{C}}$ is a \mathbb{Z}_2 -equivariant map. Then, by (1.20) and (1.21), we see that

$$(1.25) \quad i_{D,n,\mathbb{C}}(\mathbb{Q}_n^{\mathbf{D},\Sigma}(\mathbb{R})) \subset \Omega X_\Sigma(\mathbf{n})^{\mathbb{Z}_2} = \Omega X_{\Sigma,\mathbb{R}}(\mathbf{n}).$$

Thus, the restriction $i_{D,n,\mathbb{C}}|_{\mathbb{Q}_n^{\mathbf{D},\Sigma}(\mathbb{R})}$ defines the map

$$(1.26) \quad i_{D,n,\mathbb{R}} = i_{D,n,\mathbb{C}}|_{\mathbb{Q}_n^{\mathbf{D},\Sigma}(\mathbb{R})} : \mathbb{Q}_n^{\mathbf{D},\Sigma}(\mathbb{R}) \rightarrow \Omega X_{\Sigma,\mathbb{R}}(\mathbf{n})$$

such that the following diagram is commutative:

$$(1.27) \quad \begin{array}{ccccc} \mathbb{Q}_n^{\mathbf{D},\Sigma}(\mathbb{R}) & \xrightarrow{i_{D,n,\mathbb{R}}} & \Omega X_{\Sigma,\mathbb{R}}(n) & \xleftarrow[\simeq]{\Omega q_{n,\mathbb{R}}} & \Omega \mathcal{Z}_{\mathcal{K}_\Sigma}(\mathbb{R}^n, (\mathbb{R}^n)^*) \\ i_n^D \downarrow & & \Omega i_n^X \downarrow & & \Omega j_n \downarrow \\ \mathbb{Q}_n^{\mathbf{D},\Sigma}(\mathbb{C}) & \xrightarrow{i_{D,n,\mathbb{C}}} & \Omega X_\Sigma(n) & \xleftarrow{\Omega q_{n,\mathbb{C}}} & \Omega \mathcal{Z}_{\mathcal{K}_\Sigma}(\mathbb{C}^n, (\mathbb{C}^n)^*) \end{array}$$

where the $j_n : \mathcal{Z}_{\mathcal{K}_\Sigma}(\mathbb{R}^n, (\mathbb{R}^n)^*) \xrightarrow{\subset} \mathcal{Z}_{\mathcal{K}_\Sigma}(\mathbb{C}^n, (\mathbb{C}^n)^*)$ denotes the inclusion map. Moreover, note that $\Omega q_{n,\mathbb{R}}$ is a homotopy equivalence. Thus, there is a based map

$$(1.28) \quad j_{D,n,\mathbb{R}} : \mathbb{Q}_n^{\mathbf{D},\Sigma}(\mathbb{R}) \rightarrow \Omega \mathcal{Z}_{\mathcal{K}_\Sigma}(\mathbb{R}^n, (\mathbb{R}^n)^*)$$

which satisfies the following equality:

$$(1.29) \quad \Omega q_{n,\mathbb{R}} \circ j_{D,n,\mathbb{R}} = i_{D,n,\mathbb{R}} \quad (\text{up to homotopy}).$$

Definition 1.11. (i) We say that a set $S = \{\mathbf{n}_{i_1}, \dots, \mathbf{n}_{i_s}\}$ is a *primitive collection* if $\text{Cone}(S) \notin \Sigma$ and $\text{Cone}(T) \in \Sigma$ for any proper subset $T \subsetneq S$.

(ii) For each r -tuple $D = (d_1, \dots, d_r) \in \mathbb{N}^r$, define the positive integer $d(D, \Sigma, n, \mathbb{K})$ by

$$(1.30) \quad d(D; \Sigma, n, \mathbb{K}) = \begin{cases} (2nr_{\min}(\Sigma) - 2)\lfloor d_{\min}/n \rfloor - 2 & \text{if } \mathbb{K} = \mathbb{C}, \\ (nr_{\min}(\Sigma) - 2)\lfloor d_{\min}/n \rfloor - 2 & \text{if } \mathbb{K} = \mathbb{R}, \end{cases}$$

where $\lfloor x \rfloor$ denotes the integer part of a real number x , and $r_{\min}(\Sigma)$ and $d_{\min} = d_{\min}(D)$ are the positive integers given by

$$(1.31) \quad \begin{cases} r_{\min}(\Sigma) = \min\{s \in \mathbb{N} : \{\mathbf{n}_{i_1}, \dots, \mathbf{n}_{i_s}\} \text{ is a primitive collection}\}, \\ d_{\min} = d_{\min}(D) = \min\{d_1, d_2, \dots, d_r\}. \end{cases}$$

(iii) To state the main result of this paper, we need to consider the following two conditions (1.31)* and (1.31)†:

$$(1.31)^* \quad d_{\min} \geq n \geq 1.$$

(1.31)† One of the following three conditions (1.31.a), (1.31.b) and (1.31.c) holds:

$$(1.31.a) \quad d_{\min} \geq n \geq 2.$$

$$(1.31.b) \quad n = 1, d_{\min} \geq 2, \text{ and } r_{\min}(\Sigma) \geq 4.$$

$$(1.31.c) \quad n = d_{\min} = 1 \text{ and } r_{\min}(\Sigma) \geq 5.$$

Note that the condition (1.31)† holds if (1.31)* is satisfied. □

2 Main results

Now we can state the main results of this article as follows.

Theorem 2.1 ([18]; The case $\mathbb{K} = \mathbb{C}$). *Let $n \in \mathbb{N}$, $D = (d_1, \dots, d_r) \in \mathbb{N}^r$, and let X_Σ be an m dimensional simply connected smooth toric variety such that the two conditions (1.17)* and (1.31)* hold.*

(i) *If $\sum_{k=1}^r d_k \mathbf{n}_k = \mathbf{0}_m$, the map (given by (1.23))*

$$j_{D,n,\mathbb{C}} : \mathbb{Q}_n^{\mathbb{D},\Sigma}(\mathbb{C}) \rightarrow \Omega \mathcal{Z}_{\mathcal{K}_\Sigma}(\mathbb{C}^n, (\mathbb{C}^n)^*)$$

is a homotopy equivalence through dimension $d(D; \Sigma, n, \mathbb{C})$.

(ii) *If $\sum_{k=1}^r d_k \mathbf{n}_k \neq \mathbf{0}_m$, there is a map*

$$j_{D,n,\mathbb{C}} : \mathbb{Q}_n^{\mathbb{D},\Sigma}(\mathbb{C}) \rightarrow \Omega \mathcal{Z}_{\mathcal{K}_\Sigma}(\mathbb{C}^n, (\mathbb{C}^n)^*)$$

which is a homotopy equivalence through dimension $d(D; \Sigma, n, \mathbb{C})$. □

Theorem 2.2 ([18]; The case $\mathbb{K} = \mathbb{R}$). *Let $n \in \mathbb{N}$, $D = (d_1, \dots, d_r) \in \mathbb{N}^r$, and let X_Σ be an m dimensional simply connected non-singular toric variety such that the two conditions (1.17)* and (1.31)[†] hold.*

(i) *If $\sum_{k=1}^r d_k \mathbf{n}_k = \mathbf{0}_m$, the map (given by (1.28))*

$$j_{D,n,\mathbb{R}} : \mathbb{Q}_n^{\mathbb{D},\Sigma}(\mathbb{R}) \rightarrow \Omega \mathcal{Z}_{\mathcal{K}_\Sigma}(\mathbb{R}^n, (\mathbb{R}^n)^*)$$

is a homotopy equivalence through dimension $d(D; \Sigma, n, \mathbb{R})$.

(ii) *If $\sum_{k=1}^r d_k \mathbf{n}_k \neq \mathbf{0}_m$, there is a map*

$$j_{D,n,\mathbb{R}} : \mathbb{Q}_n^{\mathbb{D},\Sigma}(\mathbb{R}) \rightarrow \Omega \mathcal{Z}_{\mathcal{K}_\Sigma}(\mathbb{R}^n, (\mathbb{R}^n)^*)$$

which is a homotopy equivalence through dimension $d(D; \Sigma, n, \mathbb{R})$. \square

Corollary 2.3 ([18]). *Let $n \in \mathbb{N}$, let $D = (d_1, \dots, d_r) \in \mathbb{N}^r$, and let X_Σ be an m dimensional simply connected non-singular toric variety such that the two conditions (1.17)* and (1.31)* hold.*

(i) *If $\sum_{k=1}^r d_k \mathbf{n}_k = \mathbf{0}_m$, the map $i_{D,n,\mathbb{C}} : \mathbb{Q}_n^{\mathbb{D},\Sigma}(\mathbb{C}) \rightarrow \Omega X_\Sigma(n)$ induces an isomorphism*

$$(i_{D,n,\mathbb{C}})_* : \pi_s(\mathbb{Q}_n^{\mathbb{D},\Sigma}(\mathbb{C})) \xrightarrow{\cong} \pi_s(\Omega X_\Sigma) \cong \pi_{s+1}(X_\Sigma(n))$$

for any $2 \leq s \leq d(D; \Sigma, n, \mathbb{C})$.

(ii) *If $\sum_{k=1}^r d_k \mathbf{n}_k \neq \mathbf{0}_m$, the map $i_{D,n,\mathbb{C}} : \mathbb{Q}_n^{\mathbb{D},\Sigma}(\mathbb{C}) \rightarrow \Omega X_\Sigma(n)$ defined by*

$$(2.1) \quad i_{D,n,\mathbb{C}} := \Omega q_{n,\mathbb{C}} \circ j_{D,n,\mathbb{C}}$$

induces an isomorphism $(i_{D,n,\mathbb{C}})_ : \pi_s(\mathbb{Q}_n^{\mathbb{D},\Sigma}(\mathbb{C})) \xrightarrow{\cong} \pi_s(\Omega X_\Sigma(n)) \cong \pi_{s+1}(X_\Sigma(n))$ for any $2 \leq s \leq d(D; \Sigma, n, \mathbb{C})$. \square*

Let $\mathbb{Z}_2 = \{\pm 1\}$ be the multiplicative cyclic group of order 2, and consider the \mathbb{Z}_2 -action on the spaces $\mathbb{Q}_n^{\mathbb{D},\Sigma}(\mathbb{C})$ and $\mathcal{Z}_{\mathcal{K}_\Sigma}(\mathbb{C}^n, (\mathbb{C}^n)^*)$ induced from the complex conjugation on \mathbb{C} .

Since $\mathbb{Q}_n^{\mathbb{D},\Sigma}(\mathbb{R}) = \mathbb{Q}_n^{\mathbb{D},\Sigma}(\mathbb{C})^{\mathbb{Z}_2}$, $\mathcal{Z}_{\mathcal{K}_\Sigma}(\mathbb{C}^n, (\mathbb{C}^n)^*)^{\mathbb{Z}_2} = \mathcal{Z}_{\mathcal{K}_\Sigma}(\mathbb{R}^n, (\mathbb{R}^n)^*)$ and $j_{D,n,\mathbb{R}} = (j_{D,n,\mathbb{C}})^{\mathbb{Z}_2}$, we also obtain the following result.

Corollary 2.4 ([18]). *Let $n \in \mathbb{N}$, let $D = (d_1, \dots, d_r) \in \mathbb{N}^r$, and let X_Σ be an m dimensional simply connected non-singular toric variety satisfying the two conditions (1.17)* and (1.31)[†]. Then the map*

$$j_{D,n,\mathbb{C}} : \mathbb{Q}_n^{\mathbb{D},\Sigma}(\mathbb{C}) \rightarrow \Omega \mathcal{Z}_{\mathcal{K}_\Sigma}(\mathbb{C}^n, (\mathbb{C}^n)^*)$$

is a \mathbb{Z}_2 -equivariant homotopy equivalence through dimension $d(D; \Sigma, n, \mathbb{R})$. \square

Corollary 2.5 ([18]). *Let $n \in \mathbb{N}$, let $D = (d_1, \dots, d_r) \in \mathbb{N}^r$, and let X_Σ be a simply connected compact non-singular toric variety such that the two conditions (1.17)* and (1.31)* hold. Let $\Sigma(1)$ denote the set of all one dimensional cones in Σ , and Σ_1 any fan in \mathbb{R}^m satisfying the condition*

$$(2.2) \quad \Sigma(1) \subset \Sigma_1 \subsetneq \Sigma.$$

(i) Let $\sum_{k=1}^r d_k \mathbf{n}_k = \mathbf{0}_m$. Then the map

$$j_{D,n,\mathbb{C}} : \mathbb{Q}_n^{\mathbb{D},\Sigma_1}(\mathbb{C}) \rightarrow \Omega \mathcal{Z}_{\Sigma_1}(\mathbb{C}^n, (\mathbb{C}^n)^*)$$

is a homotopy equivalence through the dimension $d(D; \Sigma_1, n, \mathbb{C})$.

Moreover, the map $i_{D,n,\mathbb{C}} : \mathbb{Q}_n^{\mathbb{D},\Sigma_1}(\mathbb{C}) \rightarrow \Omega X_{\Sigma_1}$ induces an isomorphism

$$(i_{D,n,\mathbb{C}})_* : \pi_s(\mathbb{Q}_n^{\mathbb{D},\Sigma_1}(\mathbb{C})) \xrightarrow{\cong} \pi_s(\Omega X_{\Sigma_1}(\mathbf{n})) \cong \pi_{s+1}(X_{\Sigma_1}(\mathbf{n}))$$

for any $2 \leq s \leq d(D; \Sigma_1, n, \mathbb{C})$.

(ii) If $\sum_{k=1}^r d_k \mathbf{n}_k \neq \mathbf{0}_m$, there is a map

$$j_{D,n,\mathbb{C}} : \mathbb{Q}_n^{\mathbb{D},\Sigma_1}(\mathbb{C}) \rightarrow \Omega \mathcal{Z}_{\mathcal{K}_{\Sigma_1}}(\mathbb{C}^n, (\mathbb{C}^n)^*)$$

which is a homotopy equivalence through dimension $d(D; \Sigma_1, n, \mathbb{C})$.

Moreover, the map $i_{D,n,\mathbb{C}} : \mathbb{Q}_n^{\mathbb{D},\Sigma_1}(\mathbb{C}) \rightarrow \Omega X_{\Sigma_1}$ defined by

$$(2.3) \quad i_{D,n,\mathbb{C}} := \Omega q_{n,\mathbb{C}} \circ j_{D,n,\mathbb{C}}$$

induces an isomorphism $(i_{D,n,\mathbb{C}})_* : \pi_s(\mathbb{Q}_n^{\mathbb{D},\Sigma_1}(\mathbb{C})) \xrightarrow{\cong} \pi_s(\Omega X_{\Sigma_1}(\mathbf{n})) \cong \pi_{s+1}(X_{\Sigma_1}(\mathbf{n}))$ for any $2 \leq s \leq d(D; \Sigma_1, n, \mathbb{C})$. \square

Corollary 2.6 ([18]). Let $n \in \mathbb{N}$, let $D = (d_1, \dots, d_r) \in \mathbb{N}^r$, and let X_{Σ} be a simply connected compact non-singular toric variety such that the two conditions (1.17)* and (1.31)[†] hold. Let Σ_1 be any fan in \mathbb{R}^m satisfying the condition (2.2).

(i) Let $\sum_{k=1}^r d_k \mathbf{n}_k = \mathbf{0}_m$. Then the map

$$j_{D,n,\mathbb{R}} : \mathbb{Q}_n^{\mathbb{D},\Sigma_1}(\mathbb{R}) \rightarrow \Omega \mathcal{Z}_{\Sigma_1}(\mathbb{R}^n, (\mathbb{R}^n)^*)$$

is a homotopy equivalence through the dimension $d(D; \Sigma_1, n, \mathbb{R})$.

(ii) If $\sum_{k=1}^r d_k \mathbf{n}_k \neq \mathbf{0}_m$, there is a map

$$j_{D,n,\mathbb{R}} : \mathbb{Q}_n^{\mathbb{D},\Sigma_1}(\mathbb{R}) \rightarrow \Omega \mathcal{Z}_{\mathcal{K}_{\Sigma_1}}(\mathbb{R}^n, (\mathbb{R}^n)^*)$$

which is a homotopy equivalence through dimension $d(D; \Sigma_1, n, \mathbb{R})$. \square

3 Examples

Example 3.1. Consider the case $X_{\Sigma} = \mathbb{C}\mathbb{P}^{m-1}$ for $m \geq 2$. It is a toric variety and its associated fan Σ_m is the fan in \mathbb{R}^{m-1} given by

$$(3.1) \quad \Sigma_m = \left\{ \text{Cone}(\mathbf{n}) : \mathbf{n} \subsetneq \{ \mathbf{e}_0, \mathbf{e}_1, \dots, \mathbf{e}_{m-1} \} \right\},$$

where the set $\{\mathbf{e}_k\}_{k=1}^{m-1}$ denotes the set of standard orthogonal basis of \mathbb{R}^{m-1} given by

$$(3.2) \quad \mathbf{e}_1 = (1, 0, 0, \dots, 0), \mathbf{e}_2 = (0, 1, 0, \dots, 0), \dots, \mathbf{e}_{m-1} = (0, 0, 0, \dots, 1),$$

and we set $\mathbf{e}_0 = -\sum_{k=1}^{m-1} \mathbf{e}_k$. Thus, the underlying simplicial complex of Σ_m is given by

$$(3.3) \quad \mathcal{K}_{\Sigma_m} = \left\{ \sigma \subset \{0, 1, 2, \dots, m-1\} : \sigma \neq \{0, 1, 2, \dots, m-1\} \right\}.$$

Since $I(\mathcal{K}_{\Sigma_m}) = \left\{ \{0, 1, 2, \dots, m-1\} \right\}$, the set $\{\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_{m-1}\}$ is only a primitive collection and we obtain the following:

$$(3.4) \quad \mathcal{Z}_{\mathcal{K}_{\Sigma_m}}(\mathbb{K}^n, (\mathbb{K}^n)^*) = (\mathbb{K}^n)^m \setminus \{\mathbf{0}_{mn}\} \simeq S^{d(\mathbb{K})mn-1}, \quad r_{\min}(\Sigma_m) = m,$$

where $d(\mathbb{K}) = \dim_{\mathbb{R}} \mathbb{K} = \begin{cases} 1 & \text{if } \mathbb{K} = \mathbb{R}, \\ 2 & \text{if } \mathbb{K} = \mathbb{C}. \end{cases}$

In particular, for $D = (d_0, d_1, \dots, d_{m-1}) \in \mathbb{N}^m$,

$$(3.5) \quad \sum_{k=0}^{m-1} d_k \mathbf{e}_k = \mathbf{0}_{m-1} \Leftrightarrow d_0 = d_1 = \dots = d_{m-1} = d \Leftrightarrow D = D_m(d)$$

for some $d \in \mathbb{N}$, where $D_m(d) = (d, d, \dots, d)$ (m -times) as in (1.19).

Now let $d_{\mathbb{K}}(D; m, n)$ denote the positive integer defined by

$$(3.6) \quad d_{\mathbb{K}}(D; m, n) = \begin{cases} (2mn - 2)(\lfloor d_{\min}/n \rfloor + 1) - 1 & \text{if } \mathbb{K} = \mathbb{C}, \\ (mn - 2)(\lfloor d_{\min}/n \rfloor + 1) - 1 & \text{if } \mathbb{K} = \mathbb{R}. \end{cases}$$

Recall the following previous result for the case $X_{\Sigma} = \mathbb{C}P^{m-1}$.

Theorem 3.2 ([16]). *Let $m, n \in \mathbb{N}$ be positive integers such that $m \geq 2$, $D = (d_0, \dots, d_{m-1}) \in \mathbb{N}^m$, $d_{\min} = \min\{d_0, \dots, d_{m-1}\}$, and suppose that $d(\mathbb{K})mn \geq 4$.*

(i) *If $D = D_m(d)$ for some $d \in \mathbb{N}$, then $d_{\min} = d$ and the natural map*

$$j_{D, n, \mathbb{K}} : \mathbb{Q}_n^{D, \Sigma_m}(\mathbb{K}) \rightarrow \Omega S^{d(\mathbb{K})mn-1}$$

is a homotopy equivalence through dimension $d_{\mathbb{K}}(D_m; m, n) = (d(\mathbb{K})mn - 2)(\lfloor d/n \rfloor + 1) - 1$.

(ii) *If $D \neq D_m(d)$ for any $d \in \mathbb{N}$, then there is a map*

$$j_{D, n, \mathbb{K}} : \mathbb{Q}_n^{D, \Sigma_m}(\mathbb{K}) \rightarrow \Omega S^{d(\mathbb{K})mn-1}$$

which is a homotopy equivalence through dimension $d_{\mathbb{K}}(D; m, n)$. □

Remark 3.3. If we use Theorems 2.1 and 2.2, the homotopy stability dimension of the natural map $j_{D, n, \mathbb{K}}$ is $d(D; \Sigma_m, n, \mathbb{K}) = (d(\mathbb{K})mn - 2)\lfloor d_{\min}/n \rfloor - 2$. However, since $d(D; \Sigma_m, n, \mathbb{K}) < d_{\mathbb{K}}(D; \Sigma_m, n)$, the result of Theorem 3.2 is better than that of Theorems 2.1 and 2.2 for the case $X_{\Sigma} = \mathbb{C}P^{m-1}$. In fact, one can show that the result of Theorem 3.2 is the best possible stability dimension. □

Finally consider the case of the Hirzerbruch surface $H(k)$.

Definition 3.4 ([14], Example 2.12). For each $k \in \mathbb{Z}$, let $H(k) \subset \mathbb{CP}^2 \times \mathbb{CP}^1$ be the Hirzerbruch surface defined by

$$(3.7) \quad H(k) = \{([x_0 : x_1 : x_2], [y_1 : y_2]) \in \mathbb{CP}^2 \times \mathbb{CP}^1 : x_1 y_1^k = x_2 y_2^k\}.$$

Since there are isomorphisms

$$(3.8) \quad H(-k) \cong H(k) \text{ if } k \neq 0, \text{ and } H(0) \cong \mathbb{CP}^1 \times \mathbb{CP}^1 \text{ if } k = 0,$$

without loss of generalities, we may assume that $k \geq 1$. Then the Hirzerbruch surface $H(k)$ is a two dimensional toric variety (over \mathbb{C}), and its associated fan $\Sigma_{H(k)}$ is given by

$$(3.9) \quad \Sigma_{H(k)} = \left\{ \begin{array}{l} \text{Cone}(\mathbf{n}_1, \mathbf{n}_2), \text{Cone}(\mathbf{n}_2, \mathbf{n}_3), \text{Cone}(\mathbf{n}_3, \mathbf{n}_4), \text{Cone}(\mathbf{n}_4, \mathbf{n}_1) \\ \text{Cone}(\mathbf{n}_1), \text{Cone}(\mathbf{n}_2), \text{Cone}(\mathbf{n}_3), \text{Cone}(\mathbf{n}_4), \{\mathbf{0}_2\} \end{array} \right\},$$

where we set $\mathbf{n}_1 = (1, 0)$, $\mathbf{n}_2 = (0, 1)$, $\mathbf{n}_3 = (-1, k)$, $\mathbf{n}_4 = (0, -1)$.

Thus, the set of all one dimensional cones in $\Sigma_{H(k)}$ is given by

$$(3.10) \quad \Sigma_{H(k)}(1) = \{\text{Cone}(\mathbf{n}_1), \text{Cone}(\mathbf{n}_2), \text{Cone}(\mathbf{n}_3), \text{Cone}(\mathbf{n}_4)\}.$$

Since $\{\mathbf{n}_1, \mathbf{n}_3\}$ and $\{\mathbf{n}_2, \mathbf{n}_4\}$ are the only primitive collections,

$$(3.11) \quad r_{\min}(\Sigma_{H(k)}) = 2.$$

Moreover, for $D = (d_1, d_2, d_3, d_4) \in \mathbb{N}^4$,

$$(3.12) \quad \sum_{k=1}^4 d_k \mathbf{n}_k = \mathbf{0}_2 \Leftrightarrow (d_3, d_4) = (d_1, kd_1 + d_2). \quad \square$$

Thus, it follows from Theorems 2.1, 2.2 and Corollaries 2.3, 2.5 that we obtain the following examples.

Example 3.5 (The case $X_\Sigma = H(k)$). Let $(k, n) \in \mathbb{N}^2$, $X_\Sigma = H(k)$, $D = (d_1, d_2, d_3, d_4) \in \mathbb{N}^4$, $d_{\min} = \min\{d_1, d_2, d_3, d_4\}$, $\mathbb{K} = \mathbb{C}$ or \mathbb{R} , and assume that the following condition holds.

$$(3.13) \quad d_{\min} \geq n \geq 1 \text{ if } \mathbb{K} = \mathbb{C}, \text{ and } d_{\min} \geq n \geq 2 \text{ if } \mathbb{K} = \mathbb{R}.$$

(i) If $(d_3, d_4) = (d_1, kd_1 + d_2)$, $d_{\min} = \min\{d_1, d_2\}$ and the map

$$j_{D, n, \mathbb{K}} : Q_n^{\mathbb{D}, \Sigma_{H(k)}}(\mathbb{K}) \rightarrow \Omega \mathcal{Z}_{\mathcal{K}_{\Sigma_{H(k)}}}(\mathbb{K}^n, (\mathbb{K}^n)^*)$$

is a homotopy equivalence through dimension $(2d(\mathbb{K})n - 2) \lfloor \min\{d_1, d_2\}/n \rfloor - 2$.

Moreover, if $\mathbb{K} = \mathbb{C}$, the map $i_{D, n, \mathbb{C}}$ induces an isomorphism

$$(i_{D, n, \mathbb{C}})_* : \pi_s(Q_n^{\mathbb{D}, \Sigma_{H(k)}}(\mathbb{C})) \xrightarrow{\cong} \pi_s(\Omega X_{\Sigma_{H(k)}}(\mathbb{C})) \cong \pi_{s+1}(X_{\Sigma_{H(k)}}(\mathbb{C}))$$

for any $2 \leq s \leq (4n - 2)\lfloor \min\{d_1, d_2\}/n \rfloor - 2$.

(ii) If $(d_3, d_4) \neq (d_1, kd_1 + d_2)$, there is a map

$$j_{D,n,\mathbb{K}} : \mathbb{Q}_n^{\mathbb{D}, \Sigma_{\mathbb{H}(k)}}(\mathbb{K}) \rightarrow \Omega \mathcal{Z}_{\mathcal{K}_{\Sigma_{\mathbb{H}(k)}}}(\mathbb{K}^n, (\mathbb{K}^n)^*)$$

which is a homotopy equivalence through dimension $(2d(\mathbb{K})n - 2)\lfloor d_{\min}/n \rfloor - 2$.

Moreover, if $\mathbb{K} = \mathbb{C}$, the map $i_{D,n,\mathbb{C}} = \Omega q_{n,\mathbb{C}} \circ j_{D,n,\mathbb{C}}$ induces an isomorphism

$$(i_{D,n,\mathbb{C}})_* : \pi_s(\mathbb{Q}_n^{\mathbb{D}, \Sigma_{\mathbb{H}(k)}}(\mathbb{C})) \xrightarrow{\cong} \pi_s(\Omega X_{\Sigma_{\mathbb{H}(k)}}(n)) \cong \pi_{s+1}(X_{\Sigma_{\mathbb{H}(k)}}(n))$$

for any $2 \leq s \leq (4n - 2)\lfloor d_{\min}/n \rfloor - 2$. □

Acknowledgements. The author was supported by JSPS KAKENHI Grant Number JP22K03283. This work was also supported by the Research Institute for Mathematical Sciences, a Joint Usage/Research Center located in Kyoto University.

References

- [1] M. Adamaszek, A. Kozłowski and K. Yamaguchi, Spaces of algebraic and continuous maps between real algebraic varieties, *Quart. J. Math.* **62** (2011), 771–790.
- [2] M. F. Atiyah and J. D. S. Jones, Topological aspects of Yang-Mills theory, *Commun. Math. Phys.* **59** (1978), 97–118.
- [3] V. M. Buchstaber and T. E. Panov, Torus actions and their applications in topology and combinatorics, *Univ. Lecture Note Series* **24**, Amer. Math. Soc. Providence, 2002.
- [4] D. A. Cox, The homogenous coordinate ring of a toric variety, *J. Algebraic Geometry* **4** (1995), 17–50.
- [5] D. A. Cox, The functor of a smooth toric variety, *Tohoku Math. J.* **47** (1995), 251–262.
- [6] D. A. Cox, J. B. Little and H. K. Schenck, *Toric varieties*, Graduate Studies in Math. **124**, Amer. Math. Soc., 2011.
- [7] B. Farb and J. Wolfson, Topology and arithmetic of resultants, I, *New York J. Math.* **22** (2016), 801–821.
- [8] M. A. Guest, The topology of the space of rational curves on a toric variety, *Acta Math.* **174** (1995), 119–145.
- [9] M.A. Guest, A. Kozłowski and K. Yamaguchi, The topology of spaces of coprime polynomials, *Math. Z.* **217** (1994), 435–446.
- [10] M. A. Guest, A. Kozłowski and K. Yamaguchi, Spaces of polynomials with roots of bounded multiplicity, *Fund. Math.* **116** (1999), 93–117.

- [11] A. Kozłowski and K. Yamaguchi, Topology of complements of discriminants and resultants, *J. Math. Soc. Japan*, **52**, (2000), 949-959.
- [12] A. Kozłowski and K. Yamaguchi, The homotopy type of spaces of coprime polynomials revisited, *Topology Appl.* **206** (2016), 284-304.
- [13] A. Kozłowski and K. Yamaguchi, The homotopy type of spaces of resultants of bounded multiplicity, *Topology Appl.* **232** (2017), 112-139.
- [14] A. Kozłowski and K. Yamaguchi, The homotopy type of spaces of rational curves on a toric variety, *Topology Appl.* **249** (2018), 19-42.
- [15] A. Kozłowski and K. Yamaguchi, The homotopy type of the space of algebraic loops on a toric variety, *Topology Appl.* **300** (2021), Paper no. 107705.
- [16] A. Kozłowski and K. Yamaguchi, The homotopy type of real resultants with bounded multiplicity, *J. Math. Soc. Japan* **74** (2022), 1047-1077.
- [17] A. Kozłowski and K. Yamaguchi, Spaces of non-resultant systems of bounded multiplicity determined by a toric variety, *Topology Appl.*, **337** (2023), Paper no. 108626.
- [18] A. Kozłowski and K. Yamaguchi, Spaces of non-resultant systems of real bounded multiplicity determined by a toric variety, (to appear) *J. Math. Soc. Japan* (arXiv:2405.07372).
- [19] J. Mostovoy, Spaces of rational loops on a real projective space, *Trans. Amer. Math. Soc.*, **353** (2001), 1959-1970.
- [20] J. Mostovoy and E. Munguia-Villanueva, Spaces of morphisms from a projective space to a toric variety, *Rev. Colombiana Mat.* **48** (2014), 41-53.
- [21] G. B. Segal, The topology of spaces of rational functions, *Acta Math.* **143** (1979), 39-72.
- [22] V. A. Vassiliev, Complements of discriminants of smooth maps, *Topology and Applications*, Amer. Math. Soc., *Translations of Math. Monographs* **98**, 1992 (revised edition 1994).
- [23] K. Yamaguchi, Complements of resultants and homotopy types, *J. Math. Kyoto Univ.*, **39** (1999), 675-684.

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