Finite topological spaces and the homotopy of subgroup complexes

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1 Introduction

The study of subgroup complexes has long provided deep insights into the interplay between algebraic and topological properties of finite groups. Originating in the work of Quillen [9] and others, these simplicial complexes, constructed from the lattice of subgroups satisfying certain conditions, have played a central role in understanding the cohomology and homotopy type of groups. In particular, subgroup complexes serve as a bridge between group theory and algebraic topology, often encoding delicate information about the structure of finite groups through their homotopy-theoretic invariants.

On the other hand, the theory of finite topological spaces, initiated by Alexandroff, offers a combinatorial model of topology that is particularly well suited to the study of spaces with a finite underlying set. Every finite poset naturally gives rise to a finite topological space, and conversely, the topology of such spaces can be entirely recovered from the associated order structure. This duality has made finite spaces a powerful tool in combinatorial and algebraic topology, providing discrete models for continuous phenomena and enabling explicit calculations of homotopy-theoretic properties.

The aim of this paper is to investigate the homotopy properties of p-subgroup complexes from the viewpoint of finite topological spaces. Here, p is one prime factor of the order of a finite group. By translating subgroup complexes into finite spaces, we gain access to methods from poset topology and Alexandroff spaces, which allow for a finer analysis of their homotopy properties. In particular, we focus on how classical constructions in finite topology, such as minimal finite models and homotopy reductions, can be applied to subgroup complexes to clarify their structure and relations to group-theoretic data.

Our approach highlights the advantages of working within the category of finite spaces: it permits the application of combinatorial techniques, explicit homotopy constructions, and order-theoretic tools to problems that traditionally arise in algebraic topology and group theory. We also demonstrate how this perspective connects with known results, while providing new insights and simplifications in the study of subgroup complexes.

2 Notations and preliminaries

For a partially ordered set (simply, a poset) $\mathcal{P}$, the symbol $\Delta(\mathcal{P})$ is the *order complex* of $\mathcal{P}$; this is the simplicial complex whose k-dimensional simplices are the non-empty chains $x_0 < x_1 < \cdots < x_k$ of $\mathcal{P}$. Also, by considering the topological space $|\Delta(\mathcal{P})|$, which is the geometric realization of the order complex $\Delta(\mathcal{P})$, we can discuss homotopy on $|\Delta(\mathcal{P})|$. Throughout this paper, let G be a finite group and p be a prime number dividing its order. Here we focus on nontrivial p-subgroup posets ordered by set inclusion. In particular, $S_p(G)$ and $A_p(G)$ consisting of nontrivial p-subgroup families are fundamental:

$$S_p(G) := \{P \mid P \text{ is a nontrivial } p\text{-subgroup of } G\},$$
$$A_p(G) := \{P \mid P \text{ is a nontrivial elementary abelian } p\text{-subgroup of } G\}.$$

$\Delta(S_p(G))$ and $\Delta(A_p(G))$ are respectively called a *Brown complex* and a *Quillen complex* of G at p. Such an order complex formed from a p-subgroup poset is called a *p-subgroup complex*. The purpose of this paper is to review the research on homotopy types of p-subgroup complexes and to introduce the latest results.

Let $\mathcal{P}, \mathcal{Q}$ be finite posets. An order preserving map $f : \mathcal{P} \to \mathcal{Q}$ induces naturally a continuous map $|\Delta(f)| : |\Delta(\mathcal{P})| \to |\Delta(\mathcal{Q})|$. Also, if two order preserving maps $f, g : \mathcal{P} \to \mathcal{Q}$ satisfy $f(x) \leq g(x)$ for any $x \in P$, then $|\Delta(f)|$ is homotopic to $|\Delta(g)|$. Note that in many combinatorics papers, $|\Delta(f)|$ is often simply written as f. If a finite poset $\mathcal{P}$ has a maximal or minimal element, $|\Delta(\mathcal{P})|$ is immediately contractible. More generally, the following holds.

Proposition 2.1. Let $\mathcal{P}$ be a finite poset and let x_0 be an element of $\mathcal{P}$. If there is an order preserving self-map $f : \mathcal{P} \to \mathcal{P}$ such that $x \leq f(x) \geq x_0$ for any $x \in \mathcal{P}$, then $|\Delta(\mathcal{P})|$ is contractible.

The above contractibility is especially said to be *conically contractible*. Next, we write the *intercept* of the poset $\mathcal{P}$ with the following symbols:

$$\mathcal{P}_{\leq x} := \{y \in \mathcal{P} \mid y \leq x\}.$$

$\mathcal{P}_{<x}, \mathcal{P}_{\geq x}, \mathcal{P}_{>x}$, etc. are defined similarly. Under this symbol, the following theorem, which is fundamental in the homotopy theory of order complexes, is stated.

Proposition 2.2. Let $\mathcal{P}, \mathcal{Q}$ be finite posets. If there is an order preserving map $f : \mathcal{P} \to \mathcal{Q}$ and $|\Delta(f^{-1}(\mathcal{Q}_{\leq y})|$ (or $|\Delta(f^{-1}(\mathcal{Q}_{\geq y}))|$) is contractible for any $y \in \mathcal{Q}$, then $|\Delta(f)| : |\Delta(\mathcal{P})| \to |\Delta(\mathcal{Q})|$ is a homotopy equivalence.

This theorem was first discovered by Quillen [9] and is called Fibre Lemma. The proof is given using the ‘Acyclic Carrier Theorem’. As a direct application of this proposition, the following is well known.

Proposition 2.3. Let $\mathcal{P}$ be a finite poset and let

$$\mathcal{P}^{<} := \{x \in \mathcal{P} \,|\, |\Delta(\mathcal{P}_{<x})| \text{ is not contractible}\}.$$

Let $\mathcal{Q}$ be a subposet of $\mathcal{P}$ such that $\mathcal{P}^{<} \subseteq \mathcal{Q} \subseteq \mathcal{P}$.Then $\mathcal{Q}$ is homotopy equivalent to $\mathcal{P}$, where the inclusion map induces a homotopy equivalence map.

Now, let $\mathcal{P} = \mathcal{S}_p(G)$. By definition,

$$\mathcal{S}_p(G)^{<} = \{P \in \mathcal{S}_p(G) \,|\, |\Delta(\mathcal{S}_p(G)_{<P})| \text{ is not contractible}\}.$$

Then we have the following proposition.

Proposition 2.4. $\mathcal{A}_p(G) = \mathcal{S}_p(G)^{<}$

Proof. Let $P \in \mathcal{S}_p(G)$ is not an elementary abelian p-subgroup, then there exists a nontrivial Frattini subgroup $\Phi(P)$, and so the map $f : \mathcal{S}_p(G)_{<P} \to \mathcal{S}_p(G)_{<P}\,;\, Q \mapsto Q\Phi(P)$ is well-defined. Furthermore, $Q \subseteq Q\Phi(P) \supseteq \Phi(P)$. By Proposition 2.1, $|\Delta(\mathcal{S}_p(G)_{<P})|$ is contractible. Therefore, we obtain $\mathcal{A}_p(G) \supseteq \mathcal{S}_p(G)^{<}$. Conversely, let $P \in \mathcal{A}_p(G)$ be an elementary abelian p-subgroup, and P seems to be a vector space over $\mathbb{Z}_p$. Note that $\mathcal{S}_p(G)_{<P} = \mathcal{S}_p(P)_{<P}$, and this set is a bundle of families of proper subspaces of the vector space P. Hence, from the Solomon-Tits theorem, $|\Delta(\mathcal{S}_p(G)_{<P})|$ is homotopy equivalent to some spherical bouquet or empty set. In any case, it is non-contractible. Thus, $\mathcal{A}_p(G) \subseteq \mathcal{S}_p(G)^{<}$.

Corollary 2.5. $|\Delta(\mathcal{A}_p(G))|$ is homotopy equivalent to $|\Delta(\mathcal{S}_p(G))|$.

Needless to say, the Quillen complex $\Delta(\mathcal{A}_p(G))$ is characterized by the homotopy term ‘contractible’ in the Brown complex $\Delta(\mathcal{S}_p(G))$. The following question naturally arises.

Problem 2.6. Is there any other p-subgroup complex like this?

There are some known p-subgroup complexes on this problem. For example, there is the following p-subgroup poset by Bouc:

$$\mathcal{B}_p(G) := \{P \in \mathcal{S}_p(G) \,|\, P = O_p(N_G(P))\},$$

where the symbol $O_p(N_G(P))$ denotes the maximal normal p-subgroup of the normalizer $N_G(P)$ of P in G. As can be seen from its definition, it contains the set $\mathrm{Syl}_p(G)$ of all Sylow p-subgroups of a finite group G. The order complex $\Delta(\mathcal{B}_p(G))$ is called a *p-radical complex*, or *Bouc complex*.

Now, let $C_G(P)$ be the centralizer of P in G. Then we consider that p-subgroup poset $\mathcal{C}_p(G) := \{P \in \mathcal{S}_p(G) \,|\, \text{any } p\text{-element of } C_G(P) \text{ belongs to } P\}$. The order complex of this poset is called a *p-centric complex*: We write the intersection of $\mathcal{B}_p(G)$ and $\mathcal{C}_p(G)$ by the symbol $\mathcal{B}_p^{cen}(G)$. That is,

$$\mathcal{B}_p^{cen}(G) := \mathcal{B}_p(G) \cap \mathcal{C}_p(G).$$

The geometric realization $|\Delta(\mathcal{B}_p^{cen}(G))|$ of the order complex $\Delta(\mathcal{B}_p^{cen}(G))$ is homotopy equivalent to the geometric realization $|\Delta(\mathcal{C}_p(G))|$, but it is not homotopy equivalent to $|\Delta(\mathcal{B}_p(G))|$ in general [11].

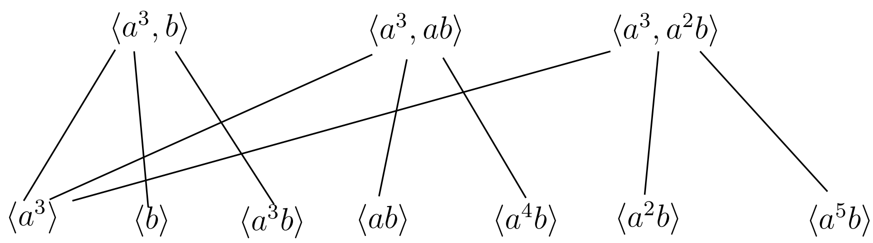
3 Examples of p-subgroup posets

We shall present some examples of p-subgroup posets.

Example 3.1. Take $G = D_6$, the dihedral group of order 12, and $p = 2$. We can give the abstract presentation of G by the generators and relations:

$$D_6 = \langle a, b \,|\, a^6 = b^2 = 1,\ b^{-1}ab = a^{-1} \rangle;$$

where these represent a rotation and a reflection, when D_6 is regarded concretely as the group of a regular hexagon. We find three Sylow 2-subgroups of order 4: $\langle a^3, b\rangle$, $\langle a^3, ab\rangle$, $\langle a^3, a^2b\rangle$, and the minimal members are generated by 7 involutions: $\langle a^3\rangle$, $\langle b\rangle$, $\langle ab\rangle$, $\langle a^2b\rangle$, $\langle a^3b\rangle$, $\langle a^4b\rangle$, $\langle a^5b\rangle$. Thus the poset diagram for $S_2(D_6)(= \mathcal{A}_2(D_6))$ is given by:

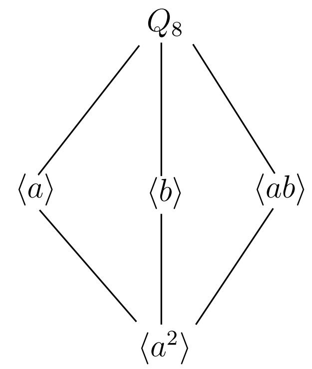


Observe that each of three Sylow 2-subgroups is not the normal subgroup of D_6 and the center $Z(D_6)$ of D_6 equals $\langle a^3\rangle$. Therefore $B_2(D_6) = \{\langle a^3, b\rangle,\ \langle a^3, ab\rangle,\ \langle a^3, a^2b\rangle,\ \langle a^3\rangle\}$. When $p = 3$, we easily have $\mathcal{S}_3(D_6) = \mathcal{A}_3(D_6) = \mathcal{B}_3(D_6) = \{\langle a^2\rangle\}$.

Example 3.2. Take $G = Q_8$, the quaternion group of order 8, and $p = 2$. We can give the abstract presentation of G by the generators and relations:

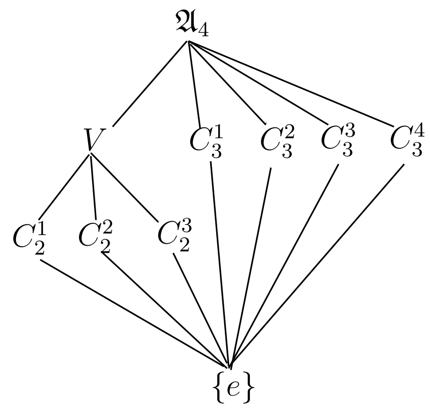
$$G = \langle a, b \,|\, a^4 = 1,\ b^2 = a^2,\ b^{-1}ab = a^{-1} \rangle.$$

We find three Sylow 2-subgroups of order 4: $\langle a \rangle$, $\langle b \rangle$, $\langle ab \rangle$, and each of these three Sylow 2-subgroups contains the unique cyclic subgroup $\langle a^2 \rangle$. Thus the poset diagram for $S_2(Q_8)$ is given by:



Since any subgroup of Q_8 is normal, so that $B_2(Q_8) = \{Q_8\}$. Note that $\mathcal{A}_2(Q_8) = \{\langle a^2 \rangle\}$.

Example 3.3. Take $G = \mathfrak{A}_4$, the alternative group of letter 4. We find one Sylow 2-subgroup of order 4 and four Sylow 3-subgroups of order 3. The subgroups diagram for $\mathfrak{A}_4$ is given by:



Here V is a Klein group, each $C_3^i\,(i = 1, 2, 3, 4)$ is a distinct cyclic group of order 3, and each $C_2^j\,(j = 1, 2, 3)$ is a distinct cyclic group of order 2. Then $\mathcal{S}_2(\mathfrak{A}_4) = \mathcal{A}_2(\mathfrak{A}_4) = \{V, C_2^1, C_2^2, C_2^3\}$ and $B_2(\mathfrak{A}_4) = \{V\}$.Furthermore, $\mathcal{S}_3(\mathfrak{A}_4) = \mathcal{A}_3(\mathfrak{A}_4) = B_3(\mathfrak{A}_4) = \{C_3^1, C_3^2, C_3^3, C_3^4\}$.

4 Finite T_0-spaces

A *finite space* is simply a finite set with a topology, and any finite space is homotopy equivalent to a finite T_0-space (namely, for any two distinct points, there exists a finite space which is an open neighborhood of one of the finite space that has one open neighborhood that does not contain the other). Hence, homotopy-theoretically, finite space theory is sufficient to target a finite T_0-space. Furthermore, since a finite T_0-space is in correspondence with a finite poset, both $S_p(G)$ and $A_p(G)$ are finite T_0 spaces. A finite space X is weakly homotopy equivalent to the corresponding compact polyhedron $|\Delta(X)|$. Here, *weak homotopy equivalence* means that the homotopy groups of every dimension are isomorphic. Let α be a point in $|\Delta(X)|$, that is, $\alpha = t_1x_1 + t_2x_2 + \cdots + t_rx_r$ where $\sum_{i=1}^{r} t_i = 1, t_i > 0$ for every $1 \leq i \leq r$ and $x_1 < x_2 < \cdots < x_r$ is a chain of X. Then $\mu_X : |\Delta(X)| \to X$ by $\mu_X(\alpha) = x_1$ is a weak homotopy equivalence. Then we obtain the following commutative diagram:

$$\begin{array}{ccc} |\Delta(X)| & \xrightarrow{|\Delta(f)|} & |\Delta(Y)| \\ \mu_X \big\downarrow & & \big\downarrow \mu_Y \\ X & \xrightarrow{f} & Y, \end{array}$$

where f is a continuous map between finite T_0 spaces X and Y, and $|\Delta(f)|$ is also a continuous map between polyhedra $|\Delta(X)|$ and $|\Delta(Y)|$. These were first discovered by McCord [7]. In the above diagram, f is a weak homotopy equivalence if and only if $|\Delta(f)|$ is a homotopy equivalence. Let us consider an application to p-subgroup complexes. Let $X = \mathcal{A}_p(G), Y = \mathcal{S}_p(G)$ and $f = \iota(=$ inclusion map). Then ι is a weak homotopy equivalence. Stong found that $|\Delta(\iota)| : |\Delta(\mathcal{A}_p(G))| \to |\Delta(\mathcal{S}_p(G))|$ is a homotopy equivalence. However the homotopy type of the Quillen complex or the Brown complex is still unknown. His result is the following.

Proposition 4.1. ([13]) A finite T_0-space $\mathcal{S}_p(G)$ is contractible if and only if $O_p(G)$ is non-trivial, where $O_p(G)$ denotes the maximal normal p-subgroup of G.

Therefore, the Quillen conjecture can be rephrased as follows: If $|\Delta(\mathcal{S}_p(G))|$ is contractible, then the finite T_0-space $\mathcal{S}_p(G)$ is contractible. From a topological point of view, this problem deals with a difference between weak homotopy equivalence and homotopy equivalence. We consider a *finite T_0-G-space* which is a finite T_0-space with a G-action. In particular, for a p-subgroup poset, we give the G-action by conjugation.

5 Main results

We shall recall the Bouc poset $\mathcal{B}_p(G) = \{P \in \mathcal{S}_p(G) \mid P = O_p(N_G(P))\}$. Note that $\mathcal{B}_p(G)$ contains $\mathrm{Syl}_p(G)$. Then we obtain the following.

Theorem 5.1. If G is a nilpotent group, then $|\Delta(\mathcal{B}_p(G))|$ is contractible.

Proof. By assumption, $\mathrm{Syl}_p(G)$ is the set consisting of a unique element. By Sylow's theorem, the Bouc poset $\mathcal{B}_p(G)$ has the maximal element. The finite T_0-space $\mathcal{B}_p(G)$ is contractible, and so $|\Delta(\mathcal{B}_p(G))|$ is contractible.

Corollary 5.2. Let p and q be prime numbers such that $p > q$. Let G be a finite group whose order is pq. Then $|\Delta(\mathcal{B}_p(G))|$ is contractible.

Proof. Similarly the proof of the above theorem, $\mathrm{Syl}_p(G)$ is the set consisting of a unique element.

Theorem 5.3. A finite T_0-space $\mathcal{B}_p(G)$ is contractible if and only if $O_p(G)$ is non-trivial.

Proof. If $O_p(G)$ is non-trivial, the Bouc poset $\mathcal{B}_p(G)$ has the minimal element. Thus a finite T_0-space $\mathcal{B}_p(G)$ is contractible. Conversely, if a finite T_0-space $\mathcal{B}_p(G)$ is contractible, $\mathcal{B}_p(G)$ has the G-invariant core which consists of a unique element. Its element is the normal subgroup of G.

Let X be a finite T_0-G-space and let x and y be points of X. If $x \in U_y$, then $gx \in gU_y = U_{gy}$. Therefore a G-action on a finite T_0-space X preserves the order. Thus $\Delta(X)$ is a G-simplicial complex (in short, G-complex). Let $\mathbb{N}_0$ be the union set of natural numbers $\{1, 2, 3, \cdots\}$ and $\{0\}$. A *CW*-complex Z with a G-action is called a *G-CW-complex* if it satisfies the following conditions:
(i) The G-action determines a cellular map, that is, for any $g \in G$, $gZ^i \subset Z^i$ for each $i \in \mathbb{N}_0$, where Z^i denotes the union of cells of dimension $\leq i$ and is called the *i-skeleton* of Z.
(ii) Let e is a cell of Z. If $g(e) = e$, then g is trivial on $\overline{e}$, that is, $Z^g \supset \overline{e}$, where $\overline{e}$ is the closure of e.

We obtained the following theorem.

Theorem 5.4. ([3]) Let X be a finite T_0-G-space. Then $|\Delta(X)|$ is a finite G-*CW*-complex.

Remark that p-sugroup posets $\mathcal{S}_p(G), \mathcal{A}_p(G)$ and $\mathcal{B}_p(G)$ are G-posets by conjugation. Therefore their order complexes are G-complexes. The three polyhedra $|\Delta(\mathcal{S}_p(G))|, |\Delta(\mathcal{A}_p(G))|$ and $|\Delta(\mathcal{B}_p(G))|$ are all G-*CW*-complexes.

Next we shall state the result about orbit space. We also got the following.

Theorem 5.5. ([3]) Let X be a finite T_0-G-space. Then $|\Delta(X)|/G$ is homotopy equivarent to $|\Delta(X)/G|$.

Since the orbit map $p : X \to X/G$ is continuous, it is an order-preserving map. It determines a simplicial map $\Delta(p) : \Delta(X) \to \Delta(X/G)$ and also a continuous map $|\Delta(p)| : |\Delta(X)| \to |\Delta(X/G)|$. Noting that $|\Delta(X/G)|$ is a G-space with a trivial G-action, we have a continuous map $\tilde{p} : |\Delta(X)|/G \to |\Delta(X/G)|$ such that the following diagram commutes:

$$\begin{array}{ccc} |\Delta(X)| & & \\ {\scriptstyle q}\downarrow & \searrow^{|\Delta(p)|} & \\ |\Delta(X)|/G & \xrightarrow[\tilde{p}]{} & |\Delta(X/G)|\,, \end{array}$$

where q is the orbit map from $|\Delta(X)| \to |\Delta(X)|/G$. By McCord's result, there is a weak homotopy equivalence $\mu_X : |\Delta(X)| \to X$. Then μ_X determines a continuous map $\tilde{\mu}_X : |\Delta(X)|/G \to X/G$ such that the following diagram commutes:

$$\begin{array}{ccc} |\Delta(X)|/G & \xrightarrow{\tilde{p}} & |\Delta(X/G)| \\ & \searrow_{\tilde{\mu}_X} & \downarrow{\scriptstyle \mu_{X/G}} \\ & & X/G\,. \end{array}$$

Therefore $\tilde{p}$ is a weak homotopy equivalence if and only if $\tilde{\mu}_X$ is ao. In general, $\tilde{\mu}_X$ is not a weak homotopy equivalence. According to Symond's result [14], $|\Delta(\mathcal{S}_p(G))|/G$ is contractible.

Let us have a terminology: a topological space is *homotopically contractible* if and only if all its homotopy groups are trivial.

Theorem 5.6. If G is a nilpotent group. If $|\Delta(\mathcal{B}_p(G))|/G$ is homotopically contractible, it is contractible. Then $|\Delta(\mathcal{B}_p(G))/G|$ is also contractible.

Proof. Let $\{*\}$ be a finite space consisting of a single point. Since $\mathcal{B}_p(G)$ is contractible, $\mathcal{B}_p(G)/G$ is also so. Hence we have a diagram:

$$\begin{array}{ccccc} |\Delta(\mathcal{B}_p(G))|/G & \xrightarrow{\tilde{p}} & |\Delta(\mathcal{B}_p(G)/G)| & \xrightarrow{|\Delta(f)|} & |\Delta(\{*\})| \\ & \searrow_{\tilde{\mu}_{\mathcal{B}_p(G)}} & \downarrow{\scriptstyle \mu_{\mathcal{B}_p(G)/G}} & & \downarrow{\scriptstyle \mu_{\{*\}}} \\ & & \mathcal{B}_p(G)/G & \xrightarrow{f} & \{*\}\,, \end{array}$$

where f is a homotopy equivalence. By McCord's theorem, there exists a homotopy equivalence $|\Delta(f)| : |\Delta(\mathcal{B}_p(G)/G)| \to |\Delta(\{*\})|$. Therefore

$|\Delta(\mathcal{B}_p(G)/G)|$ is contractible. By assumption, $\tilde{\mu}_{\mathcal{B}_p(G)}$ is a weak homotopy equivalence, and so $\tilde{p}$ is a weak homotopy equivalence. Now, since both $|\Delta(\mathcal{B}_p(G))|/G$ and $|\Delta(\mathcal{B}_p(G)/G)|$ are CW-complexes, $\tilde{p}$ is homotopy equivalence. Thus $|\Delta(\mathcal{B}_p(G))|/G$ is contractible. Then it follows immediately from theorem 5.5, $|\Delta(\mathcal{B}_p(G))/G|$ is also contractible.

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