ARCHIMEDEAN WHITTAKER FUNCTIONS ON GL_n VIA THETA LIFTING

YAO CHENG

1. Introduction

This article is based on the author's talk at the conference "Arithmetic Aspects of Automorphic Forms and Automorphic Representations," held at RIMS, Kyoto University, from January 20 to 24, 2025. We extend our gratitude to K. Namikawa for the kind invitation to this outstanding conference. The talk was based on joint work with Shih-Yu Chen, in which we investigated Archimedean Whittaker functions on GL_n using the method of theta lifting.

1.1. **Classical Whittaker functions.** The classical Whittaker function $W_{\alpha,\beta}(z)$ is a special solution to the Whittaker equation:

$$\frac{d^2W}{dz^2} + \left(-\frac{1}{4} + \frac{\alpha}{z} + \frac{\frac{1}{4} - \beta^2}{z^2}\right) W = 0 \tag{1.1}$$

which was introduced by Whittaker around 1903 ([Whi03]).

If 2β is not an integer, a fundamental system of solutions [1] of (1.1) consists of the functions $M_{\alpha,\beta}(z)$, $M_{\alpha,-\beta}(z)$, and

$$W_{\alpha,\beta}(z) = \frac{\Gamma(-2\beta)}{\Gamma\left(\frac{1}{2} - \alpha - \beta\right)} M_{\alpha,\beta}(z) + \frac{\Gamma(2\beta)}{\Gamma\left(\frac{1}{2} - \alpha + \beta\right)} M_{\alpha,-\beta}(z)$$

which is the unique (up to constant) solution of (1.1) that decreases rapidly as $z \to \infty$.

Whittaker functions have applications in various fields, including number theory. For instance, if $f(z)$ is a Maass cups form of weight k with the eigenvalue $\frac{1-s^2}{4}$, its Fourier expansion at the ∞ cusp is given by

$$f\left(x + \sqrt{-1}y\right) = \sum_{0 \neq n \in \mathbb{Z}} a_n \, W_{\frac{k}{2},\frac{s}{2}} \left(4\pi \, |n| \, y\right) \, e^{2\pi\sqrt{-1}\, nx}$$

for some coefficients $a_n \in \mathbb{C}$.

1.2. **Whittaker functions on** GL_2**.** Later, Jacquet ([Jac67]) generalized this concept to Whittaker functions associated with reductive groups over local fields. The functions studied by Whittaker correspond to the specific case where the reductive group is SL_2 and the local field is $\mathbb{R}$. To illustrate, consider the closely related group $G_2 = \mathrm{GL}_2(\mathbb{R})$. Fix an additive character $\psi : \mathbb{R} \to \mathbb{C}^\times$ defined by $\psi(x) = e^{2\pi\sqrt{-1}\,x}$.

Let $N_2 \subset G_2$ denote the subgroup of upper triangular unipotent matrices, and let $\psi_2 : N_2 \to \mathbb{C}^\times$ be the character defined by

$$\psi_2\left(\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\right) = \psi(b).$$

[1] Essentially the confluent hypergeometric functions of the first kind.

Let $\mathcal{C}^\infty(N_2\backslash G_2,\psi_2)^{\rm mg}$ denote the space of **moderate growth** smooth functions $W: G_2\to\mathbb{C}$ satisfying

$$W(ug)=\psi_2(u)W(g)\quad\text{for}\quad (u,g)\in N_2\times G_2.$$

Note that G_2 acts on $\mathcal{C}^\infty(N_2\backslash G_2,\psi_2)$ via the right translation.

Let π be an irreducible smooth representation of G_2 in the sense of Casselman-Wallach (cf. [Cas89]). It is well-known (cf. [JL70, Theorem 5.13]) that the dimension of the space

$$\operatorname{Hom}_{G_2}\left(\pi,\mathcal{C}^\infty(N_2\backslash G_2,\psi_2)^{\rm mg}\right)$$

is at most one. The representation π is said to be **generic** if this space is non-zero. In this case, the underlying space of π can be realized as a subspace $\mathscr{W}(\pi,\psi_2)$ of $\mathcal{C}^\infty(N_2\backslash G_2,\psi_2)^{\rm mg}$, called the **Whittaker model** of π, on which G_2 acts via the right translation ρ.

In many applications, there is significant interest in deriving explicit formulas for Whittaker functions within the subspace $\mathscr{W}(\pi,\psi_2)_\infty$ consisting of $\mathrm{O}_2(\mathbb{R})$-finite elements. By the Iwasawa decomposition for G_2 and the $\mathrm{O}_2(\mathbb{R})$-finiteness assumption, it suffices to determine the values of these $\mathrm{O}_2(\mathbb{R})$-finite Whittaker functions W on the diagonal elements, referred to as the A_2-**radial** part of W. Here A_2 denotes the diagonal torus of G_2.

Suppose that π is a **spherical** principal series. Then π is generic and is determined by two complex numbers s_1 and s_2 with $s:=s_1-s_2\notin 1+2\mathbb{Z}$. The subspace $\mathscr{W}(\pi,\psi_2)_\infty$ is spanned by the elements $0\neq W_k\in\mathscr{W}(\pi,\psi_2)$ for $k\in 2\mathbb{Z}$, characterized by

$$\rho(r_\theta)W_k=e^{\sqrt{-1}\,k\theta}\,W_k\quad\text{with}\quad r_\theta:=\begin{pmatrix}\cos\theta&\sin\theta\\-\sin\theta&\cos\theta\end{pmatrix}. \tag{1.2}$$

The element W_0 is referred to as a spherical Whittaker function.

We determine the A_2-radial part of W_k for $k\in 2\mathbb{Z}$ using two different methods. The first method involves solving differential equations; while the second utilizes theta lifting. Interestingly, both approaches are discussed in the book by Jacquet-Langlands ([JL70]).

1.2.1. *First method.* Define

$$w_k(t)=W_k\left(\begin{pmatrix}t\,|t|^{-\frac12}&0\\0&|t|^{-\frac12}\end{pmatrix}\right)\quad\text{for}\quad t\in\mathbb{R}^\times.$$

Using the action of the Casimir element (cf. [JL70, Theorem 5.13]), the function $w_k(t/2)$ satisfies the Whittaker equation (1.1) with $\alpha=\frac{k}{2}$, $\beta=\frac{s}{2}$ and $z=t$. Since w_k is of moderate growth, we conclude that

$$w_k(t)=W_{\frac{k}{2},\frac{s}{2}}(2t).$$

This result also justifies the term "Whittaker model".

1.2.2. *Second method.* Let ω_ψ denote the Weil representation of the group $\mathrm{SL}_2(\mathbb{R})\times\mathrm{O}_{1,1}(\mathbb{R})$ acting on the space $\mathscr{S}(\mathbb{R}^2)$ of Bruhat-Schwartz functions on $\mathbb{R}^2$. In particular, we have

$$\omega_\psi\left(\begin{pmatrix}1&b\\0&1\end{pmatrix}\right)\varphi(x,y)=\psi(bxy)\varphi(x,y)\quad\text{for}\quad\varphi\in\mathscr{S}(\mathbb{R}^2). \tag{1.3}$$

Furthermore, the action of ω_ψ can be extended from $\mathrm{SL}_2(\mathbb{R})$ to G_2 by the formula:

$$\omega_\psi\left(\begin{pmatrix}a&0\\0&1\end{pmatrix}\right)\varphi(x,y)=\varphi(ax,y) \tag{1.4}$$

for $a\in\mathbb{R}^\times$ and $\varphi\in\mathscr{S}(\mathbb{R}^2)$ (cf. [JL70, Chapter 1]).

Given $\varphi \in \mathscr{S}(\mathbb{R}^2)$ and $g \in G_2$, define

$$W_\varphi(g) = |\det(g)|^{s_1+\frac{1}{2}} \int_{\mathbb{R}^\times} \omega_\psi(g)\varphi(t,t^{-1})\,|t|^s\,d^\times t.$$

It is clear that the integral defining $W_\varphi(g)$ converges absolutely. Moreover, by (1.3), it is also easy to check that W_φ lies in $\mathcal{C}^\infty(N_2\backslash G_2,\psi_2)^{\mathrm{mg}}$. However, a non-trivial fact is that the space $\mathscr{W}(\pi,\psi_2)$ consists precisely of the functions W_φ for $\varphi \in \mathscr{S}(\mathbb{R}^2)$ (cf. [JL70, Theorem 5,13]).

By the result just mentioned, to determine the functions W_k for $k \in 2\mathbb{Z}$, it suffices to find $\varphi_k \in \mathscr{S}(\mathbb{R}^2)$ satisfying:

$$\omega_\psi(r_\theta)\,\varphi_k = e^{\sqrt{-1}\,k\theta}\,\varphi_k \quad \text{and} \quad W_{\varphi_k} \neq 0. \tag{1.5}$$

To achieve this, define the partial Fourier transform:

$$\varphi(x,y)^\sim = \int_{\mathbb{R}} \varphi(x,u)\psi(yu)du \quad \text{for} \quad \varphi \in \mathscr{S}(\mathbb{R}^2). \tag{1.6}$$

Here du is the standard Lesbesgue measure on $\mathbb{R}$. Then the following identity holds:

$$(\omega_\psi(g)\varphi)^\sim = \rho(g)\varphi^\sim \quad \text{for} \quad g \in G_2.$$

Here ρ denotes the right translation action of G_2 on $\mathscr{S}(\mathbb{R}^2)$.

The partial Fourier transform is considered because it can be easily verified that

$$\rho(r_\theta)\,\Phi_k = e^{\sqrt{-1}\,k\theta}\,\Phi_k, \quad \text{where} \quad \Phi_k(x,y) := \left(x+\sqrt{-1}\,y\right)^k e^{-\pi(x^2+y^2)}.$$

Now if $\varphi_k \in \mathscr{S}(\mathbb{R}^2)$ is the function such that $\varphi_k^\sim = \Phi_k$, then the first requirement in (1.5) is satisfied. It remains to show that $W_{\varphi_k} \neq 0$; this can also be done. For more details, see [Che21, Appendix].

Note that φ_k can be explicitly computed. It then follows from the formula (1.4) that the values

$$W_{\varphi_k}\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right) \quad \text{for} \quad a \in \mathbb{R}^\times,$$

can also be explicitly determined.

Remark 1.1. Although more complicated, it is still feasible to utilize these two methods to explicitly determine the A_2-radial part of $\mathrm{U}_2(\mathbb{R})$-finite Whittaker functions of generic representations of $\mathrm{GL}_2(\mathbb{C})$ (cf. [Jac72, Section 18], [Che21, Appendix], [HIM22]).

1.3. **Whittaker functions on** GL_n**.** The initial works extending beyond the case of GL_2 appear to be those of Vinogradov-Takhtajan ([Tak82]), Proskurin ([Pro85]), and Bump ([Bum84]). They derived explicit integral formulas for spherical Whittaker functions associated with spherical principal series representations of $\mathrm{GL}_3(\mathbb{R})$, either by evaluating Jacquet's integral (cf. [Jac67]) or by solving the corresponding differential equations.

Further investigations into spherical Whittaker functions on $\mathrm{GL}_n(\mathbb{R})$ were subsequently developed in the works of Stade and Ishii ([Sta90], [Sta01], [IS07]). In particular, Ishii and Stade developed a recursive relation, known as the **propagation formula** connecting spherical Whittaker functions on $\mathrm{GL}_n(\mathbb{R})$ and $\mathrm{GL}_{n-1}(\mathbb{R})$. By computing the Whittaker functions associated with Godement sections, Ishii and Miyazaki derived a propagation formula connecting highest weight Whittaker function in the minimal K-type (in the sense of Vogan) on GL_{n+1} to that on GL_n for non-spherical representations in [IM22].

The problem becomes significantly more challenging for non-spherical generic representations. This difficulty arises in part from the increased complexity of explicitly describing the decomposition of the $\mathrm{O}_n(\mathbb{R})$-finite part of an irreducible representation of $\mathrm{GL}_n(\mathbb{R})$ as an $\mathrm{O}_n(\mathbb{R})$-module. Additionally, the structure of non-trivial irreducible representations of $\mathrm{O}_n(\mathbb{R})$ further complicates the analysis.

For $\mathrm{GL}_3(\mathbb{R})$ or $\mathrm{GL}_3(\mathbb{C})$, this problem is addressed in the works of Hirano-Oda ([HO09]), and Hirano-Ishii-Miyazaki ([HIM12], [HIM22]). For $\mathrm{GL}_4(\mathbb{R})$, this problem is tackled in the recent preprint by Hirano-Ishii-Miyazaki ([HIM24]). To derive explicit formulas for the Whittaker function, they analyzed a system of partial differential equations satisfied by the Whittaker functions, among other techniques. These formulas have practical applications, such as computing certain zeta integrals, as demonstrated in *loc. cit.*.

In this article, we present alternative approach to attacking this problem for arbitrary n. The idea is to apply the theta lifting from GL_n to GL_{n+1}. As a result, we obtain a propagation formula relating the Whittaker functions on GL_{n+1} in the minimal K-type to those on GL_n. Notably, part of our method aligns with the approach of Ishii and Miyazaki in [IM22], but theta lifting offers us a new perspective. Furthermore, we use these formulas to explicitly compute the local zeta integrals associated with Asai L-functions. We hope this alternative perspective will shed some light on this problem.

2. Notation and Conventions

2.1. Let $F = \mathbb{R}$ or $\mathbb{C}$. If $F = \mathbb{R}$, we denote by $|x|_F$ the ordinary absolute value; while if $F = \mathbb{C}$, we set $|x|_F = x\bar{x}$. Fix a non-trivial additive character ψ of F given by $\psi(x) = e^{2\pi i x}$ when $F = \mathbb{R}$, and $\psi(x) = e^{2\pi i(x+\bar{x})}$ when $F = \mathbb{C}$. The space of n-by-m matrices with entries in F is denoted by $M_{n,m}$.

2.2. Let $n > 0$ be an integer; we put $G_n = \mathrm{GL}_n(F)$. Denote by $B_n = A_n N_n \subset G_n$ the upper triangular Borel subgroup, where A_n is Levi part of B_n, consisting of diagonal matrices; whereas N_n is the unipotent radical of B_n, consisting of upper triangular unipotent matrices. The character ψ can be extended to $\psi_n : N_n \longrightarrow \mathbb{C}^\times$ by

$$u = (u_{i,j}) \mapsto \psi(u_{1,2} + u_{2,3} + \cdots + u_{n-1,n}).$$

Let $K_n = \mathrm{O}_n(\mathbb{R})$ or $\mathrm{U}_n(\mathbb{R})$ depending on whether $F = \mathbb{R}$ or $\mathbb{C}$, respectively, so that K_n is the standard maximal compact subgroup of G_n.

2.3. By a representation of G_n we will always mean a smooth admissible Fréchet representation of moderate growth in the sense of Casselman-Wallach (cf. [Cas89]). We often abuse notation by using the same letter to denote both the representation and the space on which it acts.

2.4. Let π be a representation of G_n. A Whittaker functional of π (with respect to ψ_n) is a continuous functional $\lambda_{\psi_n} : \pi \to \mathbb{C}$ that satisfies

$$\lambda_{\psi_n}(\pi(u)v) = \psi_n(u)\lambda_{\psi_n}(v) \quad \text{for} \quad u \in N_n,\ v \in \pi.$$

If the space of Whittaker functionals on π is one-dimensional, we say that π is generic, and denote by $\mathscr{W}(\pi, \psi_n)$ the Whittaker model of π, which consists of the functions $W_v(g) = \lambda_{\psi_n}(\pi(g)v)$ for $g \in G_n$ and $v \in \pi$, with G_n acting by the right translation ρ. It is well-known that when π is irreducible, the space of Whittaker functionals on π is at most one-dimensional.

Let $\mathcal{C}^\infty(N_n\backslash G_n, \psi_n)^{\mathrm{mg}}$ denote the space of smooth functions $W : G_n \to \mathbb{C}$ that are of moderate growth and satisfy $W(ug) = \psi_n(u)W(g)$ for $u \in N_n$ and $g \in G_n$. Since π is of moderate growth, it follows that $\mathscr{W}(\pi, \psi_n) \subset \mathcal{C}^\infty(N_n\backslash G_n, \psi_n)^{\mathrm{mg}}$.

3. Theta Correspondence

3.1. Let $n, m > 0$ be integers. Then (G_n, G_m) forms a type II irreducible reductive dual pair in the sense of Howe ([How79]). The Weil representation $\omega_{n,m}$ of $G_n \times G_m$ on the the space $\mathscr{S}(M_{n,m})$ of Bruhat-Schwartz functions on $M_{n,m}$ is given by

$$\omega_{n,m}(h, g)\varphi(x) = |\det(h)|_F^{-\frac{m}{2}} |\det(g)|_F^{\frac{n}{2}} \varphi(h^{-1}xg).$$

Now let π be an irreducible representation of G_n. Assume that π appears as a quotient of $\omega_{n,m}$. Then a famous result of Howe ([How89]) asserts that there is a unique irreducible representation $\sigma := \theta_{n,m}(\pi)$ of G_m such that

$$\mathrm{Hom}_{G_n\times G_m}(\omega_{n,m}, \pi \widehat{\boxtimes} \sigma) \neq 0. \tag{3.1}$$

Here $\widehat{\boxtimes}$ denotes the completed projective tensor product. As the same assertion holds if we start with an irreducible representation σ of G_m, the condition (3.1) establishes a bijection between the sets of irreducible representations of G_n and G_m occurring as quotients of $\omega_{n,m}$.

Remark 3.1. When $(n, m) = (1, 2)$, the Weil representation $\omega_{1,2}$ described here is isomorphic to ω_ψ introduced in §1.2. The intertwining map is essentially realized by the partial Fourier transform defined in (1.6).

3.2. A more delicate problem is to describe this bijection explicitly, namely, to determine which π occurs as a quotient of $\omega_{n,m}$, and for such a π, to describe $\theta_{n,m}(\pi)$ explicitly. The term "explicitly" can have several interpretations. Ideally, the bijection could be computed in terms of some parameterization of representations, such as the Langlands classification or its variations. However, other descriptions are possible. For the dual pair (G_n, G_m), this problem was addressed by Adams and Barbasch in [AB95] and [Ada07].

In this article, we are particularly interested in the case $m = n + 1$, which we assume from this point onward. In this case, we also write

$$\omega = \omega_{n,n+1}, \quad \theta(\pi) = \theta_{n,n+1}(\pi), \quad \text{and} \quad \mathscr{S} = \mathscr{S}(M_{n,n+1}).$$

To describe $\theta(\pi)$, let

$$P_{n,1} = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in G_m \mid a \in G_n, d \in G_1 \right\}$$

be a maximal parabolic subgroup of G_{n+1}, and define an irreducible representation $\pi\widehat{\boxtimes}1$ of $P_{n,1}$ by

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mapsto \pi(a)|d|_F.$$

We then have the normalized induced representation

$$\pi \times 1 := \mathrm{Ind}_{P_{n,1}}^{G_{n+1}}(\pi\widehat{\boxtimes}1)$$

of G_{n+1}. Now we have the following result:

Theorem 3.2 (Admas-Barbasch)**.** *Every irreducible representation π of G_n occurs as a quotient of ω, and $\theta(\pi)$ is the unique subquotient of $\pi^\vee \times 1$ containing the minimal K-type of $\pi^\vee \times 1$, where $\pi^\vee$ is the contragredient representation of π.*

3.3. One approach to study $\theta(\pi)$ is to explicitly construct a non-zero element in the (one-dimensional) space (3.1). When π is generic, this was achieved by Watanabe in [Wat95], where he computed the Fourier coefficients of a global theta lift of a cusp form. In particular, he obtained a map

$$V : \mathscr{S} \otimes \mathscr{W}(\pi, \psi_n) \longrightarrow \mathcal{C}^\infty(N_{n+1}\backslash G_{n+1}, \psi_{n+1})^{\mathrm{mg}}$$

which is given by the following absolutely convergent integral:

$$V_{\varphi\otimes W}(g) = \int_{N_n\backslash G_n} \int_{N_{n+1}} \omega(h, ug)\varphi(\varepsilon_n)W(h)\psi_{n+1}^{-1}(u)dudh \tag{3.2}$$

where

$$\varepsilon_n := (I_n, 0) \in M_{n,n+1}.$$

It is easy to check that

$$V_{\omega(h,g)\varphi\otimes\rho(h)W} = \rho(g)V_{\varphi\otimes W} \quad \text{for} \quad (h, g) \in G_n \times G_{n+1}. \tag{3.3}$$

It follows that G_{n+1} acts on the space

$$\mathscr{V}(\pi, \psi_{n+1}) := \left\{ V_{\varphi\otimes W} \mid \varphi \in \mathscr{S},\ W \in \mathscr{W}(\pi, \psi_n) \right\} \subset \mathcal{C}^\infty(N_{n+1}\backslash G_{n+1}, \psi_{n+1})^{\mathrm{mg}}$$

by the right translation, and the spaces $\mathscr{V}(\pi, \psi_{n+1})$ and $\mathscr{W}(\theta(\pi^\vee), \psi_{n+1})$ should, in some way, be related. Actually, we have the following key result:

Theorem 3.3 (Chen-C). *Assume that* $\pi \times 1$ *is irreducible. Then* $\mathscr{V}(\pi \times 1, \psi_{n+1}) = \mathscr{W}(\pi, \psi_{n+1})$.

Roughly, the proof of Theorem 3.3 relies on computing the Whittaker functions associated to Godement sections, Theorem 3.2 and the fact that the space of Whittaker functionals on the "full theta lift" $\Theta(\pi^\vee)$ of $\pi^\vee$ is one-dimensional.

4. Propagation Formulas

4.1. After discussing the image of the theta lift, our next goal is to obtain a propagation formula for the Whittaker functions using the theta lift. More precisely, let π be an irreducible generic representation of G_n, as before. We assume that $\Pi := \pi \times 1$ is irreducible. Note that Π is also generic.

Let τ (resp. Υ) denote the minimal K-type of π (resp. Π), which occurs in π (resp. Π) with multiplicity one. Consequently, the subspace

$$(\mathscr{W}(\pi, \psi_n) \otimes \tau^\vee)^{K_n} = \mathbb{C} \cdot W_{\pi[\tau]} \tag{4.1}$$

of K_n-fixed vectors is one-dimensional. The basis $W_{\pi[\tau]}$ can be regarded as a $\tau^\vee$-valued function on G_n satisfying:

- $W_{\pi[\tau]}(gk) = \tau^\vee(k)^{-1}W_{\pi[\tau]}(g)$ for $(g, k) \in G_n \times K_n$;
- the $\mathbb{C}$-valued function $g \mapsto \langle v, W_{\pi[\tau]}(g)\rangle_\tau$ belongs to $\mathscr{W}(\pi, \psi_n)$ for every $v \in \tau$, where $\langle\cdot,\cdot\rangle_\tau : \tau \times \tau^\vee \to \mathbb{C}$ denotes the K_n-invariant bilinear pairing.

Moreover, $W_{\pi[\tau]}$ is uniquely characterized by these two properties.

Similarly, we also have

$$(\mathscr{W}(\Pi, \psi_{n+1}) \otimes \Upsilon^\vee)^{K_{n+1}} = \mathbb{C} \cdot W_{\Pi[\Upsilon]}$$

with $W_{\Pi[\Upsilon]}$ satisfying analogous properties. By the Iwasawa decomposition, it is clear that $W_{\pi[\tau]}$ (resp. $W_{\Pi[\Upsilon]}$) is completely determined by its values on A_n^+ (resp. A_{n+1}^+). Here $A_n^+ \subset A_n$ denotes the subgroup of diagonal matrices with all diagonal entries being positive real numbers.

4.2. We aim to express $W_{\Pi[\Upsilon]}$ in terms of $W_{\pi[\tau]}$ using the equation (3.2). To achieve this, let $\mathscr{H} \subset \mathscr{S}$ denote the subspace of joint harmonic in the sense of Howe (cf. [How89, Section 7 (b)]). This is a $(K_n \times K_{n+1})$-invariant subspace which admits a multiplicity-free decomposition:

$$\mathscr{H} \cong \bigoplus_i \mathscr{H}_{\tau_i,\Upsilon_i} \cong \bigoplus_i \tau_i \boxtimes \Upsilon_i.$$

Moreover, for each i, τ_i and Υ_i determine each other, i.e.

$$\dim_{\mathbb{C}} \mathrm{Hom}_{K_n \times K_{n+1}} \left(\mathscr{H}_{\tau_i,\Upsilon_i}, \mathscr{H}_{\tau_j,\Upsilon_j} \right) = \delta_{\tau_i,\tau_j} \cdot \delta_{\Upsilon_i,\Upsilon_j},$$

and they share the same degree defined by Howe.

Now the key observation is that $\tau^\vee \boxtimes \Upsilon$ occurs in $\mathscr{H}$ (cf. [Ada07, Proposition 7.5]). This follows from the irreducibility assumption on Π and the genericity assumption on π. As a result, the space

$$(\mathscr{H} \otimes (\tau \boxtimes \Upsilon^\vee))^{K_n \times K_{n+1}} = \mathbb{C} \cdot \varphi_{\tau^\vee,\Upsilon} \tag{4.2}$$

is one-dimensional, and the basis $\varphi_{\tau^\vee,\Upsilon}$ can be regarded as a $(\tau \otimes \Upsilon^\vee)$-valued function on $M_{n,n+1}$, whose coefficients are elements in $\mathscr{H}_{\tau^\vee,\Upsilon}$, and satisfying the property:

$$\omega(k,k')\varphi_{\tau^\vee,\Upsilon} = (\tau \boxtimes \Upsilon^\vee)(k,k')^{-1}\varphi_{\tau^\vee,\Upsilon} \quad \text{for} \quad (k,k') \in K_n \times K_{n+1}. \tag{4.3}$$

Here, $\omega(k,k')$ acts on the coefficients of $\varphi_{\tau^\vee,\Upsilon}$.

At this point, consider the $\Upsilon^\vee$-valued function $V_{\varphi_{\tau^\vee,\Upsilon}\otimes W_{\pi[\tau]}}$ on G_{n+1} defined by the integral:

$$V_{\varphi_{\tau^\vee,\Upsilon}\otimes W_{\pi[\tau]}}(g) = \int_{N_n\backslash G_n} \int_{N_{n+1}} \langle \omega(h,ug)\varphi_{\tau^\vee,\Upsilon}(\varepsilon_n), W_{\pi[\tau]}(h) \rangle_\tau \psi_{n+1}^{-1}(u) du dh.$$

Since the coefficients are given by the integral (3.2), which converges absolutely, the function $V_{\varphi_{\tau^\vee,\Upsilon}\otimes W_{\pi[\tau]}}$ is well-defined. Moreover, we have the following main result:

Theorem 4.1 (Chen-C)**.** *The function* $V_{\varphi_{\tau^\vee,\Upsilon}\otimes W_{\pi[\tau]}}$ *is non-zero and belongs to*

$$(\mathscr{W}(\Pi,\psi_{n+1}) \otimes \Upsilon^\vee)^{K_{n+1}}.$$

Hence, we can identify

$$W_{\Pi[\Upsilon]} = V_{\varphi_{\tau^\vee,\Upsilon}\otimes W_{\pi[\tau]}}.$$

Moreover, we have the following propagation formula:

(4.4)

$$W_{\Pi[\Upsilon]}(g) = c|\det(g)|_F^{\frac{n}{2}} \int_{A_n} \int_{N_{n+1}} \langle \varphi_{\tau^\vee,\Upsilon}(t^{-1}\varepsilon_n ug), W_{\pi[\tau]}(t) \rangle_\tau \psi_{n+1}^{-1}(u) \delta_{B_n}^{-1}(t) |\det(t)|_F^{-\frac{n+1}{2}} du dt,$$

where $c = \mathrm{Vol}(K_n, dk)$ *and* δ_{B_n} *is the modulus functions of* B_n*.*

Remark 4.2. By the formula (4.4), the Whittaker function $W_{\Pi[\Upsilon]}$ can be computed inductively. The problem is thus reduced to finding the test function $\varphi_{\tau^\vee,\Upsilon}$, and explicitly evaluating the integral. We note that the space of joint harmonic can be explicitly defined in terms of the Fock model of ω. This fact may assist in determining $\varphi_{\tau^\vee,\Upsilon}$.

5. Examples

In this section, we apply the propagation formula (4.4) to explicitly compute the Whittaker functions in the minimal K-type of certain representations of $\mathrm{GL}_n(\mathbb{C})$. So let $F = \mathbb{C}$ in this section.

5.1. For $\nu \in \mathbb{C}$ and $\kappa \in \mathbb{Z}$, let $\chi_{[\nu,\kappa]}$ denote the character of $\mathbb{C}^\times$ defined by

$$\chi_{[\nu,\kappa]}(z) = |z|_{\mathbb{C}}^{\nu} \left(\frac{z}{\sqrt{|z|_{\mathbb{C}}}} \right)^{\kappa}.$$

For $\underline{\nu} = (\nu_1, \cdots, \nu_n) \in \mathbb{C}^n$, define

$$|\underline{\nu}| = \nu_1 + \cdots + \nu_n \quad \text{and} \quad \tilde{\nu} = (\nu_2 - \nu_1, \cdots, \nu_n - \nu_1) \in \mathbb{C}^{n-1}.$$

For $\underline{\nu} = (\nu_1, \cdots, \nu_n) \in \mathbb{C}^n$ and $\underline{\kappa} = (\kappa_1, \cdots, \kappa_n) \in \mathbb{Z}^n$, let $\pi_{[\underline{\nu},\underline{\kappa}]}$ denote the principal series representation of G_n defined by

$$\pi_{[\underline{\nu},\underline{\kappa}]} = \chi_{[\nu_1,\kappa_1]} \times \cdots \times \chi_{[\nu_n,\kappa_n]}.$$

5.2. When $\underline{\kappa} = 0$, we write $\pi_{\underline{\nu}} = \pi_{[\underline{\nu},0]}$. In this case, we have the following Mellin–Barnes integral representation of a normalized spherical Whittaker function $W_{\underline{\nu}} \in \mathscr{W}(\pi_{\underline{\nu}}, \psi_n)^{K_n}$ due to Ishii and Stade (cf. [Sta95, Proposition 2.1], [IS07, Theorem 12]): The formula is inductively represented. For $a = (a_1, \cdots, a_{n-1}) \in \mathbb{R}_+^{n-1}$, let $t(a) = \mathrm{diag}(\alpha_1, \cdots, \alpha_n) \in T_n^+$ with $\alpha_i = a_i a_{i+1} \cdots a_{n-1}$ for $1 \le i \le n-1$ and $\alpha_n = 1$. Define $f_{\underline{\nu}} : \mathbb{R}_+^{n-1} \to \mathbb{C}$ by

$$f_{\underline{\nu}}(a) = \delta_{B_n}(t(a))^{-1/2} W_{\underline{\nu}}(t(a)).$$

Then $f_{\underline{\nu}}(a) = a^{\nu_1+\nu_2} K_{\nu_1-\nu_2}(4\pi a)$ for $n = 2$, and

$$\begin{aligned} f_{\underline{\nu}}(a) = &\int_{z_1,\cdots,z_{n-2}} \frac{dz}{(2\pi\sqrt{-1})^{n-2}} (\mathcal{M} f_{\underline{\tilde{\nu}}})(z_1, \cdots, z_{n-2}) \\ &\times \int_{s_1,\cdots,s_{n-1}} \frac{ds}{(2\pi\sqrt{-1})^{n-1}} \prod_{i=1}^{n-2} a_i^{-s_i} \Gamma_{\mathbb{C}}\left(\frac{s_i - z_{i-1}}{2} + i\nu_1 \right) \Gamma_{\mathbb{C}}\left(\frac{s_i - z_i}{2} + i\nu_1 \right) \\ &\times a_{n-1}^{-s_{n-1}} \Gamma_{\mathbb{C}}\left(\frac{s_{n-1} - z_{n-2}}{2} + (n-1)\nu_1 \right) \Gamma_{\mathbb{C}}\left(\frac{s_{n-1}}{2} + |\underline{\tilde{\nu}}| + (n-1)\nu_1 \right) \end{aligned} \tag{5.1}$$

for $n \ge 3$, where we put $z_0 = 0$, and $\mathcal{M} f_{\underline{\tilde{\nu}}}$ is the Mellin transform of $f_{\underline{\tilde{\nu}}}$. Moreover, the path of integration in each z_i or s_j being a vertical line in the complex plane, of sufficiently large real part. The formula (5.1) of Ishii–Stade can be obtained by using the propagation formula (4.4).

5.3. More generally, assume $\pi_{[\underline{\nu},\underline{\kappa}]}$ is irreducible and

$$\underline{\kappa} = (\kappa, 0, \cdots, 0) \quad \text{with} \quad \kappa \ge 0. \tag{5.2}$$

We use the propagation formula (4.4) to deduce a Mellin–Barnes type integral representations for Whittaker functions of $\pi_{[\underline{\nu},\underline{\kappa}]}$ in the minimal K_n-type.

Let ρ_κ be the minimal K_n-type of $\pi_{[\underline{\nu},\underline{\kappa}]}$. Then it has $(\kappa, 0, \cdots, 0)$ as its highest weight. The action of ρ_κ is recalled as follows: Let $\mathbb{C}[X_1, \cdots, X_n]_\kappa$ be the space of homogeneous complex polynomials of degree κ in variables $X_1, \cdots, X_n$. We write $X = (X_1, \cdots, X_n)$ as row vector variable. Then K_n acts on $\mathbb{C}[X_1, \cdots, X_n]_\kappa$ by

$$(\rho_\kappa(k)P)(X) := P(Xk) \quad \text{for} \quad k \in K_n,\, P \in \mathbb{C}[X_1, \cdots, X_n]_\kappa. \tag{5.3}$$

5.4. Now we use Theorem 4.1 to define a generator

$$W_{[\underline{\nu},\underline{\kappa}]} \in \left(\mathscr{W}(\pi_{[\underline{\nu},\underline{\kappa}]}, \psi_n) \otimes \overline{\rho}_\kappa\right)^{K_n}.$$

To do so, let $\pi := \pi_{[\underline{\tilde{\nu}},(-\kappa,\cdots,-\kappa)]}$ and $\Pi := \pi \times 1$. Then we have

$$\pi_{[\underline{\nu},\underline{\kappa}]} = \Pi \otimes \chi_{[\nu_1,\kappa]}.$$

Let Υ and τ be the minimal K_n-type and K_{n-1}-type of Π and π respectively. Note that

$$\Upsilon = \rho_\kappa \otimes \det{}^{-\kappa} \quad \text{and} \quad \tau = \det{}^{-\kappa}.$$

The distinguished Bruhat-Schwartz function $\varphi_{\tau^\vee,\Upsilon}$ in (4.2) is defined by

$$\varphi_{\tau^\vee,\Upsilon}(x) = (-1)^{\kappa(n-1)} 2^{4(n-1)} \cdot \det \begin{pmatrix} X \\ \overline{x} \end{pmatrix}^{\kappa} e^{-2\pi \mathrm{tr}(x^t \overline{x})}.$$

It is clear that $\varphi_{\tau^\vee,\Upsilon}$ satisfies (4.3).

We normalized the Whittaker function $W_{\pi[\tau]}$ in (4.1) (with n replaced by $n-1$) by

$$W_{\pi[\tau]} := W_{\underline{\tilde{\nu}}} \otimes \chi_{[0,-\kappa]}.$$

By Theorem 4.1, we have a generator

$$W_{\Pi[\Upsilon]} := V_{\varphi_{\tau^\vee,\Upsilon} \otimes W_{\pi[\tau]}} \in \left(\mathscr{W}(\Pi, \psi_n) \otimes \overline{\Upsilon}\right)^{K_n}.$$

We then define

$$W_{[\underline{\nu},\underline{\kappa}]} := W_{\Pi[\Upsilon]} \otimes \chi_{[\nu_1,\kappa]}.$$

5.5. For $\underline{\ell} = (\ell_1, \cdots, \ell_n) \in \mathbb{Z}_{\geq 0}^n$ with $\ell_1 + \cdots + \ell_n = \kappa$, we write $X^{\underline{\ell}} = X_1^{\ell_1} \cdots X_n^{\ell_n}$. It is clear that the weight vectors in the minimal K_n-type of $\pi_{[\underline{\nu},\underline{\kappa}]}$ are indexed by these $\underline{\ell}$. In the Whittaker model of $\pi_{[\underline{\nu},\underline{\kappa}]}$, we normalize these weight vectors $W_{[\underline{\nu},\underline{\kappa}],\underline{\ell}}$ so that

$$W_{[\underline{\nu},\underline{\kappa}]} = \sum_{\ell_1+\cdots+\ell_n=\kappa} \frac{\kappa!}{\ell_1! \cdots \ell_n!} (\sqrt{-1})^{\sum_{i=1}^{n-1} \tilde{\ell}_i} X^{\underline{\ell}} \cdot W_{[\underline{\nu},\underline{\kappa}],\underline{\ell}}. \tag{5.4}$$

Here $\tilde{\ell}_i := \ell_1 + \cdots + \ell_i$.

For each $\underline{\ell}$ with $\ell_1 + \cdots + \ell_n = \kappa$, define $f_{[\underline{\nu},\underline{\kappa}],\underline{\ell}} : \mathbb{R}_+^{n-1} \to \mathbb{C}$ by

$$f_{[\underline{\nu},\underline{\kappa}],\underline{\ell}}(a) = \delta_{B_n}(t(a))^{-1/2} W_{[\underline{\nu},\underline{\kappa}],\underline{\ell}}(t(a)).$$

The we have the following result:

Theorem 5.1 (Chen-C)**.** *Let $\underline{\ell} = (\ell_1, \cdots, \ell_n) \in \mathbb{Z}_{\geq 0}^n$ with $\ell_1 + \cdots + \ell_n = \kappa$. We have*

$$\begin{aligned} & f_{[\underline{\nu},\underline{\kappa}],\underline{\ell}}(a_1, \cdots, a_{n-1}) \\ &= \int_{z_1,\cdots,z_{n-2}} \frac{dz}{(2\pi\sqrt{-1})^{n-2}} (\mathcal{M} f_{\underline{\tilde{\nu}}})(z_1, \cdots, z_{n-2}) \\ &\quad \times \int_{s_1,\cdots,s_{n-1}} \frac{ds}{(2\pi\sqrt{-1})^{n-1}} \prod_{i=1}^{n-2} a_i^{-s_i} \Gamma_{\mathbb{C}}\left(\frac{s_i + \kappa - \tilde{\ell}_i - z_{i-1}}{2} + i\nu_1\right) \Gamma_{\mathbb{C}}\left(\frac{s_i + \tilde{\ell}_i - z_i}{2} + i\nu_1\right) \\ &\quad \times a_{n-1}^{-s_{n-1}} \Gamma_{\mathbb{C}}\left(\frac{s_{n-1} + \kappa - \tilde{\ell}_{n-1} - z_{n-2}}{2} + (n-1)\nu_1\right) \Gamma_{\mathbb{C}}\left(\frac{s_{n-1} + \tilde{\ell}_{n-1}}{2} + |\underline{\tilde{\nu}}| + (n-1)\nu_1\right). \end{aligned}$$

Here we put $z_0 = 0$ and $\tilde{\ell}_i = \ell_1 + \cdots + \ell_i$ for $1 \leq i \leq n-1$. The path of integration in each z_i or s_j being a vertical line in the complex plane, of sufficiently large real part.

6. Asai local zeta integrals

In this section, we use the Mellin–Barnes type integral representation to compute the local zeta integrals introduced by Flicker ([Fli93]) and Kable ([Kab04]).

6.1. Let E/F be a quadratic extension of local fields of characteristic zero, and $c \in \mathrm{Gal}(E/F)$ the non-trivial automorphism. Let ψ_F be a non-trivial additive character of F and $\psi_E = \psi_F \circ \mathrm{Tr}_{E/F}$. Let π be an irreducible admissible generic representation of $G_n(E)$. For $\varphi \in \mathscr{S}(M_{1,n}(F))$ and $W \in \mathscr{W}(\pi, \psi_E)$, we have the local zeta integral

$$Z(s,\varphi,W) = \int_{N_n(F)\backslash G_n(F)} \varphi(e_n g) W\left(\mathrm{diag}(\delta^{n-1},\delta^{n-2},\cdots,1)g\right) |\det(g)|_F^s \, dg. \tag{6.1}$$

Here $e_n = (0,\cdots,0,1)$ and $\delta \in E$ is a fixed element with $\mathrm{Tr}_{E/F}(\delta) = 0$.

The integral converges absolutely for $\Re(s)$ sufficiently large and admits meromorphic continuation to the whole complex plane (cf. [BP21, Theorem 1-(i)], see also [Fli93, Appendix] and [Kab04, Theorem 2] when F is non-archimedean).

6.2. Let $\phi_\pi : WD_E \to \mathrm{GL}_n(\mathbb{C})$ be the Langlands parameter of π. Denote by $\mathrm{As}(\phi_\pi) : WD_F \to \mathrm{GL}_{n^2}(\mathbb{C})$ the multiplicative induction from WD_E to WD_F (cf. [Pra92, § 7]). Then the Asai L-factor $L(s,\pi,\mathrm{As})$ associated to π is defined by

$$L(s,\pi,\mathrm{As}) := L(s,\mathrm{As}(\phi_\pi)).$$

In particular, if π is a principal series representation

$$\pi = \chi_1 \times \cdots \times \chi_n,$$

then we have (cf. [Pra92, Lemma 7.1-(d)])

$$L(s,\pi,\mathrm{As}) = \prod_{i=1}^n L(s,\chi_i|_{F^\times}) \prod_{1\le i<j\le n} L(s,\chi_i\chi_j^c). \tag{6.2}$$

6.3. It is expected that $L(s,\pi,\mathrm{As})$ is a "greatest common divisor" of the local zeta integrals. More precisely, we expect that there exist finitely many φ_i and W_i such that

$$L(s,\pi,\mathrm{As}) = \sum_i Z(s,\varphi_i,W_i).$$

When F is non-archimedean, this is proved by Matringe in [Mat11, Theorem 5.3]. When $F = \mathbb{R}$, as far as the authors are aware, no relevant results have been proved in the existing literature. In this section, we apply the result in the previous section to compute the local zeta integral (6.1).

6.4. Let $\pi = \pi_{[\underline{\nu},\underline{\kappa}]}$ satisfying (5.2). Recall the minimal K_n-type ρ_κ of $\pi_{[\underline{\nu},\underline{\kappa}]}$ whose model is described in (5.3). Let $\langle\cdot,\cdot\rangle_{\rho_\kappa}$ be a K_n-invariant hermitian pairing on $\rho_\kappa \times \rho_\kappa$ normalized so that

$$\langle X_n^\kappa, X_n^\kappa\rangle_{\rho_\kappa} = 1.$$

Define the Bruhat-Schwartz function $\varphi_\kappa \in \mathscr{S}(M_{1,n}(\mathbb{R})) \otimes \rho_\kappa$ by

$$\varphi_\kappa(x) = (x{}^tX)^\kappa \, e^{-\pi x {}^t x}.$$

Note that we have

$$\varphi_\kappa(xk) = (\rho_\kappa(k)^{-1}\varphi_\kappa)(x) \quad \text{for} \quad k \in \mathrm{O}_n(\mathbb{R}). \tag{6.3}$$

Let $W_{[\underline{\nu},\underline{\kappa}]}$ be the $\overline{\rho}_\kappa$-valued Whittaker function of $\pi_{[\underline{\nu},\underline{\kappa}]}$ normalized in (5.4). The coefficients of $W_{[\underline{\nu},\underline{\kappa}]}$ are weight vectors in the minimal K_n-type ρ_κ. In Theorem 6.1 below, we compute the local zeta integral

$$\begin{aligned} &Z(s,\varphi_\kappa,W_{[\underline{\nu},\underline{\kappa}]}) \\ &:= \int_{N_n(\mathbb{R})\backslash G_n(\mathbb{R})} \left\langle \varphi_\kappa(e_n g),\, W_{[\underline{\nu},\underline{\kappa}]}\left(\mathrm{diag}((\sqrt{-1})^{n-1},(\sqrt{-1})^{n-2},\cdots,1)g\right)\right\rangle_{\rho_\kappa} |\det(g)|_{\mathbb{R}}^s\,dg. \end{aligned}$$

This is a finite sum of local zeta integrals of the form (6.1).

Theorem 6.1 (Chen-C)**.** *We have*

$$Z(s,\varphi_\kappa,W_{[\underline{\nu},\underline{\kappa}]}) = 2^{n(n-2)-4}\frac{\Gamma_{\mathbb{R}}(s+2\nu_1+\kappa)}{\Gamma_{\mathbb{R}}(s+2\nu_1+\epsilon)}\cdot L(s,\pi_{[\underline{\nu},\underline{\kappa}]},\mathrm{As}).$$

Here $\epsilon\in\{0,1\}$ *such that* $\kappa\equiv\epsilon\,(\mathrm{mod}\,2)$.

Remark 6.2. Based on the results for $n=2,3$, we expect that

$$Z(s,\varphi'_\kappa,W_{[\underline{\nu},\underline{\kappa}]}) = c\cdot L(s,\pi_{[\underline{\nu},\underline{\kappa}]},\mathrm{As})$$

for some non-zero constant c, where

$$\varphi'_\kappa(x) = (X^tX)^{\frac{\kappa}{2}}\,e^{-\pi x^t x} \quad\text{or}\quad \varphi'_\kappa(x) = (x^tX)\,(X^tX)^{\frac{\kappa-1}{2}}\,e^{-\pi x^t x}$$

depending on whether κ is even or odd, respectively. Unfortunately, we are unable to verify this expection at present.

References

[AB95] J. Adams and D. Barbasch. Reductive dual pair correspondence for complex groups. *Journal of Functional Analysis*, 132:1–42, 1995.

[Ada07] J. Adams. The theta correspondence over $\mathbb{R}$. In Jian-Shu Li, Eng-Chye Tan, Nolan Wallach, and Chen-Bo Zhu, editors, *Harmonic Analysis, Group Representations, Automorphic Forms And Invariant Theory*, volume 12 of *Lecture note series, Institute for Mathematical Sciences, National University of Singapore*. World Scientific, 2007.

[BP21] R. Beuzart-Plessis. Archimedean theory and ϵ-factor for the Asai Rankin-Selberg integrals. In *Relative Trace Formulas*, pages 1–50. Springer International Publishing, 2021.

[Bum84] D. Bump. *Automorphic Forms on* $\mathrm{GL}(3,\mathbb{R})$, volume 1083 of *Lecture Notes in Mathematics*. Springer Berlin Heidelberg, 1984.

[Cas89] W. Casselman. Canonical extensions of Harish-chandra modules to representations of G. *Canadian Journal of Mathematics*, 41(3):385–438, 1989.

[Che21] Y. Cheng. Special value formula for the twisted triple product L-function and an application to the restricted L^2-norm problem . *Forum Mathematicum*, 33:59–108, 2021.

[Fli93] Y. Flicker. On zeros of the twisted tensor L-function. *Mathematische Annalen*, 297(2):199–219, 1993.

[HIM12] M. Hirano, T. Ishii, and T. Miyazaki. The archimedean Whittaker functions on $GL(3)$. In *Geometry and analysis of automorphic forms of several variables*, volume 7, pages 77–109, 2012.

[HIM22] M. Hirano, T. Ishii, and T. Miyazaki. *Archimedean zeta integrals for* $GL(3)\times GL(2)$, volume 278. Memoirs of the American Mathematical Society, 2022.

[HIM24] M. Hirano, T. Ishii, and T. Miyazaki. Whittaker functions on $\mathrm{GL}(4,\mathbb{R})$ and archimedean Bump-Friedberg integrals. *arXiv:2409.00401v1*, 2024.

[HO09] M. Hirano and T. Oda. Calculus of principal series Whittaker functions on $GL(3,\mathbb{C})$. *Journal of Functional Analysis*, 256(7):2222–2267, 2009.

[How79] R. Howe. θ-series and invariant theory. In *Automorphic forms, representations, and L-functions, Part 1*, volume 33, pages 275–285. Preceedings of Symposium in Pure Mathematics, Oregon State University, 1979.

[How89] R. Howe. Transcending classical invariant theory. *Journal of American Mathematics Society*, 2(3):535–552, 1989.

[IM22] T. Ishii and T. Miyazaki. Calculus of archimedean Rankin-Selberg integrals with recursive relations. *Representation Theory*, 26:714–763, 2022.

[IS07] T. Ishii and E. Stade. New formulas for Whittaker functions on $GL(n,\mathbb{R})$. *Journal of Functional Analysis*, 244(1):289–314, 2007.

[Jac67] H. Jacquet. Fonctions de Whittaker associées aux groupes de Chevalley. *Bulletin de la Société Mathématique de France*, 95:243–309, 1967.

[Jac72] H. Jacquet. *Automorphic Forms on* GL(2) II, volume 278. Springer-Verlag, Berlin and New York, 1972.

[JL70] H. Jacquet and R. Langlands. *Automorphic Forms on* GL(2), volume 114 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin and New York, 1970.

[Kab04] A Kable. Asai L-functions and Jacquet's conjecture. *American Journal of Mathematics*, 126(4):789–820, 2004.

[Mat11] N. Matringe. Derivatives and asymptotics of Whittaker functions. *Representation Theory*, 15:646–669, 2011.

[Mœg89] C. Mœglin. Correspondance de Howe pour les paires reductives duales: quelques calculs dans le cas archimédien. *Journal of Functional Analysis*, 85(1):1–85, 1989.

[Pra92] D. Prasad. Invariant linear forms for representations of GL(2) over a local field. *American Journal of Mathematics*, 114:1317–1363, 1992.

[Pro85] N. V. Proskurin. Automorphic functions and the Bass-Milnor-Serre homomorphism, I, II. *Journal of Soviet Mathematics*, 29(2):1160–1191; 1192–1219, 1985.

[Sta90] E. Stade. On explicit integral formulas for $GL(n,\mathbb{R})$-Whittaker functions. *Duke Mathematical Journal*, 60(2):313–362, 1990.

[Sta95] E. Stade. Mellin transforms of Whittaker functions on $GL(4,\mathbb{R})$ and $GL(4,\mathbb{C})$. *Manuscripta Mathematica*, 87:511–526, 1995.

[Sta01] E. Stade. Mellin transforms of $GL(n,\mathbb{R})$ Whittaker functions. *American Journal of Mathematics*, 123(1):121–161, 2001.

[Tak82] A. I. Vinogradovand L. A. Takhtadzhyan. Theory of Eisenstein series for the group $\mathrm{SL}(3,\mathbb{R})$ and its application to a binary problem. *Journal of Soviet Mathematics*, 18(3):293–324, 1982.

[Wat95] T. Watanabe. The local theta correspondence of irreducible type 2 dual reductive pairs. *Tohoku Mathmetical Journal*, 47:521–540, 1995.

[Whi03] E. T. Whittaker. An expression of certain known functions as generalized hypergeometric functions. *Bulletin of the American Mathematical Society*, 10:125–134, 1903.

No. 151, Yingzhuan Road, Tamsui District, New Taipei City 251, Taiwan (R.O.C), Lui-Hsien Memorial Science Hall.

Email address: briancheng@o365.tku.edu.tw